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A note on the adjustment to hydrostatic balance

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ABSTRACT

This note investigates the problem of the relation between the initial conditions for the primitive equations and the full set of initial conditions for the Euler equations of adiabatic motion. We formulate for the primitive equations a problem analogous to the one that was considered by Rossby (1938) concerning the quasi geostrophic approximation; a problem which is now well known as the adjustment to geostrophy. The major conclusion is that the initial conditions for the primitive equations may be derived from a full set of initial conditions by solving a one-dimensional unsteady problem of vertical motion.

1. Introduction

During the fifties when short range numerical weather prediction was based on the quasi geostrophic approximation, there was raised the problem of adjustment to geostrophy formulated by Rossby (1938) and it was examined by several authors, for instance Cahn (1945), Bolin (1953), Kibel (1955) and recently by Guiraud and Zeytounian (1980). For a review of this aspect Phillips (1963), Monin (1969) and Blumen (1972) are well suited. Nowadays the numerical prediction models mostly rely on the primitive equations, which are derived from the hydrostatic approximation to the Euler equations for adiabatic motion. Therefore, the corresponding adjustment problem of filtering out of acoustic waves has become important to solve. We will show that the solution to this problem is a necessary step when one tries to formulate proper initial conditions for the primitive equations. The problem we consider is one of initial conditions for the primitive equations in relation to initial data which do not fit the hydrostatic approximation. It is quite different from the initialization problem for the primitive equations designed to filter out the high-frequency gravity oscillations of relatively short duration which are present in the primitive equation models. In some sense we formulate a problem for the primitive equations analogous to the one which was considered by Rossby (1938) concerning the quasi geostrophic approximation, now well known as adjustment to geostrophy. Accordingly, our problem is one of *adjustment to hydrostatic balance*, and its solution shows the way by which the acoustic waves are filtered out during the process of adjustment. In Section 2 the full equations are given for the adiabatic motion of the atmosphere, considered as a perfect gas with constant specific heats of ratio γ, under the tangent plane assumption, written in nondimensional form containing a few basic nondimensional parameters. In Section 3 we recall the classical hydrostatic approximation associated with the limiting process when ε, the quotient between the height scale and the length scale, goes to zero ($\varepsilon \to 0$), with time t and position fixed. We stress that initial conditions for the primitive equations are different from the ones for the full Euler equations. In Section 4, introducing a fast time $\hat{t} = t/\varepsilon$, we formulate the initial value problem whose solution allows us to explain how these two sets of initial conditions are related. As this problem is nonlinear and not easy to solve, except by numerical techniques, we examine in Section 5 the final phase of adjustment by a process of linearization from the hydrostatic balance. This phase is described by the equation of telegraphy with coefficients depending on the

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altitude. Assuming that the temperature of the standard (hydrostatic) atmosphere is constant, the corresponding initial boundary value problem is solved by standard techniques, and it may be shown that the acoustic waves described by the equations of adjustment to hydrostatic balance decay in time like $\hat{t}^{-3/2}$. This analysis does *not* explain how gravity waves with short horizontal wave length are filtered out when one goes from the Euler to the primitive equations.

2. Formulation

We use dimensionless variables and the equations of adiabatic motion of the atmosphere, considered as a perfect gas with constant specific heats of ratio γ, under the tangent plane assumption, assuming an advective time scale,

$$\rho\left\{\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{R_0}\mathbf{k}\times\mathbf{u}\right\} + \frac{1}{\gamma M^2}\left(\nabla p + \frac{1}{\varepsilon}\rho\mathbf{e}\right) = 0, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{2}$$

$$\left\{\frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla)\right\}(p\rho^{-\gamma}) = 0, \tag{3}$$

where we use the notation p, ρ, $\mathbf{u}$ for dimensionless pressure, density and velocity, respectively. The unit vectors $\mathbf{k}$ and $\mathbf{e}$ are in the direction of the axis of rotation of the earth and of the local vertical (taking into account the effect of rotation on gravity), respectively. The following dimensionless parameters appear: the Rossby number $R_0 = U_0/2\Omega_0 L_0$, the Mach number $M = U_0/a_0$, and the length scales ratio $\varepsilon = H_0/L_0$. Here U_0 is the characteristic velocity, L_0 the characteristic horizontal length (much smaller than the earth's radius, but much larger than H_0, the height of the troposphere, which is used as a characteristic vertical length), Ω_0 is the constant rate of rotation of the earth and a_0 is the speed of sound at ground level. The dimensionless velocity and gradient operator are decomposed according to the following scheme

$$\mathbf{u} = \mathbf{V} + \varepsilon w\mathbf{e}, \quad \mathbf{v}\cdot\mathbf{e} = 0, \tag{4}$$

$$\nabla = \mathbf{D} + \frac{1}{\varepsilon}\frac{\partial}{\partial z}\mathbf{e}, \quad \mathbf{D}\cdot\mathbf{e} = 0. \tag{5}$$

3. The hydrostatic approximation

The hydrostatic approximation is obtained through the limiting process $\varepsilon \to 0$ (keeping t, z and the horizontal position fixed during the process), applied to the eqs. (1)–(3). This leads to the so-called primitive equations (see for instance Monin (1969) or Phillips (1970)). We have to specify initial conditions for eqs. (1)–(3). We observe that the initial data need not fit the hydrostatic balance and, in particular, the vertical velocity need not be $O(\varepsilon)$ with respect to the horizontal one, so that in order to consider the most general case, we assume that, at the initial time, εw is of order $O(1)$. Accordingly we get as initial conditions, for the Euler eqs. (1)–(3),

$$t = 0: \quad \mathbf{v} = \mathbf{V}^0, \quad \varepsilon w = W^0, \quad p = P^0, \quad \rho = R^0, \tag{6}$$

where $\mathbf{V}^0$, W^0, P^0 and R^0 are given functions of z and of horizontal positions. On the other hand, when considering the primitive equations we must give only the initial values of $\mathbf{v}$ and ρ; they have nothing to do with the corresponding initial conditions for the Euler equations and, consequently, we get, as initial conditions for the primitive equations

$$t = 0: \mathbf{v} = \mathbf{v}^0, \quad \rho = \rho^0, \tag{7}$$

where $\mathbf{v}^0$ is different from $\mathbf{V}^0$ and ρ^0 is different from R^0. Two of the initial conditions have been lost during the process and two questions arise:

(i) How have these initial conditions been lost?
(ii) How are $\mathbf{v}^0$ and ρ^0 related to $\mathbf{V}^0$, W^0, P^0 and R^0?

Regarding the first question, the answer is simple. According to the hydrostatic model, p is related to ρ by the equation of hydrostatic balance, while w, as noticed for the first time by Richardson (1922) in Chap. V of his book, is computed by the process of solution of the primitive equations; all this holds true at the initial time as well. We intend here to address ourselves to the second question.

4. The nonlinear initial value problem describing the adjustment to hydrostatic balance

A number of adjustment problems occur in fluid mechanics being related to loss of initial conditions as a consequence of loss of time derivatives during some limiting process leading to a simplified set of equations. We refer to just one of them which is most celebrated and has intrigued many fluid dynamicists. It is the loss of initial conditions for the full distribution function when one goes from the Boltzmann equation to the Navier-Stokes equations by letting the ratio of the mean free path to macroscopic length scale go to zero (see, for example, Cercignani (1975), Chap. V, §5). To the best of our knowledge these problems are solved by rescaling the time and possibly some dependent variables leading to a so-called initial layer problem. Depending on the kind of problem, we may have mainly two kinds of behavior when the rescaled time goes to infinity. Either one may have a tendency towards a limiting steady state or an undamped set of oscillations (think for example of the inertial waves in the inviscid problem of spin-up for a rotating fluid; see for example Greenspan (1968), §2.4). For the terminology of the initial layer as adapted to this kind of singular perturbation problem we refer to Nayfeh (1973, p. 23). For a general reference to singular perturbation techniques as applied to fluid mechanics see Van Dyke (1975). In the present case it is fairly obvious that the proper rescaling is

$$t = \varepsilon\hat{t}, \quad z \equiv \hat{z}, \quad \mathbf{u} = \hat{\mathbf{v}} + \hat{w}\mathbf{e}, \quad p \equiv \hat{p}, \quad \rho \equiv \hat{\rho}, \tag{8}$$

and we use the limiting process

$$\varepsilon \to 0, \quad \hat{t}, \quad \hat{z} \text{ fixed.} \tag{9}$$

The horizontal position is also fixed during the process. We note that through the process $w = \varepsilon^{-1}\hat{w}$ is $O(1)$. Let us set $\hat{\mathbf{v}}_0$, $\hat{w}_0$, $\hat{p}_0$, and $\hat{\rho}_0$ for the limiting values. It is straightforward to derive the set of limiting equations, namely:

$$\gamma M^2 \hat{\rho}_0 \left(\frac{\partial \hat{w}_0}{\partial \hat{t}} + \hat{w}_0 \frac{\partial \hat{w}_0}{\partial \hat{z}}\right) + \frac{\partial \hat{p}_0}{\partial \hat{z}} + \hat{\rho}_0 = 0; \tag{10}$$

$$\frac{\partial \hat{\rho}_0}{\partial \hat{t}} + \frac{\partial}{\partial \hat{z}}(\hat{\rho}_0 \hat{w}_0) = 0; \tag{11}$$

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{w}_0 \frac{\partial}{\partial \hat{z}}\right)(\hat{p}_0 \hat{\rho}_0^{-\gamma}) = 0; \tag{12}$$

$$\frac{\partial \hat{\mathbf{v}}_0}{\partial \hat{t}} + \hat{w}_0 \frac{\partial \hat{\mathbf{v}}_0}{\partial \hat{z}} = 0. \tag{13}$$

These equations are decoupled. The system (10)–(12) is identical to the equations for one-dimensional vertical motion in the atmosphere. Once $\hat{w}_0$ has been obtained through the solution of (10)–(12) with the initial conditions (see, (6)):

$$\hat{t} = 0: \quad \hat{w}_0 = W^0, \quad \hat{p}_0 = P^0, \quad \hat{\rho}_0 = R^0, \tag{14}$$

and proper boundary conditions on the ground and at infinity, we may use the transport eq. (13) in order to compute $\hat{\mathbf{v}}_0$ using the initial condition

$$\hat{t} = 0: \quad \hat{\mathbf{v}}_0 = \mathbf{V}^0. \tag{15}$$

5. The ultimate phase of adjustment to hydrostatic balance

Now we may understand the relation of (7) to (6). As a matter of fact, we should consider (7) as matching conditions between the hydrostatic approximation, at $t = 0$, and the initial boundary layer one, at $\hat{t} = \infty$. The matching requires

$$\lim_{\hat{t} \to \infty} (\hat{w}_0, \hat{p}_0, \hat{\rho}_0) = (0, p^0, \rho^0), \tag{16}$$

$$\lim_{\hat{t} \to \infty} \hat{\mathbf{v}}_0 = \mathbf{v}^0, \tag{17}$$

where p^0 and ρ^0 are related by the equation of hydrostatic balance.

Due to nonlinearity (quasi linearity) it is difficult to make rigorous statements concerning the behavior of the solution to (10)–(12) and (13) as $\hat{t} \to \infty$. Nevertheless it is quite clear, on physical grounds, that the solution, besides entropy waves ruled by eq. (12), consists in acoustic waves radiating towards infinity, leaving behind them a state of hydrostatic equilibrium $\hat{p}_{0,\infty}(\hat{z})$, $\hat{\rho}_{0,\infty}(\hat{z})$, $\hat{T}_{0,\infty}(\hat{z})$ with no vertical velocity and some distribution of horizontal velocity $\mathbf{v}_{0,\infty}(\hat{z})$, that we call the limiting state of the initial layer. Accordingly we expect that the problem of adjustment to hydrostatic balance pertains to the first class of initial layer problems referred to above. Of course we cannot rely on any mathematical theorem for a justification of our assertion. Our argument is a physical one and relies on a test of internal consistency. To apply this test we assume decay and study the way in which the remaining perturbation with respect to the assumed limiting state behaves when $\hat{t}$ goes to infinity according to a linearized theory obtained by perturbing (10)–(12) around the limiting state and retaining only the linear terms. Setting

$$\hat{w}_0 = \phi \exp\left\{\frac{1}{2}\int^{\hat{z}} \frac{d\hat{z}_1}{\hat{T}_{0,\infty}\hat{z}_1}\right\}, \tag{18}$$

we get, after linearization, the following equation

$$M^2 \frac{\partial^2 \phi}{\partial \hat{t}^2} - \hat{T}_{0,\infty} \frac{\partial^2 \phi}{\partial \hat{z}^2} + \frac{1}{2\hat{T}_{0,\infty}}\left(\frac{d\hat{T}_{0,\infty}}{d\hat{z}} + \frac{1}{2}\right)\phi = 0, \tag{19}$$

which is nothing else but the well-known equation of telegraphy provided that $\hat{T}_{0,\infty}$ does not depend on $\hat{z}$. Assuming this we may refer to Favard (1963) to find the solution of (19) corresponding to any initial condition and examine its behavior as $\hat{t} \to \infty$. In this way one may prove that, for any finite altitude $\hat{z}$, one has

$$\phi = O(\hat{t}^{-3/2}), \tag{20}$$

when $\hat{t} \to \infty$.

In view of this, if we define $\mathscr{W}(\hat{t}, \hat{z}^0)$ as the solution of:

$$\frac{\partial \mathscr{W}}{\partial \hat{t}} = \hat{w}(\hat{t}, \mathscr{W}(\hat{t}, \hat{z}^0)), \tag{21}$$

where $\mathscr{W}(0, \hat{z}^0) \equiv \hat{z}^0$, then the limiting value $\mathscr{W}_\infty(\hat{z}^0) = \lim_{\hat{t}\to\infty} \mathscr{W}(\hat{t}, \hat{z}^0)$ exists and we may define a function $Z^0(\hat{z})$ in the following way:

$$\mathscr{W}_\infty(\hat{z}^0) \equiv \hat{z}_\infty \Rightarrow \hat{z}^0 = \mathscr{W}_\infty^{-1}(\hat{z}_\infty) \equiv Z^0(\hat{z}_\infty). \tag{22}$$

Then, from (10)–(12) we get for the limiting values:

$$\frac{\partial \hat{p}_{0,\infty}}{\partial \hat{z}} + \hat{\rho}_{0,\infty} = 0, \tag{23}$$

$$\frac{\hat{p}_{0,\infty}(\hat{z})}{[\hat{\rho}_{0,\infty}(\hat{z})]^\gamma} = \frac{P^0(Z^0(\hat{z}))}{[R^0(Z^0(\hat{z}))]^\gamma} \equiv \exp\{\Sigma^0(Z^0(\hat{z}))\}, \tag{24}$$

where Σ^0 is the (Eulerian) initial value of the specific entropy. This allows us to compute $\hat{p}_{0,\infty}(\hat{z})$ and $\hat{\rho}_{0,\infty}(\hat{z})$ and then, from (16), to obtain the initial values:

$$p^0 = \hat{p}_{0,\infty}(z) = \left\{1 - \frac{\gamma - 1}{\gamma}\int_0^z \exp\left[-\frac{1}{\gamma}\Sigma^0(Z^0(z_1))\right]dz_1\right\}^{\gamma/(\gamma-1)}; \tag{25}$$

$$\rho^0 = \hat{\rho}_{0,\infty}(z) - (p^0)^{1/\gamma}\exp\left[-\frac{1}{\gamma}\Sigma^0(Z^0(z))\right]. \tag{26}$$

In a similar way, from (13) and (17) we get:

$$\mathbf{v}^0(z) = \hat{\mathbf{v}}_{0,\infty}(\hat{z}) = \mathbf{V}^0(Z^0(z)). \tag{27}$$

The result may be stated very simply by saying that the horizontal components of the velocity and the specific entropy in the two sets of initial conditions are merely *shifted vertically* by the amount of vertical displacement during the whole process of the vertical, one-dimensional, unsteady motion. Finally we mention that the time derivative in eq. (10), the momentum equation, is multiplied by the Mach number M. In any realistic situation M is rather small and we have again to deal with a new singular perturbation problem as is even more clearly revealed from (19). As a consequence the true time of adjustment is $M\hat{t} = M\varepsilon t$, at least under the limiting process which is considered here, that is $\varepsilon \to 0$, with M fixed and then $M \to 0$. It would be enlightening to examine the simultaneous limit process $\varepsilon \to 0$, $M \to 0$, with some similarity relation $M = \hat{M}\varepsilon^a$, where a is some real, positive exponent to determine, and $\hat{M}$ is $O(1)$. Usually a realistic meteorological situation corresponds to $a = \frac{1}{2}$.

6. Conclusions

We have shown that the initial conditions for the primitive equations may be derived from a full set of initial conditions, corresponding to the full Euler equations, by solving a one-dimensional, unsteady problem of vertical motion. It should be stressed that the initial conditions dealt with here have L_0 as the horizontal scale. The much more difficult problem of adjustment for initial conditions having a horizontal scale comparable to H_0 remains to be solved. One expects that the initial situation associated with the Euler equations will be quasi hydrostatic in most realistic situations, and this is one of the reasons why the linear version considered in Section 5 may be realistic for the whole process and not only for the final process of adjustment to hydrostatic balance. To some extent the result of Section 5 is one of stability with respect to small deviations from hydrostatic balance. According to a linear analysis the deviations are damped out with time. Nevertheless it would be interesting to solve (numerically) the full quasi linear problem of Section 4 in order to see how an initial set (eventually far from hydrostatic) adjusts itself to the primitive equations. That is outside the aim of this note, however, our intention being to demonstrate only that we have to deal with a singular perturbation problem.

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ОДНА ЗАМЕТКА О ПРИСПОСОЬЛЕНИИ К КВАЗИСТАТИЧЕСКОМУ ВАЛАНСУ

В зтой заметке изучается пробиема о связи между начальными условиями для примитивных уравнений и истиннцми начальными условиями для уровнений адиабатических движений атмосферы в форме Зйлера. Зто нас лриводит к формулировке задаче о приспособлении к квазистатическому балансу. Зта задача описывает одномерное, нестационарное, вертикальное движение адиабатической атмосферы.