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The behaviour of the sea surface temperature (SST) as a response to stochastic latent- and sensible heat forcing

By D. BLAAUBOER[1], *Institute of Theoretical Physics, University of Utrecht, P.O. Box 80006, 3508 TA Utrecht, The Netherlands,* and G. J. KOMEN and J. REIFF, *Royal Netherlands Meteorological Institute (KNMI), P.O. Box 201, 3730 AE De Bilt, The Netherlands*

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ABSTRACT

Following Hasselmann's work on climate variability we consider a stationary stochastic process with two different time scales for the forcing (the weather variables) and the response (the climate variable). We extend the work of Hasselmann, and Frankignoul and Hasselmann by allowing for a more general equation for the sea surface temperature (SST) containing, besides a stochastic forcing term, a term with a stochastic coefficient (feedback), in order to describe wet bulb temperature and wind velocity as two stochastic variables. From this equation, expressions for the average, the variance and the spectrum of the climate variable are derived by means of a cumulant expansion. So the SST is thought to be the result of a stochastic forcing by latent and sensible heat fluxes. The expressions are applied to data. For this application the daily observations of the lightvessel *Texel* (53°00′N, 4°23′E) measured during the winters of 1965–1974 are used. It turns out that the SST variance near the lightvessel *Texel* is described better by our extended stochastic equation than by the equation of Frankignoul and Hasselmann, while the shape of the spectrum is almost the same. It is the wet-bulb temperature rather than the wind velocity that largely determines the variation of the SST near the Dutch coast in winter.

1. Introduction

In the papers of Hasselmann (1976) and Frankignoul and Hasselmann (1977) the concept of a stochastic climate model is introduced and developed. In such a model the fluctuations of so-called climate variables (for instance SST, extent of polar ice-sheets, extent of vegetation and so on) are considered as a response to the fluctuations of the so-called weather variables. (For instance air temperature, wind velocity and wind direction, air pressure and so on.) It is assumed that these weather variables are stochastic variables with short correlation times (i.e. of the order of a few days) compared to the response of the climate variables (of the order of a few weeks). Furthermore it is important that the *stochastic character* and not only the averages of the weather variables determines the fluctuations of the climate variables. The resulting theory is analogous to the theory of Brownian motion (Wang and Uhlenbeck, 1945): the random walk of big particles (= climate variables) is a result of the interaction with a quantity of little particles (= weather variables).

The model of Hasselmann has been applied to SST anomalies by Frankignoul and Hasselmann (1977). They consider an equation

$$\dot{X}(t) + A_0 X(t) = Y(t), \tag{1}$$

in which A_0 is a constant and the random forcing $Y(t)$ containing the heat flux is the only stochastic weather variable determining the climate variable $X(t)$, in this case the SST. A time series for the heat flux is simulated and by means of numerical integration the response of the SST is determined. The results obtained in this manner are compared with the observations. In this paper we study a more general equation:

$$\dot{X}(t) + A(t)X(t) = Y(t), \tag{2}$$

[1] Presently at KNMI.

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where $A(t)$ is now a stochastic coefficient. So the extension with respect to Frankignoul and Hasselmann is rather a statistical refinement than an introduction of a new physical concept. Two weather variables instead of one are included in the stochastic coefficients $A(t)$ and $Y(t)$. The term $A(t)X(t)$ makes it necessary to employ a cumulant expansion, a mathematical tool which will be explained in Section 2. A different generalization was given by Oerlemans (1979) who considered a second-order differential equation with random forcing in a climatic study of ice sheet behaviour.

In Section 2 an equation like (2) is derived from the equations, which describe the latent and sensible heat flux from the sea to the atmosphere. The relative humidity (actually the wet-bulb-temperature) and the wind velocity are considered as stochastic variables. The variables are allowed to correlate with each other and with themselves, with correlation times of the order of a few days; their stochastic character is considered to be known. So these are the weather variables mentioned earlier. Just as Frankignoul and Hasselmann, we are interested in the stochastic response of the SST (climate variable). Expressions are derived for the average and higher moments of the SST. These expressions are compared with the observations of the lightvessel *Texel*, which is situated in the North Sea 53°00′N, 4°53′E, and also with the results that may be obtained using the model of Frankignoul and Hasselmann. The theoretical background is treated in Section 2; in Section 3 we apply this model to the SST near the lightvessel *Texel*. Section 4 gives the comparison with the results of Frankignoul and Hasselmann while in Section 5 some ideas on possible extensions of the model are given.

2. Theoretical background

2.1. *An equation for the change in the SST*

We want to study the change of temperature of the sea water as a result of the latent and sensible heat flux. This change is determined by (Kraus, 1972):

$$h\rho_w c_{pw} \frac{dT}{dt} = H + LQ. \tag{3}$$

The left-hand side of (3) is actually the change of heat content per unit area of the sea with depth h (in general h is the thickness of the mixing layer, i.e. the layer over which the absorbed heat is spread out), ρ_w = specific mass of sea water, c_{pw} = specific heat of water, T = sea surface temperature (SST). On the right-hand side of (3) the heat flux has been split into a sensible heat flux H and a latent heat flux LQ in which Q is the mass flux of water vapour and L is the specific heat of evaporation. We use (Kraus, 1972):

$$H = -\rho_a c_{pa}(T - T_a)C_D\, u, \tag{4}$$

$$Q = -\rho_a(q_0 - q)C_D\, u, \tag{5}$$

in which ρ_a is the density and c_{pa} the specific heat of the air; T_a is the air temperature, C_D is the drag coefficient, u the wind speed at 10 m, q the specific humidity of the air at temperature T_a and q_0 is the specific humidity of saturation at sea temperature T. There is a relation between the specific humidity and the wet-bulb-temperature (Keijman, 1974):

$$q_0 - q = s(T - T_w) + \frac{c_{pa}}{L}(T_a - T_w), \tag{6}$$

with T_w = wet bulb temperature, and $s = (dq_s/dT)_{T = T_w}$. When we insert (4), (5) and (6) into (3), we get

$$\frac{dT}{dt} = -\lambda\{T(t) - T_w(t)\}\, u(t), \tag{7}$$

in which

$$\lambda = \frac{C_D}{h}\frac{\rho_a c_{pa}}{\rho_w c_{pw}}\left(1 + \frac{Ls}{c_{pa}}\right). \tag{8}$$

Time dependence is only assumed for T, T_w and u. A discussion of the implicit or explicit time dependence of the other parameters can be found in Section 3.

2.2. *Derivation of first and second moment of SST*

In (7) $u(t)$ and $T_w(t)$ are considered as stationary stochastic processes, i.e. they are independent of the choice of an origin in time. Their mutual and autocorrelation time is thought to be about 5 days. We now write (cf. (2))

$$\dot{X}(t) + A(t)X(t) = Y(t), \tag{9}$$

with

$$X(t) = T(t), \tag{10a}$$

$$Y(t) = \lambda T_w(t)u(t), \tag{10b}$$

$$A(t) = \lambda u(t). \tag{10c}$$

Eq. (9) is an ordinary, first order, linear differential equation with stochastic coefficients. Its formal solution is easily found:

$$X(t) = X_0 \exp[-\int_0^t A(t')\mathrm{d}t'] + \int_0^t \mathrm{d}s \exp[-\int_s^t A(t')\mathrm{d}t']Y(s), \tag{11}$$

with

$$X_0 = X(0). \tag{12}$$

In order to get some insight into the behaviour of the variable X, we will compute the moments of X. We start with the first moment, $\langle X(t)\rangle$:

$$\langle X(t)\rangle = X_0 \langle \exp[-\int_0^t A(t')\mathrm{d}t']\rangle + \langle \int_0^t \mathrm{d}s \exp[-\int_s^t A(t')\mathrm{d}t']Y(s)\rangle. \tag{13}$$

The calculation of an average of an exponential of a stochastic variable poses a problem. To proceed we have to expand the exponential in some way. The most obvious way would be a straightforward expansion:

$$\langle \exp[-\int_0^t A(t')\mathrm{d}t']\rangle = 1 - \int_0^t \langle A(t')\rangle \mathrm{d}t' + \tfrac{1}{2}\int_0^t\int_0^t \langle A(t_1)A(t_2)\rangle \mathrm{d}t_1\,\mathrm{d}t_2 + \dots \tag{14}$$

However, this method fails, because we are interested in the stationary character of the solution, i.e. in the behaviour for $t \to \infty$. It turns out that in the expansion of (14) each term becomes infinite for $t \to \infty$.

Writing

$$A(t) = \alpha A_1(t), \tag{15}$$

with α a typical magnitude of the amplitude of the fluctuations in A, we may say that the abovementioned expansion has been carried out in powers of αt and this is not valid for large t. The assumption that higher moments of A can be expressed in first and second moments would give us the possibility to derive an integral equation for $\langle X\rangle$ (Bourret, 1962).

However, it is not necessary to make such an assumption, because there exists an expansion which is asymptotic and which in addition may be used for large t: the cumulant expansion (Kubo, 1963; Freed, 1968; Van Kampen, 1976). This is in essence an expansion of the exponent of the exponential and not of the exponential itself. In this way it is made possible to carry out the integrations in the exponential. The cumulants are defined by a generating function:

$$\langle \mathrm{e}^{tB}\rangle \equiv \exp\left[\sum_{m=1}^{\infty} \frac{t^m}{m!}\langle\langle B^m\rangle\rangle\right], \tag{16}$$

in which B is any stochastic variable. Using this function it turns out that:

$$\langle\langle B\rangle\rangle = \langle B\rangle, \tag{17a}$$

$$\langle\langle B_1 B_2\rangle\rangle = \langle B_1 B_2\rangle - \langle B_1\rangle\langle B_2\rangle. \tag{17b}$$

So the cumulants, denoted by double brackets, are certain combinations of the moments (Van Kampen, 1976).

We now substitute the cumulant expansion (16) into (13):

$$\langle X(t)\rangle = X_0 \exp[-\int_0^t \langle\langle A(t')\rangle\rangle \mathrm{d}t' + \tfrac{1}{2}\int_0^t\int_0^t \langle\langle A(t_1)A(t_2)\rangle\rangle \mathrm{d}t_1\,\mathrm{d}t_2 + \dots] + \langle \int_0^t \mathrm{d}s \exp[-\int_s^t A(t')\mathrm{d}t']Y(s)\rangle. \tag{18}$$

The mth term in the series in the exponent is proportional to $\alpha^m t\tau_c^{m-1}$ with α as in (15), and τ_c the autocorrelation time of the variable A. If the correlation time is short compared to the response time ($\simeq \langle A\rangle^{-1}$) we may expand with respect to the ratio of those two. In this case this is justified by data of the lightvessel *Texel* to be used later on (see also Section 3). In the following we will neglect terms of order τ_c^2 (or better $\alpha^3\tau_c^2$) and higher powers. The autocorrelation time of Y and the cross correlation times of A and Y are set equal to τ_c. In order to treat the second term of (18) in the same way as the first we use a trick (Soong, 1973). We define

$$Z(t) \equiv \int_0^t \mathrm{d}s \exp[-\int_s^t A(t')\mathrm{d}t' - \varepsilon Y(s)], \tag{19}$$

from which follows:

$$-\frac{\partial}{\partial \varepsilon}\langle Z(t)\rangle \Big|_{\varepsilon=0} = \int_0^t \mathrm{d}s \exp\left[-\int_s^t A(t')\mathrm{d}t'\right] Y(s), \tag{20}$$

In computing $\langle Z(t)\rangle$ we apply again the cumulant expansion:

$$\langle Z(t)\rangle = \left\langle \int_0^t \mathrm{d}s \exp\left[-\int_s^t A(t')\mathrm{d}t' - \varepsilon Y(s)\right]\right\rangle$$

$$= \int_0^t \mathrm{d}s \cdot \exp\left[-\int_s^t \langle\langle A(t')\rangle\rangle \mathrm{d}t' - \varepsilon\langle\langle Y(s)\rangle\rangle\right.$$

$$\left.+ \tfrac{1}{2}\langle\langle\{\int_s^t A(t')\mathrm{d}t' + \varepsilon Y(s)\}^2\rangle\rangle\right]. \tag{21}$$

The first cumulant is equal to the average (17a). For a stationary process B, we have furthermore $\langle B(t)\rangle = \langle B\rangle$, i.e. the average is time independent. In the appendix, expressions are derived for integrated second cumulants of stationary processes with a short (auto)correlation time. Using this we find for the third term in the exponent of (21):

$$\tfrac{1}{2}\langle\langle\{\int_s^t A(t')\mathrm{d}t' + \varepsilon Y(s)\}^2\rangle\rangle$$

$$= \tfrac{1}{2}\int_s^t\int_s^t \langle\langle A(t_1)A(t_2)\rangle\rangle \mathrm{d}t_1\,\mathrm{d}t_2$$

$$+ \varepsilon\int_s^t \langle\langle A(t')Y(s)\rangle\rangle \mathrm{d}t' + \mathrm{O}(\varepsilon^2)$$

$$= \tfrac{1}{2}(t-s)M_{AA} + \tfrac{1}{2}\varepsilon M_{AY} + \mathrm{O}(\varepsilon^2), \tag{22}$$

with

$$M_{AA} \equiv \int_{-\infty}^{\infty} \mu_{AA}(t')\mathrm{d}t' = \int_{-\tau_c}^{\tau_c} \langle\langle A(t)A(t+t')\rangle\rangle \mathrm{d}t' \tag{23}$$

$$M_{AY} \equiv \int_{-\infty}^{\infty} \mu_{AY}(t')\mathrm{d}t' = \int_{-\tau_c}^{\tau_c} \langle\langle A(t)Y(t+t')\rangle\rangle \mathrm{d}t', \tag{24}$$

(see appendix). Substitution of (22) in (21) gives

$$\langle Z(t)\rangle = \int_0^t \mathrm{d}s \exp\left[-(t-s)\langle A\rangle - \varepsilon\langle Y\rangle\right.$$

$$\left.+ \tfrac{1}{2}(t-s)M_{AA} + \tfrac{1}{2}\varepsilon M_{AY} + \mathrm{O}(\varepsilon^2)\right], \tag{25}$$

and so

$$-\frac{\partial}{\partial \varepsilon}\langle Z(t)\rangle \Big|_{\varepsilon=0} = (\langle Y\rangle - \tfrac{1}{2}M_{AY})$$

$$\times \int_0^t \mathrm{d}s \, \exp\left[-(t-s)(\langle A\rangle - \tfrac{1}{2}M_{AA})\right]$$

$$= \frac{\langle Y\rangle - \frac{1}{2}M_{AY}}{\langle A\rangle - \frac{1}{2}M_{AA}}\{1 - \exp\left[-t(\langle A\rangle - \tfrac{1}{2}M_{AA})\right]\}. \tag{26}$$

Insertion in (18) of the results obtained above gives:

$$\langle X(t)\rangle = X_0 \exp\left[-t(\langle A\rangle - \tfrac{1}{2}M_{AA})\right]$$

$$+ \frac{\langle Y\rangle - \frac{1}{2}M_{AY}}{\langle A\rangle - \frac{1}{2}M_{AA}}\{1 - \exp\left[-t(\langle A\rangle - \tfrac{1}{2}M_{AA})\right]\}. \tag{27}$$

Note that this expansion is only valid for sufficiently large t (i.e. $t > \tau_c$); this limit was used to integrate the second cumulants (see also the appendix). Note furthermore that M_{AA} and M_{AY} are of order $\alpha^2\tau_c$. In (27) there are also terms of order $\alpha^3\tau_c^2$ that we will neglect in the computation later on. In an analogous way we may now derive expressions for $\langle X^2(t)\rangle$ and $\langle X(t_1)X(t_2)\rangle$. The same method could be used to derive expressions for higher moments of X. Here we confine ourselves to the first and second moments, however.

From the second moment $\langle X(t_1)X(t_2)\rangle$ we derive an expression for the correlation coefficient function defined by

$$\rho_{xx}(t_1, t_2) = \frac{\langle(X(t_1) - \langle X(t_1)\rangle)(X(t_2) - \langle X(t_2)\rangle)\rangle}{\sigma_x(t_1)\,\sigma_x(t_2)} \tag{28}$$

where

$$\sigma_x(t) = (\langle X^2(t)\rangle - \langle X(t)\rangle^2)^{\frac{1}{2}} \tag{29}$$

is the standard deviation. Assuming stationarity this reduces to

$$\rho_{xx}(\tau) = \frac{\mu_{xx}(\tau)}{\sigma^2}, \tag{30}$$

where $\tau = t_2 - t_1$. From this we define the power spectral density function

$$S_{xx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i\omega\tau}\rho_{xx}(\tau)\mathrm{d}\tau. \tag{31}$$

We will now take the limit t to infinity in the expression for $\langle X(t)\rangle$ and $\langle X^2(t)\rangle$; thus we obtain the equilibrium values of these quantities. For $\langle X(t)\rangle$ we find from (27):

$$\langle X\rangle_e = \lim_{t\to\infty}\langle X(t)\rangle = \frac{\langle Y\rangle - \frac{1}{2}M_{AY}}{\langle A\rangle - \frac{1}{2}M_{AA}}. \tag{32}$$

Neglecting all terms of order $\alpha^3\tau_c^2$, we find

$$\langle X\rangle_e = \frac{\langle Y\rangle}{\langle A\rangle}\left(1 + \frac{1}{2}\frac{M_{AA}}{\langle A\rangle} - \frac{1}{2}\frac{M_{AY}}{\langle Y\rangle}\right). \tag{33}$$

Taking the same limit in the expression for $\langle X^2(t)\rangle$ and neglecting terms of order $\alpha^3\tau_c^2$ we find for this,

$$\langle X^2\rangle_e = \frac{\langle Y\rangle^2}{\langle A\rangle^2}\left(1 + \frac{3}{2}\frac{M_{AA}}{\langle A\rangle}\right) - \frac{2M_{AY}\langle Y\rangle}{\langle A\rangle^2} + \frac{1}{2}\frac{M_{YY}}{\langle A\rangle}, \tag{34}$$

with

$$M_{YY} \equiv \int_{-\infty}^{\infty}\mu_{YY}(s)\mathrm{d}s = \int_{-\infty}^{\infty}\langle\langle Y(t)Y(t+s)\rangle\rangle\mathrm{d}s. \tag{35}$$

We also compute the variance of X,

$$\sigma_e^2 \equiv \langle\langle X^2\rangle\rangle_e = \langle X^2\rangle_e - \langle X\rangle_e^2 = \frac{1}{2}\frac{M_{AA}\langle Y\rangle^2}{\langle A\rangle^3} - \frac{M_{AY}\langle Y\rangle}{\langle A\rangle^2} + \frac{1}{2}\frac{M_{YY}}{\langle A\rangle}, \tag{36}$$

and the power spectral density function

$$S_{xx}(\omega) = \frac{\langle A\rangle - \frac{1}{2}M_{AA}}{\pi}\,\frac{1}{\omega^2 + (\langle A\rangle - \frac{1}{2}M_{AA})^2}. \tag{37}$$

When we insert the definitions (10a–c) into (33), (34) and (36), we find for the sea surface temperature T,

$$\langle T\rangle_e = \frac{\langle T_w u\rangle}{\langle u\rangle}\left(1 + \frac{M_{AA}}{2\lambda\langle u\rangle} - \frac{M_{AY}}{2\lambda\langle T_w u\rangle}\right), \tag{38}$$

$$\langle T^2\rangle_e = \frac{\langle T_w u\rangle^2}{\langle u\rangle^2}\left(1 + \frac{3M_{AA}}{2\lambda\langle u\rangle}\right) - \frac{2M_{AY}\langle T_w u\rangle}{\lambda\langle u\rangle^2} + \frac{M_{YY}}{2\lambda\langle u\rangle}, \tag{39}$$

$$\sigma_e^2 = M_{AA}\frac{\langle T_w u\rangle^2}{2\lambda\langle u\rangle^3} - \frac{M_{AY}\langle T_w u\rangle}{\lambda\langle u\rangle^2} + \frac{M_{YY}}{2\lambda\langle u\rangle}. \tag{40}$$

These equations show how the lowest moments of the sea surface temperature, thought of as a response to atmospheric forcing, are given in terms of the lowest order moments of the forcing variables A and Y (cf. 10b and 10c). They are quantitative relations, which can be verified against experimental data. Such a verification is the subject of Section 3.

3. Application to observations of the lightvessel *Texel*

We have applied the model of Section 2 to daily observations (KNMI, 1965–1974) of the lightvessel *Texel* (53°00′ N, 4°23′ E) for the winters of 1965–1974. The lightvessel is situated in the North Sea about 30 km off the Dutch coast (Fig. 1). The observations we used are those of wet-bulb-temperature, wind speed (T_w, u) and sea temperature (T). The time series of T_w and u are used to calculate the integrated auto- and cross correlations for each winter. Then with the help of (38) and (40) the average sea temperature $\langle T\rangle$ and its variance σ^2 are calculated. The calculated values can then be compared with the experimental values, as determined from the SST.

As an example of the data involved we give in Fig. 2 part of the time series for u, T_w and T. The observations are those made at 18.00 GMT. We took these because we restrict ourselves to fluctuations of at least one day. The data of Fig. 2 was taken in January 1965. Selection of this month was quite arbitrary. Fig. 2 gives an idea of the stochastic behaviour of weather variables and the SST. Note the slow change of the SST as compared to the relatively fast fluctuations of u and T_w.

For the physical constants entering in the expressions for $\langle T\rangle$ and σ^2 we took the following values:

$\rho_a = 1.25$ kg/m³, $\rho_w = 1025$ kg/m³,

$c_{pa} = 1008$ J/(kg °C),

$c_{pw} = 4032$ J/(kg °C),

$h = 25$ m, $C_D = 0.0011$,

$L = 2.47 \times 10^6$ J/kg,

$s = 3.44 \times 10^{-4}$/°C.

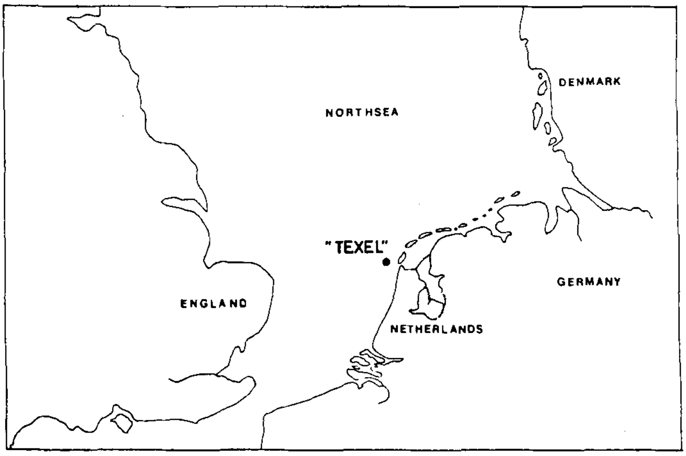

Fig. 1. Position of the lightvessel *Texel* in the North Sea.

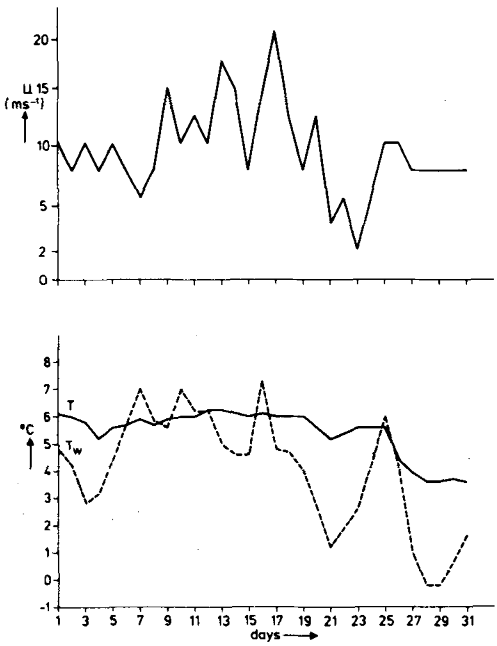

Fig. 2. Time series of u, T_w and T for the period 1-1-1965 till 31-1-1965 from the data of lightvessel *Texel*.

The mixed layer depth h was taken equal to the total depth of the water column, because it is known from Visser's (1977) description of the oceanographic situation near *Texel* that the water is well-mixed over the entire depth during winter time by the action of tide and wind. Actually s is T_w dependent, but the variations are small (less than 10% of its mean value), so we considered it as a constant.

The choice of the drag coefficient C_D poses a problem, because it is known to depend on the stability, i.e. on the difference of air and water temperature. Garratt (1977) used the following parameterization

$$
\begin{aligned}
C_D &= 0.001 \times (1 + 0.1 f(T - T_a)) \\
f(\tau) &= \tau \quad \tau \geqslant 0 \qquad\qquad (41) \\
&= 0 \quad \tau < 0.
\end{aligned}
$$

However, this temperature dependence complicates the differential equation seriously (it would become quadratic in T). Therefore, we took (41), with the average value of $T - T_a$ inserted. This average value (1 °C) then leads to $C_D = 0.0011$.

For a realistic comparison we had to estimate additional terms in the energy balance. Horizontally, heat is advected as a result of a non-vanishing residual current (Otto, 1964) plus a temperature gradient and also possibly by turbulent heat diffusion. In the vertical, radiation plays a role. The radiation balance is dominated by the incoming solar radiation and the outgoing infrared radiation. In the period January–March the sum of these terms is small, as compared to the other seasons. Each year separately we have taken the advection + radiation constant in the measuring period. However, it is allowed to vary from year to year. In the equation the advection and the net radiation are expressed by a constant term C_0 on the right-hand side of (3). This equation then reads

$$\frac{dT}{dt} = -\lambda\{T(t) - T_w(t)\}u(t) + C_0. \tag{42}$$

When C_0 is absorbed in $Y(t)$ (see eq. 10b) we arrive at the same expressions as before. In order to determine the value of C_0, we calculated for each period of three months the total change of heat content ΔQ in the water column at the position of the lightvessel. This could be done on the basis of the actual observations of the sea water temperature. The total flux of latent and sensible heat (SLH) was then determined on the basis of daily observations by integrating (7). When both ΔQ and SLH are expressed as an average change of temperature/day we have

$$C_0^{\text{exp}} = \Delta Q - \text{SLH}. \tag{43}$$

We used this value of C_0 to correct Y.

It is now straightforward to calculate $\langle T \rangle$ and σ with the help of (38) and (40). The integrals needed for the computation of M_{AA}, M_{AY} and M_{YY} (A11a–c) are evaluated with time steps of one day. The averages appearing in the integrands, like all averages in this model, are carried out over one cycle of observations (1st of January until 31st of March). The maximum correlation time τ_c in (23), (24) and (35) is taken equal to 7 days. In this way $\langle T \rangle$ and σ were obtained for 10 different winters (1965–1974).

First we will consider results for the average water temperature. Our values for $\langle T \rangle_{\text{th}}$, $\langle T \rangle_{\text{exp}}$, SLH, ΔQ and C_0^{exp} are given in Table 1. The agreement between $\langle T \rangle_{\text{th}}$ and $\langle T \rangle_{\text{exp}}$ is very poor. In an attempt to understand this we realize that two of our assumptions (stationarity and separation of time scales) are not always met in practice. This may be illustrated by two extreme cases. In 1974 strong warming by solar radiation occurred in March, especially in the second half of the month. This explains the large value of Q. That this leads to an overestimate of $\langle T \rangle$ can be understood, if one retains the term $\langle \partial T/\partial t \rangle$ in the averaged version of (42). In 1969 on the other hand, strong cooling took place during the month of February. This may have violated both the assumption of stationarity and the assumed short correlation time for the weather variables.

However, as we consider a stochastic model, more valuable results are to be expected from the higher moments calculated. Therefore, we give results for the variance of the SST in Table 2. These results were obtained from the observed statistics of the weather variables with the help of (40). C_0 was now chosen such that the mean value of the SST was reproduced. It turns out that the resulting variances do not strongly depend on the choice of C_0; in fact the correction of C_0 caused a change in the SST variance of at most 10%. We find that a notable part of the variance (square of the standard deviation) of the SST is determined by the (co)-variances of the wet-bulb-temperature and the wind velocity. Moreover, it is found that the year-to-year tendency of the experimental values of the variance is well reflected in the theoretical values.

Table 1. *Surview of heat fluxes obtained with data of lightvessel Texel. ΔQ = mean net heat increase (°C/day). SLH = mean sensible and latent heat flux (°C/day). C_0^{exp} = mean heat flux due to radiation and advection (°C/day)*

Year	$\langle T \rangle_{\text{th}}$ (°C)	$\langle T \rangle_{\text{exp}}$ (°C)	ΔQ (°C/day)	SLH (°C/day)	C_0^{exp} (°C/day)
1965	4.4	4.7	−0.009	−0.019	0.010
1966	6.2	4.8	−0.001	−0.036	0.035
1967	5.8	5.8	0.001	−0.021	0.022
1968	3.5	4.6	−0.014	−0.020	0.005
1969	5.0	4.7	−0.024	−0.033	0.009
1970	5.4	4.1	0.003	−0.031	0.034
1971	6.9	5.6	0.004	−0.027	0.031
1972	6.4	5.0	−0.010	−0.034	0.024
1973	6.6	6.2	−0.002	−0.019	0.017
1974	7.7	6.0	0.030	−0.013	0.043

Table 2. *Percentage of the total computed variance at constant u and T_w respectively (lightvessel Texel)*

		σ_T^2 (th)		
Year	σ_T^2 (exp)	Total	u° = const. (%)	T_w = const. (%)
1965	0.64	0.33	55	3
1966	1.20	0.73	77	1
1967	0.44	0.17	100	0
1968	0.46	0.02	100	50
1969	1.68	0.58	79	2
1970	0.24	0	—	—
1971	0.28	0.20	100	5
1972	0.78	0.58	19	5
1973	0.31	0.19	32	5
1974	0.21	0.04	100	0

Assuming a linear regression the correlation coefficient is 0.84, which is substantially above the significance level of 0.30 (see Fig. 3). The angle $\phi = 57°$ of the best fit is different from 45°. This reflects the neglect of other sources of variance. It may be concluded that the errors introduced by neglecting these other sources such as advection, radiation and variation of C_D are not dominant. A rough estimate of these errors indicates that they are of the expected order of magnitude. A precise determination has not been possible. However, the contribution of the measuring error to the observed standard deviation is estimated to be 0.2 °C.

In Fig. 4 the calculated spectrum of the SST fluctuations is given, averaged over 10 years. As found earlier by Frankignoul and Hasselmann (1977) it shows an inverse square law for high frequencies, while the flattening in the low frequency domain corresponds to a feedback factor $\langle A \rangle - \frac{1}{2}M_{AA} \sim (2.2 \text{ months})^{-1}$, which is of the expected order of magnitude (see also Frankignoul and Hasselmann, 1977, and Section 4). Here the importance of $\frac{1}{2}M_{AA}$ is negligible, being less than about 1% of $\langle A \rangle$. It is not possible to compare these results with experimental data: for this the record length (3 months) was insufficient.

The assumption of neglecting terms of order τ_c^2 and higher powers of τ_c in the cumulant expansion of (16), implicitly proved to be valid by the results of Table 2, is checked explicitly. Therefore the integrated third cumulants are calculated with the help of the measurements of 1965 from the lightvessel *Texel*. We obtained values which are 3 orders of magnitude smaller than the second cumulants.

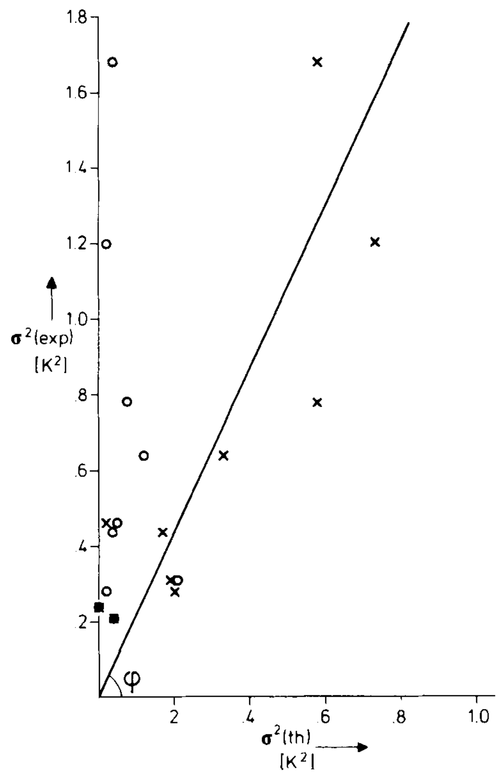

Fig. 3. Calculated against experimental values of the variance of the SST ($\tan \phi = 2.17$). × = calculated with our model. O = calculated with the model of Frankignoul and Hasselmann.

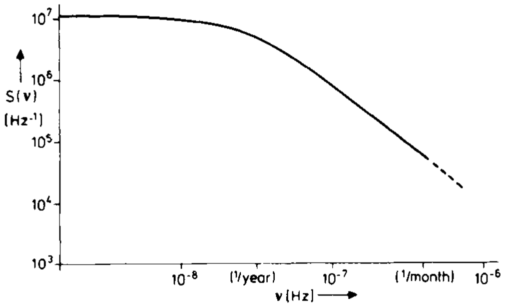

Fig. 4. Calculated spectrum of the SST anomalies as calculated from (37) with $\langle A \rangle - \frac{1}{2}M_{AA} = (2.2 \text{ months})^{-1}$.

4. Comparisons with the model of Frankignoul and Hasselmann

The correlation between the year-to-year tendency in σ^2 (exp) and σ^2 (th) calculated from our model proved to be significant. A crucial question remains: is the extension of the model of Frankignoul and Hasselmann necessary to obtain this result?

Frankignoul and Hasselmann started from the equation

$$\dot{X}'(t) + A'X'(t) = Y'(t), \tag{44}$$

where

$$X'(t) = T'(t) \quad \text{(SST)}, \tag{45a}$$

$$A' = \lambda'\langle u \rangle, \quad \lambda' = \frac{C_D}{h}\frac{\rho_a c_{pa}}{\rho_w c_{pw}}(1 + B), \tag{45b}$$

$$Y'(t) = \lambda'\langle T_a \rangle u(t). \tag{45c}$$

This deviates from (9, 10) in three respects:

(a) a constant Bowen ratio was assumed (this is not realistic in our case, De Bruin, private communication);
(b) the heat flux is parameterized in terms of $T - T_a$: the wet-bulb-temperature is not involved;
(c) the wind velocity u in Y' is the only stochastic variable in the model.

The first two assumptions have been made for convenience of their numerical simulation. The third one implies a change in the stochastic character. We want to compare our results with those of Frankignoul and Hasselmann, just with respect to the difference in stochastic treatment. Therefore a comparison is made between our model and a model with the same physical parameterization but with the stochastic simplification made by Frankignoul and Hasselmann:

$$\dot{X}''(t) + A''X''(t) = Y''(t), \tag{46}$$

where

$$A'' = \lambda\langle u \rangle, \quad \lambda = \frac{C_D}{h}\frac{\rho_a c_{pa}}{\rho_w c_{pw}}\left(1 + \frac{Ls}{c_{pa}}\right), \tag{47a}$$

$$Y''(t) = \lambda\langle T_w \rangle u(t). \tag{47b}$$

Evaluation in the same way as in Section 2 yields

$$\langle T'' \rangle = \langle T_w \rangle, \tag{48}$$

$$(\sigma'')^2 = \frac{1}{2}\frac{M_{Y''Y''}}{\langle A'' \rangle}. \tag{49}$$

In Fig. 3 the results for the variance are indicated by open circles. Much less agreement with the experimental values of the variance is found (correlation coefficient of only 0.14). Therefore we conclude that the stochastic character of the wet-bulb-temperature and the wind velocity in the feedback parameter is important.

In order to estimate the influence of each of the stochastic weather variables separately we have also performed our calculations with successively u = constant ($M_{AA} = M_{AY} = 0$) and T_w = constant ($M_{YY} = \langle T_w \rangle M_{AY} = \langle T_w \rangle^2 M_{AA}$). From Table 2 the influence of the stochastic character of T_w on σ^2(th) appears to be much larger than that of u.

5. Conclusions

We extended the work of Frankignoul and Hasselmann (1977) on stochastic forcing by allowing for a more general equation containing a linear term with a stochastic coefficient. Following Hasselmann we considered stationary processes and assumed a different time scale for the forcing (weather variables) and response (climate variables). With these assumptions we obtained our results (38–40), which express the average values and the variance of the climate variable in terms of the stochastic properties (cumulants) of the weather variables. In the derivation the cumulant expansion proved to be a powerful tool.

The model was applied to observations of the lightvessel *Texel*. It has problems in reproducing the observed averaged SST for reasons which have been discussed in Section 3, and which are related to the possible non-stationarity of the time-series and the occasional occurrence of weather behaviour with longer correlation times. Correcting for this, we see that a notable fraction of the experimental values of the variance is determined by the (co)variances of the wet-bulb-temperature and the wind velocity and, moreover, that the year-to-year tendency of the experimental variance is well reflected in the theoretical values. A comparison between our model and the model of Frankignoul and Hasselmann shows that our extensions are useful: our calculated values of the variance do reflect the experimental values better. This means that in winter time near the coast, the variations in the feedback parameter and in the wet-bulb-temperature, and thus in the Bowen ratio, cannot be neglected. The overall shape of the spectrum was the same in both models. Due to the record length of our data sets (3 months) it was not possible to compare this with an observed spectrum.

The model is very general in its framework. It is essentially a linear (heat) flux balance. Therefore, it may be applicable to various other problems, for example to the description of the extent of the polar ice sheets (Oerlemans, 1979), but in that case radiation would be very important. Further extensions of this model are easily guessed. One may generalize to space dependent equations in order to obtain a model of an ocean as a whole, with currents and coastal effects included. One may look at the influence of radiation. It is also possible to conceive more weather variables as stochastic variables.

6. Acknowledgements

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7. Appendix

Method for integration of second cumulants of stationary, stochastic processes with short correlation time

In applying the cumulant expansion we encounter integrals of the following type:

$$I_0 \equiv \int_{u_2}^{s_2}\int_{u_1}^{s_1} \langle\langle B(t_1)C(t_2)\rangle\rangle \mathrm{d}t_1\,\mathrm{d}t_2, \quad u_1 < s_1,\, u_2 < s_2. \tag{A.1}$$

In general, it is not possible to say much about I_0. Here, we assumed that B and C are stationary and have a short correlation time (in this case short in comparison with the length of the integration interval) and from this more can be said about I_0.

We consider first a simpler case:

$$I_1 \equiv \int_0^t\int_0^t \langle\langle B(t_1)B(t_2)\rangle\rangle \mathrm{d}t_1\,\mathrm{d}t_2$$

$$= \int\int \langle\langle B(t_1)B(t_1+s)\rangle\rangle \mathrm{d}t_1\,\mathrm{d}s. \tag{A.2}$$

Because B is stationary, the integrand of (A.2) only depends on the time-difference $t_2 - t_1 = s$, in which case we find only contributions to the integral for $|t_1 - t_2| < \tau_c$:

$$I_1 = t\int_{-\tau_c}^{\tau_c} \mu_{BB}(s)\mathrm{d}s = t\int_{-\infty}^{\infty} \mu_{BB}(s)\mathrm{d}s \equiv tM_{BB}. \tag{A.3}$$

with M_{BB} = constant. The double integral reduces to a simple one. This is illustrated in Fig. A1.

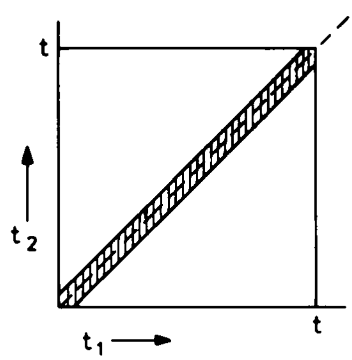

Fig. A1. Integration region of I_1; contributions in the shaded area around $t_1 = t_2$ only.

We subsequently consider

$$I_2 \equiv \int_u^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \langle\langle B(t_1)B(t_2)\rangle\rangle. \quad \text{(A.4)}$$

For this we find (Fig. A2):

$$I_2 = (t-u)\int_{-\tau c}^{\tau c} \mu_{BB}(s)\mathrm{d}s = (t-u)M_{BB}. \quad \text{(A.5)}$$

One now may also find an expression for I_0 from (A.3) (Fig. A3):

$$I_0 = (\min(s_1, s_2) - \max(u_1, u_2))\int_{-\tau c}^{\tau c} \mu_{BC}(s)\mathrm{d}s$$

$$= (\min(s_1, s_2) - \max(u_1, u_2))M_{BC}. \quad \text{(A.6)}$$

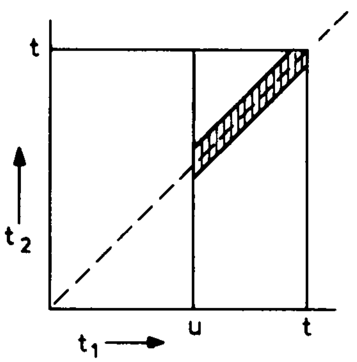

Fig. A2. Integration region of I_2; contributions in the shaded area around $t_1 = t_2$ only.

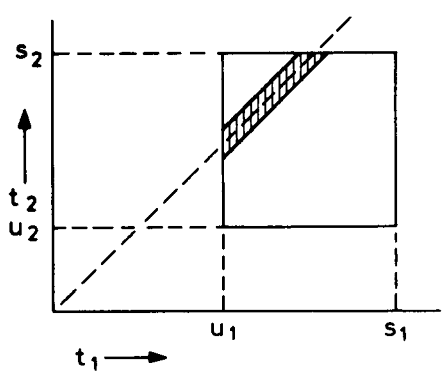

Fig. A3. Integration region of I_3; contributions in the shaded area around $t_1 = t_2$ only.

We also encounter integrals of the type:

$$I_3 \equiv \int_{t_1}^{t_2} \langle\langle B(s)C(t_3)\rangle\rangle \mathrm{d}s, \quad t_1 < t_2, \quad \text{(A.7)}$$

There are several possibilities:

$$t_3 < t_1 \text{ or } t_3 > t_2: \quad I_3 = 0, \quad \text{(A.8a)}$$

$$t_3 = t_1 \text{ or } t_3 = t_2: \quad I_3 = \tfrac{1}{2}M_{BC}, \quad \text{(A.8b)}$$

$$t_1 < t_3 < t_2,\ t_3 - t_1 > \tau_c,\ t_2 - t_3 > \tau_c: \quad I_3 = M_{BC}. \quad \text{(A.8c)}$$

In the case of (A.8b), one should note that we assume that $\mu_{BC}(s)$ is symmetrical, which means

$$\int_{-\tau c}^{0} \mu_{BC}(s)\mathrm{d}s = \int_0^{\tau c} \mu_{BC}(s)\mathrm{d}s = \tfrac{1}{2}M_{BC}. \quad \text{(A.9)}$$

If in (A.8c) $t_1 < t_3 < t_1 + \tau_c$ or $t_2 - \tau_c < t_3 < t_2$, nothing can be said about I_3 without more knowledge of $\mu_{BC}(s)$. In the cases where that takes place, there is in our situation always an integration with respect to t_1 or t_2 too; in that case the errors made are of the order $\alpha^2\tau_c^2$. For example:

$$\int_0^t \mathrm{d}uF(t,u)\int_0^t \langle\langle B(u)C(t')\rangle\rangle \mathrm{d}t'$$

$$= \int_0^t \mathrm{d}uF(t,u)\int_{-u}^{t-u} \mu_{BC}(s)\mathrm{d}s$$

$$= \int_{\tau c}^{t-\tau c} \mathrm{d}uF(t,u)\int_{-u}^{t-u} \mu_{BC}(s)\mathrm{d}s + \int_0^{\tau c} \ldots$$

$$+ \int_{t-\tau c}^{t} \ldots = M_{BC}\int_0^t \mathrm{d}uF(t,u) + O(\tau_c^2). \quad \text{(A.10)}$$

The occurring integrated second cumulants are in our case:

$$M_{AA} \equiv \int_{-\infty}^{\infty} \mu_{AA}(s)\mathrm{d}s = \int_{-\tau c}^{\tau c} \langle\langle A(t)A(t+s)\rangle\rangle \mathrm{d}s, \quad \text{(A.11a)}$$

$$M_{AY} \equiv \int_{-\infty}^{\infty} \mu_{AY}(s)\mathrm{d}s = \int_{-\tau c}^{\tau c} \langle\langle A(t)Y(t+s)\rangle\rangle \mathrm{d}s, \quad \text{(A.11b)}$$

$$M_{YY} \equiv \int_{-\infty}^{\infty} \mu_{YY}(s)\mathrm{d}s = \int_{-\tau c}^{\tau c} \langle\langle Y(t)Y(t+s)\rangle\rangle \mathrm{d}s. \quad \text{(A.11c)}$$

REFERENCES

Bourret, R. C. 1962. Propagation of randomly perturbed fields. *Can. J. Phys. 40*, 782.

Frankignoul, C. and Hasselmann, K. 1977. Stochastic climate models, Part II. *Tellus 29*, 289–305.

Freed, J. H. 1968. Generalized cumulant expansions and spin-relaxation theory. *J. Chem. Phys. 49*, 376.

Garratt, J. R. 1977. Review of drag coefficients over oceans and continents. *Mon. Wea. Rev. 105*, 915–929.

Hasselmann, K. 1976. Stochastic climate models, Part I. *Tellus 28*, 473–485.

Keijman, J. Q. 1974. The estimation of the energy balance of a lake from simple weather data. *Boundary-Layer Meteorol. 7*, 399–407.

KNMI 1965–1974. *Meteorologische en Oceanografische waarnemingen verricht aan boord van Nederlandse lichtschepen in de Noordzee*, KNMI, De Bilt.

Kraus, E. B. 1972. *Atmosphere-ocean interaction.* Oxford: Clarendon Press.

Kubo, R. 1963. Stochastic Liouville equations. *J. Math. Phys. 4*, 174.

Oerlemans, J. 1979. A model of a stochastically driven ice sheet with planetary wave feedback. *Tellus 31*, 469–477.

Otto, L. 1964. Results of current observations at the Netherlands lightvessels over the period 1910–1939. *Med. & Verh. 85*, KNMI, De Bilt.

Soong, T. T. 1973. *Random differential equations in science and engineering*, ch. 8. New York and London: Academic Press.

Van Kampen, N. G. 1976. Stochastic differential equations. *Phys. Reports 24*, No. 3, 171–228.

Visser, M. 1977. Oceanografische omstandigheden in het zeegebied bij Texel. KNMI, De Bilt, Scientific report WR 77-1.

Wang, M. C. and Uhlenbeck, G. E. 1945. On the theory of the Brownian motion. *Rev. Mod. Phys. 17*, 323–342.