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Dynamics of quasi-geostrophic systems with cumulus convection

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ABSTRACT

The dynamic effects of a cumulus ensemble on a quasi-geostrophic weather system in the midlatitude region are examined, in addition to the heating due to release of latent heat. It has been found that the effects of the eddy vertical advection of momentum, and the eddy dynamic pressure due to the eddy kinetic energy associated with cumulus clouds are rather small. The only important dynamic effect comes from the ability of clouds to produce a horizontal eddy flux of vertical vorticity. In the momentum equation, this effect appears in the form of an eddy Coriolis force, which may be obtained from the horizontal eddy vorticity flux by rotating it by 90° clockwise. In the vorticity equation, it appears in the form of generation of positive vorticity by convective elements in the lower levels, and generation of negative vorticity in the upper levels.

Using Ekman-layer pumping theory as the closure condition, the effects of cumulus convection on baroclinic instability are investigated. It has been found that the growth rate of a baroclinically unstable wave is increased appreciably by the cloud scale processes.

1. Introduction

The effects of cumulus convection in midlatitude weather systems have been a rather controversial subject. Although undoubtedly most synoptic scale disturbances in midlatitude regions are baroclinic in origin, possible cumulus effects have been suggested. Results obtained in many studies, however, have been less than conclusive (e.g., Danard and Ellenton, 1980; Rasmussen, 1979). The extent of the influence of cumulus clouds in extratropical systems is not yet well understood.

One of the examples where cumulus convection may play a significant role is the phenomenon of cyclogenesis on the east coast of the North American continent. Rapidly deepening storms affecting eastern states of the U.S. and the maritime provinces of Canada are often observed during the late winter seasons. The amounts of precipitation observed in these storms are often as high as 35–40 mm/day (Danard and Ellenton, 1980), although it is not clear how much of it is produced by cumulus convection, and how much by stratiform clouds. One explanation for these cyclogeneses is heat and moisture input from the warm ocean surface, and subsequently induced moist convection because the patterns of heat exchange between water and air over the north Atlantic Ocean often resemble those of cyclone location and genesis (Bunker and Worthington, 1976; Reitan, 1974). However, results from some recent studies seem to suggest (e.g. Danard and Ellenton, 1980) that fluxes of sensible and latent heat from the ocean are of little consequence.

Another example of midlatitude weather systems in which cumulus convection may play a role is the phenomenon of polar lows. These are sub-synoptic scale disturbances usually observed in the polar air stream northward of the polar front away from the major baroclinic zone. They are characterized in their mature stage by a comma-shaped cloud pattern in satellite photographs. Two theories have been offered to explain the dynamic origin of these disturbances. Based on a number of studies of these disturbances over the northeast Atlantic, it has been suggested (Harold and Browning, 1969;

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Mansfield, 1974; Duncan, 1977) that polar lows are basically shallow baroclinic waves with precipitation produced mainly in stratiform clouds associated with the so-called "slantwise convection" within a narrow tongue of moist air ascending steadily in the disturbance. On the other hand, Rasmussen (1979) suggested that polar lows are thermal instability phenomena and can be explained by the CISK mechanism. Recently, Reed (1979) found from a number of cases that polar lows over the north Pacific have considerable vertical extent. Furthermore, the comma-shape cloud patterns appear to evolve from regions of convection in which the elements have become larger and deeper than in the surroundings. Although the structures of these disturbances appear to indicate they are baroclinic in origin, possible influence by cumulus convection cannot be ruled out.

In the past, most of the studies of cumulus effects in the midlatitude regions are based on thermodynamic considerations. Houze (1973) was one of the earlier investigators who first pointed out the importance of cumulus momentum transport. Recently, a number of diagnostic studies have been reported, indicating the possible importance of cumulus convection in the kinetic energy budgets of extratropical synoptic scale disturbances (Ward and Smith, 1976; Fuelberg and Scoggins, 1978; Vincent and Schlatter, 1979; Robertson and Smith, 1980).

The purpose of this paper is to examine, in addition to thermodynamic considerations, the possible dynamic processes through which cumulus convection may influence the larger-scale flow. As a first attempt, the large-scale flow in which clouds are embedded is assumed to be quasi-geostrophic. This will place certain limitations on the applicability of the results to smaller-scale systems. Because of the lack of a proper closure condition to determine cloud amount, a level of cloud activity will also be assumed in the analysis. This is not completely satisfactory since the amount of cloud activity is likely to vary considerably from one system to another. However, the amount of cloud activity characterized by a few cloud ensemble parameters will be specified at as reasonable a level as possible, and, as will become clear in the analysis, the conclusions will be applicable to any system in which the amount of cloud is not larger than the specified cloud amount by one order of magnitude.

The organization of the paper is as follows. The dynamic processes through which cumulus clouds may influence the large-scale flow will first be examined in Section 2. The quasi-geostrophic omega and tendency equations, including cumulus effects, will then be derived and discussed in Section 3. Using the formulation derived, a baroclinic instability analysis will be made in Section 4 in the framework of a two-level model to illustrate the magnitude of the influence of convection. The paper will then be concluded by a few remarks in Section 5.

2. Governing equations

The effects of cumulus convection on large-scale flow have been studied extensively for tropical systems. Simple representations of several cloud effects have been derived. Although these representations were derived originally for the tropical atmosphere, they apply equally well to midlatitude systems. The major limitations of these theories are:

(1) they represent the effects of convective clouds only;
(2) the fractional area covered by active cores of clouds is about 10% or less.

Neither of these two conditions will be a serious restraint for the purpose of this investigation.

Condition (1) implies that the effect of latent heat release realized in stratiform clouds is not included. Although this form of clouds is very important in midlatitude weather systems, no attempts will be made to treat its effects separately because in this study we are concerned only with the effects of convective clouds.

From observations it has been found that the second condition is satisfied in tropical systems. It is likely to be true in midlatitude systems as well. Using the slice method to determine the criterion of conditional instability, one can show that the fractional area occupied by the active cores of convective clouds cannot exceed 10–20% of the total area, depending on the thermal stratification of the atmosphere.

The total effects of cumulus clouds are proportional to the cloud amount. Before we begin to examine the various possible effects of convection on quasi-geostrophic systems, we shall specify as a reference point in our discussions a set of values for certain parameters of the cumulus ensemble embedded in the larger-scale system. While the values of these parameters that we choose will be reasonable, they are not meant to be typical. There may be more convective clouds in some systems, and may be considerably less in some others. The effects of clouds should be scaled proportionally in each individual case. The reference values are meant only to give a reasonable order of magnitude for various processes.

Keeping this purpose in mind, we shall assume that the rate of precipitation P produced by cumulus clouds is of the order of 20 mm/day:

$$P \sim 20\ \text{mm/day}. \tag{1}$$

This corresponds to a mean tropospheric heating rate of the order of

$$H \sim 5\ ^{\circ}\text{C/day}. \tag{2}$$

The fractional area covered by active cloud (Σ) will be assumed to be

$$\Sigma \sim \text{a few \%}. \tag{3}$$

The active lifetime (τ) for individual clouds is estimated to be

$$\tau \sim 1\ \text{h} \sim 10^{3}\ \text{s}. \tag{4}$$

This gives the parameter Σ/τ, called the recycling rate, a value

$$\Sigma/\tau \sim 10^{-5}\ \text{s}^{-1}. \tag{5}$$

The total cloud mass flux M_c will be defined here as $M_c = \Sigma\bar{\rho} g \bar{w}_c$, where $\bar{w}_c$ is the mean cloud vertical velocity. From thermodynamic considerations to be discussed later, the mean heating rate given in (2) corresponds to a total cloud mass flux of the order of

$$M_c = 10^{-3}\ \text{mb s}^{-1}. \tag{6}$$

Using these values, the importance of various effects of clouds, especially those related directly to dynamic processes, will be estimated. We begin first with a brief discussion of the thermodynamic equation.

2.1. *The thermodynamic equation*

In terms of the potential temperature θ, the thermal dynamic equation can be written as

$$\frac{\partial\bar{\theta}}{\partial t} + \bar{\mathbb{V}}\cdot\nabla\bar{\theta} + \bar{\omega}\frac{\partial\bar{\theta}}{\partial p} = -\nabla\cdot\overline{\mathbb{V}'\theta'} - \frac{\partial\overline{\omega'\theta'}}{\partial p} + \frac{L}{C_p\pi}(\bar{C}-\bar{e}). \tag{7}$$

Here, clouds are treated as subscale eddies. The overbar denotes areal average. C and e are the rates of condensation and evaporation, respectively. L is the latent heat of vaporization of water and $\pi = (p/1000)^{R/C_p}$ is the non-dimensional pressure.

It is generally assumed that (e.g., Ooyama, 1971),

$$\nabla\cdot\overline{\mathbb{V}'\theta'} = 0. \tag{8}$$

Furthermore, it has been shown that (Ooyama, 1971; Arakawa and Schubert, 1974; Cho, 1977)

$$-\frac{\partial\overline{\omega'\theta'}}{\partial p} + \frac{L}{C_p\pi}(\bar{C}-\bar{e}) = -M_c\frac{\partial\bar{\theta}}{\partial p} \tag{9}$$

where M_c is the total cloud mass flux. The thermodynamic equation, including cloud effects, can therefore be written as

$$\frac{\partial\bar{\theta}}{\partial p} + \bar{\mathbb{V}}\cdot\nabla\bar{\theta} + \bar{\omega}\frac{\partial\bar{\theta}}{\partial p} = -M_c\frac{\partial\bar{\theta}}{\partial p}. \tag{10}$$

The effects of cloud in this formulation is represented completely in terms of M_c.

2.2. *The equation of motion*

The horizontal equation of motion may be written in either one of the following two forms:

$$\frac{\partial\mathbb{V}}{\partial t} + \mathbb{V}\cdot\nabla\mathbb{V} + \omega\frac{\partial\mathbb{V}}{\partial p} + f\hat{k}\times\mathbb{V} = -\nabla\Phi \tag{11}$$

$$\frac{\partial\mathbb{V}}{\partial t} + \nabla(\tfrac{1}{2}\mathbb{V}\cdot\mathbb{V} + \Phi) + \omega\frac{\partial\mathbb{V}}{\partial p} + \hat{k}\times\eta\mathbb{V} = 0. \tag{12}$$

In eq. (11), Φ is the geopotential and f is the Coriolis parameter. $\hat{k}$ is the vertical unit vector. In eq. (12), $\eta = f + \zeta$ is the absolute vorticity. $\zeta = \hat{k}\cdot\nabla\times\mathbb{V}$ is the relative vertical vorticity. The term $\hat{k}\times\eta\mathbb{V}$ may be interpreted as the Coriolis force in a local co-ordinate system rotating with the angular velocity of the air parcel. The term $\frac{1}{2}\mathbb{V}\cdot\mathbb{V}$ is referred to as the dynamic pressure.

Treating cumulus clouds as eddies, an area average of eq. (11) or (12) gives

$$\frac{\partial\bar{\mathbb{V}}}{\partial t} + \bar{\mathbb{V}}\cdot\nabla\bar{\mathbb{V}} + \bar{\omega}\frac{\partial\bar{\mathbb{V}}}{\partial p} + \hat{k}\times f\bar{\mathbb{V}} = -\nabla\bar{\Phi} - \nabla\tfrac{1}{2}\overline{\mathbb{V}'\cdot\mathbb{V}'} - \overline{\omega'\frac{\partial\mathbb{V}'}{\partial p}} - \hat{k}\times\overline{\mathbb{V}'\eta'}. \tag{13}$$

The eddy terms on the right-hand side of the equation describe the effects of clouds.

The large-scale system in which cumulus clouds are embedded will be assumed to have typical velocity and length scales given by

$$U \sim 10 \text{ m s}^{-1}, \quad L \sim 10^6 \text{ m}. \tag{14}$$

The Coriolis parameter will be assumed to be of the order

$$f \sim 10^{-4} \text{ s}^{-1}. \tag{15}$$

These give a Rossby number

$$R_o \sim 10^{-1}. \tag{16}$$

The Richardson number of the mean environment will be assumed to be such that

$$R_i R_o \gtrsim 10. \tag{17}$$

In other words, the large-scale system we are considering would be quasi-geostrophic if cumulus effects were not dominant. In such a system, the Coriolis force has a magnitude

$$f\bar{V} \sim 10^{-3} \text{ m s}^{-2} \tag{18}$$

and the inertial term

$$\frac{\mathrm{d}\bar{\mathbb{V}}}{\mathrm{d}t} = \frac{\partial \bar{\mathbb{V}}}{\partial t} + \bar{\mathbb{V}} \cdot \nabla \bar{\mathbb{V}} \sim 10^{-4} \text{ m s}^{-2}. \tag{19}$$

With the characteristic dimensions defined for the large-scale system, we will now evaluate the importance of eddy processes relative to the processes of the mean flow:

(i) *Eddy vertical advection of momentum.* From the definition of areal averaging, it can be shown (e.g. Ooyama, 1971) that the eddy vertical advection of momentum can be expressed as

$$\overline{\omega' \frac{\partial \mathbb{V}'}{\partial p}} = -M_c \frac{\partial}{\partial p} (\bar{\mathbb{V}}_c - \bar{\mathbb{V}}). \tag{20}$$

$\bar{\mathbb{V}}_c$ here is the mean cloud horizontal velocity, weighted with respect to cloud vertical velocity. If one assumes that the changes in $\bar{\mathbb{V}}_c$ and $\bar{\mathbb{V}}$, $\Delta\bar{\mathbb{V}}_c$ and $\Delta\bar{\mathbb{V}}$, respectively, across the depth of the troposphere are of the order of 10 m s^{-1}, and that $M_c \sim 10^{-3}$ mb s^{-1}, the eddy advection has a magnitude

$$\overline{\omega' \frac{\partial \mathbb{V}'}{\partial p}} \sim M_c(\Delta\bar{\mathbb{V}}_c - \Delta\bar{\mathbb{V}})/\Delta p \sim 10^{-5} \text{ m s}^{-2}. \tag{21}$$

It is two orders of magnitude smaller than the Coriolis force, and one order smaller than the inertia of the mean flow.

(ii) *Gradient of eddy dynamic pressure.* If one assumes that the cloud-induced velocity perturbations are confined only to areas occupied by clouds, the eddy dynamic pressure can be estimated according to

$$\overline{\tfrac{1}{2}\mathbb{V}' \cdot \mathbb{V}'} \sim \tfrac{1}{2}\Sigma\, (\bar{\mathbb{V}}_c - \bar{\mathbb{V}}) \cdot (\bar{\mathbb{V}}_c - \bar{\mathbb{V}}). \tag{22}$$

Assuming $\mathbb{V}_c \sim 10$ m s^{-1} and $\Sigma \sim 1\%$, the gradient of eddy dynamic pressure would have a magnitude given by

$$\begin{aligned} \nabla \overline{\tfrac{1}{2}\mathbb{V}' \cdot \mathbb{V}'} &\sim \tfrac{1}{2}\Sigma(\bar{\mathbb{V}}_c - \bar{\mathbb{V}}) \cdot (\bar{\mathbb{V}}_c - \bar{\mathbb{V}})/L \\ &\sim 10^{-6} \text{ m s}^{-2} \end{aligned} \tag{23}$$

which is two orders smaller than the inertia of mean flow and three orders smaller than the Coriolis force.

Unfortunately, cloud-induced wind fluctuations are not likely to be confined just to the cloud areas. Cloud-induced wind can often be observed far away from a cloud. This may be taken into account by replacing Σ in eq. (22) by an effective fractional coverage Σ_e. If one assumes again $\mathbb{V}_c \sim 10$ m s^{-1} and $\Sigma_e \sim 10\%$, the gradient of eddy dynamic pressure would still be one order smaller than the inertia of the mean large-scale flow. The eddy dynamic pressure can be important only if the cloud-induced wind perturbation has a mean value of 10 m s^{-1} over the entire large-scale area or, equivalently, only if $\Sigma_e \sim 100\%$. This is not very likely. Therefore, it appears reasonable to assume that

$$\nabla \overline{\tfrac{1}{2}\mathbb{V}' \cdot \mathbb{V}'} \lesssim 10^{-5} \text{ m s}^{-2}. \tag{24}$$

(iii) $-\hat{k} \times \overline{\mathbb{V}'\eta'}$: *the eddy Coriolis force.* This term may perhaps be referred to as the eddy Coriolis force. It represents a vector which is perpendicular to $\overline{\mathbb{V}'\eta'}$, which in turn can be interpreted as the horizontal eddy flux of vertical vorticity.

Cho and Cheng (1980) suggested in their attempt to explain the A/B-scale vorticity budget observed in GATE that cumulus clouds are capable of producing significant horizontal eddy flux of vorticity. The eddy flux is envisaged to occur through the action of the cloud-scale convergence on a vortex couplet, which has been produced by the vertical advection and tilting of an originally horizontal vortex tube associated with the background flow.

Because of the small scales of cumulus clouds, point values of vertical vorticity in a cloud tend to be very large. Both positive and negative vorticities can be found. They are usually arranged in the form of vortex couplets (Knopfli and Miller, 1976; Ray, 1976; Klemp and Wilhelmson, 1978a, b; Schlesinger, 1978). It has been shown that these vortex couplets are initiated through the tilting of horizontal vortex lines. Their magnitudes are comparable to that of vertical wind shear of the cloud environment, which is about $10^{-3}\ s^{-1}$.

As has been shown by Cho and Cheng (1980), the horizontal eddy flux of vorticity is produced entirely by the irrotational component of cloud-induced wind perturbations. The correlation between $\mathbb{V}'$ and η' needed to produce a horizontal eddy flux can be illustrated by superimposing a cloud-induced converging or diverging wind field on a vortex couplet, as shown in Fig. 1. If one assumes that the vertical wind shear vector of the cloud background flow is in the eastward direction, then the orientation of the vortex couplet should be mainly in the north–south direction. The eddy flux $\overline{\mathbb{V}'\eta'}$ in this case will be northward if there is a net convergence and will be southward if there is a net divergence of air out of clouds.

It is important to point out that this eddy flux of vorticity should not be considered as a real transport of vorticity over a large distance by the motion of air parcels induced by cumulus convection. The schematic illustration presented in Fig. 1 merely demonstrates the existence of a correlation between $\mathbb{V}'$ and ζ' that has to be taken into account when explaining the behaviour of the mean flow. Conceptually it is perhaps easier to consider this eddy flux in terms of the eddy Coriolis force $-\hat{k} \times \overline{\mathbb{V}'\eta'}$. Through the interaction of the vortex couplet with the irrotational wind field, a net local eddy Coriolis force is produced over the cloud area. This eddy Coriolis force should be parallel to the vertical wind shear vector at the cloud entrainment levels, and antiparallel to the wind shear at the cloud detrainment levels. Although this force is confined mainly to the areas occupied by the clouds, its effect on the mean flow has to be taken into account if one considers an area that includes a large number of cumulus clouds, and if the eddy Coriolis force in each cloud is rather large in magnitude.

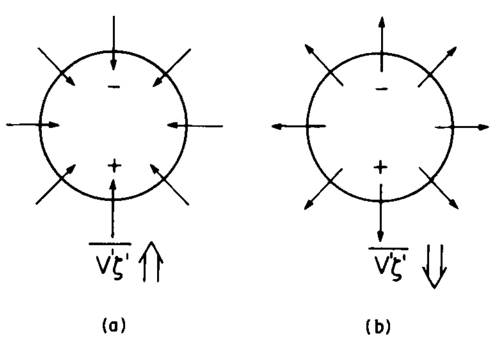

Fig. 1. The production of the horizontal eddy flux of vertical vorticity by cumulus convection.

The magnitude of the eddy vorticity flux may be estimated according to

$$\overline{\mathbb{V}'\eta'} \sim \Sigma(\mathbb{V}_c - \bar{\mathbb{V}})(\eta_c - \bar{\eta}) \sim 10^{-4}\ \mathrm{m\ s^{-1}} \tag{25}$$

if one assumes that $\Sigma \sim 1\%$, and $\mathbb{V}_c \sim 10\ \mathrm{m\ s^{-1}}$. The corresponding eddy Coriolis force is therefore also of the order of $10^{-4}\ \mathrm{m\ s^{-1}}$, which is of the same order as the inertial force of the large-scale flow.

From the estimates presented above in (i), (ii), and (iii), it appears that the most important eddy process in the equation of motion is the eddy Coriolis force. While the absolute magnitudes of the three eddy terms are proportional to the intensity of cloud activity, the relative importance between them should remain the same, no matter what the cloud amount is. As long as the strength of cloud activity is not one order of magnitude larger than what has been assumed at the beginning of this analysis, the equation of motion may be reduced to

$$\frac{\partial \bar{\mathbb{V}}}{\partial t} + \bar{\mathbb{V}}\cdot\nabla\bar{\mathbb{V}} + \bar{\omega}\frac{\partial \bar{\mathbb{V}}}{\partial p} + f\hat{k} \times \bar{\mathbb{V}} = -\hat{k} \times \overline{\mathbb{V}'\eta'} - \nabla\bar{\Phi} \tag{26}$$

as a first approximation.

Since the Coriolis force is of the order 10^{-3} m s^{-2}, it must be balanced approximately by the gradient of the geopotential, if the cloud activity is not one order stronger than that which has been assumed in this study. In this case, the geostrophic wind remains a good approximation. Furthermore, from the thermodynamic equation it can be shown that

$$\bar{\omega} + M_c \sim P\cdot\frac{\bar{U}}{L}\cdot\frac{1}{R_1 R_0} \lesssim 10^{-3}\ \mathrm{mb\ s^{-1}} \tag{27}$$

where P is the surface pressure. Since M_c is assumed to be $\sim 10^{-3}$ mb s^{-1}, one finds

$$\bar{\omega} \lesssim 10^{-3} \text{ mb s}^{-1} \tag{28}$$

which is of the same order of magnitude as in an ordinary quasi-geostrophic system.

2.3. *The vorticity equation*

Within the context of the quasi-geostrophic approximation, the vorticity equation, which is obtained by taking the curl of eq. (26), can be reduced to

$$\frac{\partial \bar{\zeta}}{\partial t} + \bar{\mathbb{V}} \cdot \nabla(\bar{\zeta} + f) - f_0 \frac{\partial \bar{\omega}}{\partial p} = - \nabla \cdot \overline{\mathbb{V}' \eta'} \tag{29}$$

where f_0 is the value of the Coriolis parameter at a reference latitude.

An explicit expression, in terms of cloud ensemble properties, has been derived by Cho and Cheng (1980) for the divergence of the horizontal eddy flux of vorticity given on the right-hand side of eq. (29). Since this is an important term in the mean vorticity equation, their derivation will be summarized here briefly.

Cho and Cheng's formulation was based on a number of properties of the cloud vorticity field first suggested by Cho et al. (1979). Using a simple scale analysis, these authors showed that (i) in the cloud vertical vorticity equation, the vertical advection and tilting effects balance one another when averaged over a cloud horizontal cross-sectional area; (ii) the areal mean vorticity $\bar{\eta}_c$ of a cloud is of the same order of magnitude as $\bar{\eta}$, the large-scale mean absolute vorticity, although the point values of cloud vorticity are usually much larger; (iii) the cloud mean vorticity $\bar{\eta}_c$ is governed by a simple conservation equation

$$\frac{\partial \eta_c}{\partial t} = -\frac{1}{a} \int \nabla \cdot \mathbb{V}_c \, \eta_c \, da \tag{30}$$

where a is the cross-sectional area of the cloud.

Since the cloud mean vorticity is comparable in magnitude to the large-scale mean vorticity, the assumption that the fractional cloud coverage Σ is very small implies that the large-scale mean vorticity is essentially the same as the mean vorticity in the cloud environment, i.e.

$$\bar{\eta} \approx \tilde{\eta}$$

where $\tilde{\eta}$ is the mean vorticity of the cloud environment. One can also draw from eq. (30) the conclusion that the cloud mean vorticity $\bar{\eta}_c$ should be larger than $\bar{\eta}$ at the lower entrainment levels of clouds, and smaller than $\bar{\eta}$ at the upper detrainment levels if the mean vorticity around the cloud boundary is positive. Or, in other words,

$$\bar{\eta}_c - \bar{\eta} \begin{cases} >0 & \text{at entrainment levels} \\ <0 & \text{at detrainment levels.} \end{cases} \tag{31}$$

In addition to these cloud properties just discussed, Cho and Cheng (1980) also assumed that the cloud-induced vorticity fluctuations are confined mainly to the areas occupied by the clouds. This implies that in the cloud environment, $\eta' = \tilde{\eta} - \bar{\eta} \approx 0$ if no other types of eddies are present. The eddy correlation between $\mathbb{V}'$ and η' must then exist only inside cumulus clouds and $\mathbb{V}'\eta' = 0$ in the cloud environment.

In order to derive an explicit expression for the divergence of the eddy flux of vorticity, let us recall that in cumulus parameterization studies, cumulus clouds are treated as eddies by averaging over an area A which is assumed to contain many clouds, but is small compared to the dimension of the synoptic-scale disturbance. If N is the total number of cumulus clouds in such an area, the mean eddy flux of vorticity is given by

$$\overline{\mathbb{V}' \eta'} = \frac{1}{A} \int_A \mathbb{V}' \eta' \, da = \frac{1}{A} \sum_{i=1}^{N} \int_{a_i} \mathbb{V}' \eta' \, da$$

where a_i is the horizontal cross-sectional area of the ith cloud in the area. The assumption that $\eta' = 0$ in the cloud environment has been used in deriving the above equation. Using the identity

$$\nabla \cdot \overline{\mathbb{V}' \eta'} = \overline{\nabla \cdot \mathbb{V}' \eta'}$$

one can also obtain for the divergence of the horizontal eddy flux of vorticity

$$\nabla \cdot \overline{\mathbb{V}' \eta'} = \frac{1}{A} \sum_i \int_{a_i} \nabla \cdot \mathbb{V}' \eta' \, da$$

$$= \frac{1}{A} \sum_i \int_{a_i} \nabla \cdot (\mathbb{V}_c - \bar{\mathbb{V}})(\eta_c - \bar{\eta}) \, da.$$

Since $\nabla \cdot \mathbb{V}_c$ is about two orders of magnitude larger than $\nabla \cdot \bar{\mathbb{V}}$ the above equation may be simplified to get (Cho and Cheng, 1980)

$$\nabla \cdot \overline{\mathbb{V}' \eta'} \approx \frac{1}{A} \sum_i \int_{a_i} \nabla \cdot \mathbb{V}_c(\eta_c - \bar{\eta}) \, da. \tag{32}$$

We note from the relation

$$\nabla \cdot \overline{V' \eta'} = \nabla \times (k \times \overline{V' \eta'})$$

that the divergence of the horizontal eddy flux of vorticity can be interpreted as the effect of the eddy Coriolis force on the vorticity field. This eddy Coriolis force is assumed to exist only inside cumulus clouds. Therefore it modifies only the cloud vorticities.

To examine how the eddy Coriolis force may influence the mean vorticity inside cumulus clouds, let us rewrite the mean cloud vorticity eq. (30) as follows:

$$\frac{\partial \bar{\eta}_{ci}}{\partial t} = -\frac{1}{a_i}\int_{ai} \nabla \cdot V_c\, \eta_c \,\mathrm{d}a$$
$$= -\frac{1}{a_i}\int_{ai} \nabla \cdot V_c(\eta_c - \bar{\eta})\,\mathrm{d}a - \frac{1}{a_i}\bar{\eta}\int_{ai} \nabla \cdot V_c \,\mathrm{d}a$$
$$= -\frac{1}{a_i}\int_{ai} \nabla \cdot V'\, \eta' \,\mathrm{d}a - \frac{1}{a_i}\bar{\eta}\int_{ai} \nabla \cdot V_c \,\mathrm{d}a.$$

The subscript i is used here to denote the ith cloud in the area of averaging. The physical interpretation of this equation is clear. The mean cloud vorticity is influenced by two processes: one is due to the effect of the eddy Coriolis force inside the cloud, and the other the convergence of large-scale mean vorticity into the cloud. The net effect of the eddy Coriolis force in a cloud can be expressed as

$$-\frac{1}{a_i}\int_{ai} \nabla \cdot V'\, \eta' \,\mathrm{d}a = \frac{\partial \bar{\eta}_{ci}}{\partial t} + \frac{1}{a_i}\bar{\eta}\int_{ai} \nabla \cdot V_c \,\mathrm{d}a.$$

When summed over all clouds, one obtains the total effect of the eddy Coriolis force due to the cumulus cloud ensemble:

$$-\nabla \cdot \overline{V' \eta'} = +\frac{1}{A}\sum_i a_i \frac{\partial \bar{\eta}_{ci}}{\partial t} + \bar{\eta}\frac{\partial M_c}{\partial p}. \tag{33}$$

The first term on the right-hand side of eq. (33) represents the ensemble average of the time rate of change of cloud vorticity. Cho (1977) showed that it can be expressed in terms of the parameter Σ/τ, the recycling rate of atmospheric air by cumulus clouds:

$$\frac{1}{A}\sum_i a_i \frac{\partial \bar{\eta}_{ci}}{\partial t} = \frac{\Sigma}{\tau}(\bar{\eta}_{c\tau} - \bar{\eta}) \tag{34}$$

where τ is the mean lifetime of clouds and $\bar{\eta}_{c\tau}$ is the mean absolute vorticity over cloud areas at the end of cloud life-cycles. The term is referred to as the cloud life-cycle effect because $(\bar{\eta}_{c\tau} - \bar{\eta})$ represents the net increase of vorticity over cloud areas during the active periods of clouds. At the end of the active periods of cumulus clouds, the areas occupied by the clouds become part of the large-scale environment. The excessive vorticity in the cloud areas then becomes part of the large-scale vorticity, and represents a vorticity source for the mean flow. The rate of generation of vorticity for the mean flow through this cloud life-cycle effect, as indicated in eq. (34), should be proportional to the fractional cloud coverage Σ, and inversely proportional to the cloud life span τ because this generation process takes place once per cloud lifetime.

Substituting eq. (34) into eq. (33), we obtain Cho and Cheng's (1980) formulation for the divergence of the horizontal eddy flux of vorticity:

$$-\nabla \cdot \overline{V' \eta'} = \frac{\Sigma}{\tau}(\bar{\eta}_{c\tau} - \bar{\eta}) + \bar{\eta}\frac{\partial M_c}{\partial p}. \tag{35}$$

Insertion of this relation into eq. (29) gives the quasi-geostrophic vorticity equation, including the effects of cumulus clouds:

$$\frac{\partial \bar{\zeta}}{\partial t} + V \cdot \nabla(\bar{\zeta} + f) - f_0 \frac{\partial \bar{\omega}}{\partial p} = \frac{\Sigma}{\tau}(\bar{\eta}_{c\tau} - \bar{\eta}) + f_0 \frac{\partial M_c}{\partial p}. \tag{36}$$

3. The omega and the tendency equations

The release of latent heat through condensation of water vapour is realized within individual convective elements. The heat energy is imparted to the large-scale flow only through the slow compensating subsidence induced in the cloud environment. This process may be illustrated by slightly rearranging the thermodynamic equation given by eq. (10). If one uses $\tilde{\omega}$ to denote the mean vertical p-velocity in the cloud environment,

$$\tilde{\omega} = \bar{\omega} + M_c, \tag{37}$$

eq. (10) can be expressed as

$$\frac{\partial \bar{\theta}}{\partial t} + V \cdot \nabla \bar{\theta} + \tilde{\omega}\frac{\partial \bar{\theta}}{\partial p} = 0. \tag{38}$$

The actual warming of an air parcel in the mean flow depends on the vertical motion field in the cloud environment. The capability of clouds to produce real warming in the mean depends on their ability to induce environmental subsidence.

Because of the tendency to maintain geostrophic balance in a quasi-geostrophic system, the vertical velocity field is also influenced by dynamic processes. This clearly indicates the importance of the dynamic effects of clouds. It is not sufficient to consider the thermodynamic effects of clouds alone.

Using eq. (37), the vorticity eq. (36) may be rewritten as

$$\frac{\partial\bar{\zeta}}{\partial t} + \bar{\mathbb{V}}\cdot\nabla(\bar{\zeta}+f) - f_0\frac{\partial\tilde{\omega}}{\partial p} = \frac{\Sigma}{\tau}(\bar{\eta}_{c\tau} - \bar{\eta}). \tag{39}$$

From eqs. (38) and (39), one may conclude that when these equations are written with respect to the vertical velocity in the cloud environment, the only apparent effect of clouds is the generation of vorticity through the cloud life-cycles.

Since the geostrophic wind remains a good approximation as long as the intensity of cloud activities is not one order larger than what has been assumed in this study, the relative vorticity may be related directly to the geopotential:

$$\bar{\zeta} = \nabla^2\bar{\Phi}/f_0. \tag{40}$$

A diagnostic equation may be derived for the mean vertical motion in the cloud environment:

$$\begin{aligned}\left[\nabla^2 + \frac{\partial}{\partial p}\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right]\tilde{\omega} &= \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\bar{\mathbb{V}}\cdot\nabla\left(\frac{1}{f_0}\nabla^2\bar{\Phi} + f\right)\right] \\ &+ \frac{1}{\sigma}\nabla^2\left[\bar{\mathbb{V}}\cdot\nabla\left(-\frac{\partial\bar{\Phi}}{\partial p}\right)\right] \\ &- \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\frac{\Sigma}{\tau}(\bar{\eta}_{c\tau} - \bar{\eta})\right].\end{aligned} \tag{41}$$

In this equation, σ is the stability parameter:

$$\sigma = -\frac{1}{\bar{\rho}}\frac{1}{\bar{\theta}}\frac{\partial\bar{\theta}}{\partial p}.$$

Similarly, an equation may also be derived for the geopotential tendency:

$$\begin{aligned}\left[\nabla^2 + \frac{\partial}{\partial p}\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right]\frac{\partial\bar{\Phi}}{\partial t} &= -f_0\bar{\mathbb{V}}\cdot\nabla[\nabla^2\bar{\Phi}/f_0 + f] \\ &+ \frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}\bar{\mathbb{V}}\cdot\nabla\frac{\partial\bar{\Phi}}{\partial p}\right] + f_0\frac{\Sigma}{\tau}(\bar{\eta}_{c\tau} - \bar{\eta}).\end{aligned} \tag{42}$$

From this equation, one may conclude that the net effect of cumulus convection on the development of quasi-geostrophic systems is through the generation of vorticity by the convective elements. Since clouds generate positive vorticity ($\bar{\eta}_c > \bar{\eta}$) at their lower entrainment levels, and generate negative vorticity ($\bar{\eta}_c < \bar{\eta}$) at the upper detrainment levels, they may provide a potent mechanism for the intensification of cyclones if located at a favourable location. This will be studied further in an instability analysis in the next section.

4. Baroclinic instability with cumulus convection

With the formulation that has been presented, it seems of some interest to examine the effects of cumulus convection on baroclinic instability. This will be done in this section. For the sake of simplicity, the analysis will be made in the framework of a two-level model. The notations used in the model are illustrated in Fig. 2.

The background mean state of the model will be assumed to be a uniform westerly wind $\bar{U}$ that increases with height. Ψ will be used to denote the stream function of a synoptic scale disturbance.

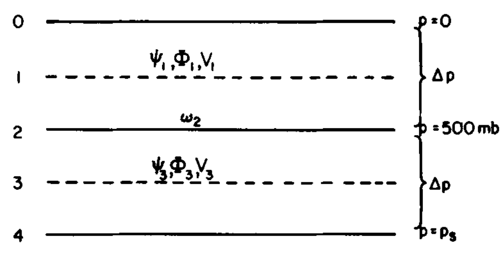

Fig. 2. The two-layer model.

The total wind vector is therefore

$$\bar{V} = \hat{x}\left(\bar{U} - \frac{\partial \Psi}{\partial y}\right) + \hat{y}\frac{\partial \Psi}{\partial x}, \tag{43}$$

where $\hat{x}$ is the unit vector in the east direction, and $\hat{y}$ in the north direction. Assuming that Ψ depends only on x and t, the vorticity and thermodynamic equations of the model can be written as

$$\left(\frac{\partial}{\partial t} + \bar{U}_1\frac{\partial}{\partial x}\right)\frac{\partial^2 \Psi_1}{\partial x^2} + \beta\frac{\partial \Psi_1}{\partial x} - \frac{f_0}{\Delta p}\tilde{\omega}_2 = \left[\frac{\Sigma}{\tau}(\bar{\eta}_{c\tau} - \bar{\eta})\right]_1$$

$$\left(\frac{\partial}{\partial t} + \bar{U}_3\frac{\partial}{\partial x}\right)\frac{\partial^2 \Psi_3}{\partial x^2} + \beta\frac{\partial \Psi_3}{\partial x} + \frac{f_0}{\Delta p}\tilde{\omega}_2 = \left[\frac{\Sigma}{\tau}(\bar{\eta}_{c\tau} - \bar{\eta})\right]_3 \tag{44}$$

$$\left(\frac{\partial}{\partial t} + \bar{U}_{\mathrm{m}}\frac{\partial}{\partial x}\right)(\Psi_1 - \Psi_3) - \bar{U}_{\mathrm{T}}\frac{\partial}{\partial x}(\Psi_1 + \Psi_3) - \frac{\sigma\,\Delta p}{f_0}\tilde{\omega}_2 = 0$$

where $\bar{U}_{\mathrm{m}} = \frac{1}{2}(\bar{U}_1 + \bar{U}_3)$ and $U_{\mathrm{T}} = \frac{1}{2}(\bar{U}_1 - \bar{U}_3)$. Subscripts are used here to denote the level in the model.

To complete the set of model equations, one still needs a closure condition to relate the overall strength of cloud activities to the state of the large-scale flow. For this we shall adopt the Ekman layer pumping hypothesis introduced by Charney and Eliassen (1964). According to this theory, the amount of cumulus cloud activities is proportional to the amount of moisture convergence in the boundary layer, which in turn is proportional to the mean vorticity at the top of the boundary layer. In the present model the cloud ensemble parameter needed is the recycling rate Σ/τ. In the context of the two-layer model, we shall assume, for the sake of simplicity, that it is proportional to the mean vorticity of the lower layer, i.e., at level 3:

$$\frac{\Sigma}{\tau} = \alpha\frac{\partial^2 \Psi_3}{\partial x^2}. \tag{45}$$

The value of α depends, among other things, on the eddy diffusivity and the moisture content in the Ekman layer.

The value of cloud vorticity in excess of the environmental vorticity

$$\Delta\eta = \eta_{c\tau} - \bar{\eta} \tag{46}$$

depends on the rate of convergence of air into cumulus clouds. As was discussed earlier, in the lower layer of a cumulus cloud, where there is a net convergence of air into cloud, $\Delta\eta > 0$. In the upper levels where cloud air diverges into the cloud environment, $\Delta\eta < 0$. Accordingly, the assumption made in this study wil be $\Delta\eta_3 > 0$ and $\Delta\eta_1 < 0$.

Assuming that the synoptic disturbance has a functional form given by

$$\begin{bmatrix}\Psi_1\\ \Psi_2\\ \tilde{\omega}_2\end{bmatrix} = \begin{bmatrix}D\\ E\\ F\end{bmatrix}\exp\,[ik(x-ct)], \tag{47}$$

where k is the wave number and c the speed of propagation, and defining

$$\begin{aligned}\lambda^2 &= f_0^2/(\sigma\Delta p^2), \quad B = \beta/(\lambda^2\,\bar{U}_{\mathrm{T}})\\ A &= \alpha\Delta\eta/(\lambda\bar{U}_{\mathrm{T}}), \quad \gamma = k/\lambda\end{aligned} \tag{48}$$

the existence of a non-trivial solution requires that

$$\frac{c}{\bar{U}_{\mathrm{T}}} = \frac{\bar{U}_{\mathrm{m}}}{\bar{U}_{\mathrm{T}}} - B\frac{1+\gamma^2}{\gamma^2(2+\gamma^2)} + i\frac{A_1 + A_3(1+\gamma^2)}{2\gamma(2+\gamma^2)} \pm \frac{1}{\gamma^2(2+\gamma^2)}X^{1/2} \tag{49}$$

where

$$\begin{aligned}X = {}& B^2 + \gamma^4(\gamma^4 - 4) - \tfrac{1}{4}\gamma^2[(A_1 + A_3) + A_3\,\gamma^2]^2\\ &+ i\{\gamma^3(2+\gamma^2)[(A_1 + A_3) - A_3\,\gamma^2]\\ &- B\gamma[(A_1 + A_3) + A_1\,\gamma^2]\}.\end{aligned} \tag{50}$$

The disturbance will be unstable if c is a complex number. In that case, the growth rate of the disturbance is given by kc_i where c_i is the imaginary part of c.

To estimate the typical magnitudes of the parameters A and B, the following values will be assumed for f_0 and β:

$$f_0 \sim 10^{-4}\,\mathrm{s}^{-1}, \quad \beta \sim 10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}.$$

The value of the stability parameter σ will be assumed to be such that (Holton, 1979)

$$\lambda^2 \sim 2 \times 10^{-12}\,\mathrm{m}^{-2}.$$

If a value 5 m s^{-1} is assumed for the thermal wind U_T, one finds

$$B \sim 1, \quad A \sim \alpha\Delta\eta \times 10^5,$$

both in SI units. $\Delta\eta$ should be of the order of 10^{-5} s^{-1}, the same as the large-scale relative vorticity (Cho et al., 1979). The value $\Delta\eta \sim 1 \times 10^{-5}$ s^{-1} will be adopted, which is probably a slight underestimate. This then gives $A \sim \alpha$. From eq. (45), one can see that a value $\alpha = 1$ means that a vorticity of the order of 10^{-5} s^{-1} at level 3 will produce through Ekman layer pumping a cloud population with a recycling rate of the order of 1 day^{-1}.

As an example, numerical solutions have been obtained for eq. (49) with $B = 1$ and two sets of values of A:

(1) $A_3 = -A_1 = 1$, (2) $A_3 = -A_1 = 1/2$.

The growth rates, expressed in the form $kc_i/\lambda \bar{U}_T$ are plotted in Fig. 3 as functions of the wavenumber expressed in the form k/λ. The two sets of values of A correspond to two different intensities of cloud activity under the same background conditions.

There are two solutions given in eq. (49). One solution is essentially stable with negative growth rate except in the region of very small wavenumbers. The other solution is unstable for all wavenumber. For comparison, the growth rates for

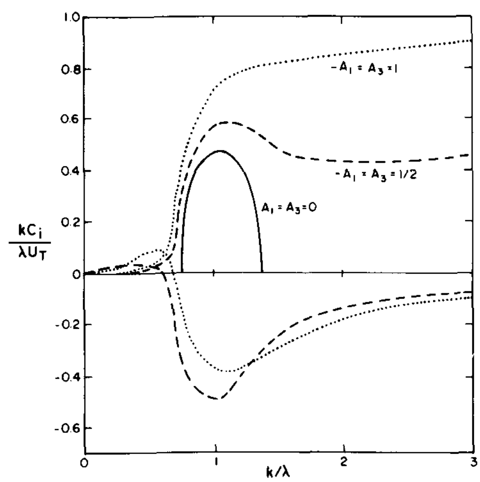

Fig. 3. The growth rate of quasi-geostrophic disturbance as a function of wavenumber. The solid curve is for the purely baroclinic case. The dashed and dotted curves include the effects of cumulus convection.

the purely baroclinic unstable wave without the effects of cumulus convection ($A_1 = A_3 = 0$) are shown in Fig. 3 as the solid curve. Overall, the effect of convection is to increase the growth rate at all wavenumbers.

With $A_3 = -A_1 = +\frac{1}{2}$, the wavelength of the most unstable mode is reduced slightly, in comparison with the purely baroclinic case. The maximum growth rate is increased by about 20%. With $A_3 = -A_1 = 1$, the effect of convection becomes very strong, and the growth rate increases monotonically with increasing wavenumber. The disturbance with the smallest wavelength has the largest growth rate, in so far as the present formulation remains valid.

5. Conclusions

We have investigated in this paper the effects of cumulus convection on the dynamics of mid-latitude quasi-geostrophic weather systems. In addition to the heating effect due to release of latent heat, we have examined the possible dynamic processes through which cumulus clouds may influence the large-scale flow.

It has been found that the main dynamic effect of a cumulus ensemble embedded in a quasi-geostrophic flow is produced by a process we referred to as the eddy Coriolis force. This process generates, over the region where the cloud ensemble is located, a positive vorticity for the large-scale flow in the lower half of the troposphere where there is a net convergence of air into the clouds, and a negative vorticity for the large-scale flow in the upper troposphere where cloud air diverges into the cloud environment. It may therefore be a very effective mechanism for the intensification of extratropical cyclones if the clouds are located at a favourable position. Consequently, an instability analysis has been made for quasi-geostrophic disturbances, including these effects of cumulus convection. It has been found that with a reasonable level of cloud activity, the growth rates of baroclinic disturbances are increased significantly.

One of the weaknesses in the evaluation of the various cloud dynamic effects presented in this paper is that an intensity of cumulus activity was assumed. This is not completely satisfactory because the amount of cumulus convection depends upon, among other things, the atmospheric moisture content and the vertical stabilities of the air masses, which may vary considerably from one system to another. Furthermore, it is not yet well understood how the cloud activities depend on the state of the synoptic-scale disturbance, in addition to the thermodynamic characteristics of the air masses. Some of the results obtained in the instability analysis, such as the maximum growth rate and the most preferred scale, could be sensitive to the closure condition used. No definitive conclusions should be drawn from these results, other than the possible magnitudes of cloud effects, until the closure problem is resolved.

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