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SHORT CONTRIBUTION

Available potential energy and simplifications to the adiabatic equation

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In this paper the known property of the "quasi-geostrophic" form of the adiabatic equation to conserve rather than convert mean total potential energy (TPE) or available potential energy (APE) is analysed by tracing the contributions to the conversion to kinetic energy (KE), made by individual terms of the adiabatic equation. The approach and notation of a previous paper (Burger and Riphagen, 1979, hereafter referred to as BR1) are used, and that paper's final example is amplified. Application of the same approach to the Lorenz (1955) form of APE then demonstrates the analytical mechanism which enables an acceptable energy behaviour to be obtained in this context.

As was shown in eq. (5.2) of BR1, for a scalar E,

$$\frac{d}{dt}\int E\,dM = \int\left[\frac{\partial E}{\partial t} - \frac{\partial}{\partial t}\left(E\frac{\partial z}{\partial t}\Big/\frac{\partial z}{\partial p}\right)\right]dM$$

where the integration is over the mass of the atmosphere. The rate of change of mean TPE (for a quasi-hydrostatic atmosphere) may now be expressed as

$$\begin{aligned}\frac{d}{dt}&\int (c_v T + gz)\,dM\\ &= \frac{d}{dt}\int\left[c_p T + \frac{\partial}{\partial p}(pgz)\right]dM\\ &= \int\Bigg(\frac{\partial}{\partial t}c_p T + \frac{\partial}{\partial p}\frac{\partial}{\partial t}(pgz)\\ &\qquad - \frac{\partial}{\partial p}\left\{\left[c_p T + \frac{\partial}{\partial p}(pgz)\right]\frac{\partial z}{\partial t}\Big/\frac{\partial z}{\partial p}\right\}\Bigg)dM\\ &= \int\Bigg\{\frac{\partial}{\partial t}c_p T\\ &\qquad - \frac{\partial}{\partial p}\left[(c_p T + gz)\frac{\partial z}{\partial t}\Big/\frac{\partial z}{\partial p}\right]\Bigg\}dM \qquad (1)\end{aligned}$$

Also, for any suitably well-behaved scalar A, the Gauss divergence theorem with the kinematic boundary condition that the component of velocity normal to the surface of the earth must vanish, gives

$$\int\left[\nabla\cdot\{A\mathbf{V}\} + \frac{\partial}{\partial p}\{A\omega\} + \frac{\partial}{\partial p}\left\{A\frac{\partial z}{\partial t}\Big/\frac{\partial z}{\partial p}\right\}\right]dM = 0 \qquad (2)$$

[See eq. (5.4) of BR1.]

In what follows we shall apply the simpler boundary condition $\omega = 0$ at $p = p_0 =$ constant, on a level earth, in which case the first two (divergence) terms of (2) vanish separately under integration, and thus also its last term and that of (1). The energy conversion is therefore completely characterized by the expression $\partial c_p T/\partial t$, and can thus be straightforwardly evaluated from the thermodynamic equation.

For an atmosphere which is quasi-hydrostatic, adiabatic, frictionless quasi-geostrophic ($Ro \ll 1$), stratified, of limited static stability and of moderate

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horizontal scale, the thermodynamic equation may be written,

$$\frac{\partial}{\partial t} c_p T = -\omega g \frac{dz_s}{dp} - \omega \frac{dc_p T_s}{dp} - \mathbf{V}_\psi \cdot \nabla c_p T$$

$$A\ell(Ro^{-1}) \quad A_1(Ro^{-2}) \quad A_2(Ro^{-2}) \quad A_3(Ro^{-1})$$

$$-\omega g \frac{\partial(z - z_s)}{\partial p} - \mathbf{V}_x \cdot \nabla c_p T - \omega \frac{\partial c_p(T - T_s)}{\partial p}$$

$$A_4(1) \quad A_5(1) \quad A_6(1) \qquad (3)$$

where each term has been labelled for future reference, while the characterizing relative *Ro*-order of magnitude is given in parentheses.

For later use the potential temperature form of the equation is written with labelling of terms according to their origin in (3) and relative orders of magnitude. Noting that the sum of the two largest terms A_1 and A_2 is an order of magnitude smaller than the terms themselves, we have

$$\frac{\partial \theta}{\partial t} + \mathbf{V}_\psi \cdot \nabla \theta + \mathbf{V}_x \cdot \nabla \theta + \omega \frac{d\theta_s}{dp}$$

$$A\ell(Ro^{-1}) \quad A_3(Ro^{-1}) \quad A_5(1) \quad A_1, A_2(Ro^{-1})$$

$$+ \left[\omega \frac{\partial(\theta - \theta_s)}{\partial p} + \omega \frac{\kappa}{p}(\theta - \theta_s) \right] - \omega \frac{\kappa}{p}(\theta - \theta_s) = 0$$

$$A_6(1) \qquad A_4(1) \qquad (4)$$

Due to the important role of A_4, we have not here simplified the last terms by combining A_4 and A_6.

With the aid of the continuity equation, (3) becomes

$$\frac{\partial}{\partial t} c_p T = -\omega \frac{\partial}{\partial p} \{g(z - z_s)\}$$

$$A\ell(Ro^{-1}) \qquad A_4(1)$$

$$- \nabla \cdot \{g z_s \mathbf{V}\} - \frac{\partial}{\partial p} \{g z_s \omega\}$$

$$A_1(Ro^{-2})$$

$$- \nabla \cdot \{c_p T_s \mathbf{V}\} - \frac{\partial}{\partial p} \{c_p T_s \omega\}$$

$$A_2(Ro^{-2})$$

$$- \nabla \cdot \{c_p(T - T_s)\mathbf{V}_\psi\}$$

$$A_3(Ro^{-1})$$

$$- \nabla \cdot \{c_p(T - T_s)\mathbf{V}_x\} - \frac{\partial}{\partial p} \{c_p(T - T_s)\omega\}$$

$$A_5, A_6(1) \qquad (5)$$

The rate of change of integrated TPE (and therefore of APE) is thus

$$\frac{d}{dt} \int (c_v T + gz)\, dM = \int \frac{\partial}{\partial t} c_p T\, dM$$

$$= -\int \omega \frac{\partial}{\partial p} \{g(z - z_s)\}\, dM$$

$$A_4(1) \qquad (6)$$

which (with the aid of the continuity equation) balances the rate of change of integrated KE, namely $-\int \mathbf{V} \cdot \nabla g(z - z_s)\, dM$.

We have thus identified A_4 as the sole contributor to the conversion from TPE to KE, and thus from APE to KE.

Since we apply the simpler boundary condition $\omega = 0$ at $p = p_0$, A_1 to A_3 may be retained or omitted individually in the thermodynamic equation without altering the energy total, but A_5 must be retained or omitted with A_6 since this pair of terms was brought into divergence form by the addition of the continuity equation, both terms of which we naturally retain. The omission of the term A_4 would mean that TPE (or APE) would be conserved, and thus the sum of mean KE and mean APE would no longer be conserved [see also BR1, following (9.11)].

Lorenz (1960) and Haltiner (1971, pages 72–79), working with the same boundary condition and an approximate form for APE derived by Lorenz (1955), indicate that the omission of terms A_4 to A_6 from the thermodynamic equation (3) or (4) to yield the "quasi-geostrophic thermodynamic equation",

$$\frac{\partial}{\partial t} c_p T = -\omega g \frac{dz_s}{dp} - \omega \frac{dc_p T_s}{dp} - \mathbf{V}_\psi \cdot \nabla c_p T$$

$$A\ell(Ro^{-1}) \quad A_1(Ro^{-2}) \quad A_2(Ro^{-2}) \quad A_3(Ro^{-1}) \qquad (7)$$

or

$$\frac{\partial \theta}{\partial t} + \mathbf{V}_\psi \cdot \nabla \theta + \omega \frac{d\theta_s}{dp} = 0,$$

$$A\ell(Ro^{-1}) \quad A_3(Ro^{-1}) \quad A_1, A_2(Ro^{-1}) \qquad (8)$$

while leading to the erroneous conservation of TPE, still allows for conversion of APE to KE.

We now examine the approximate expression for APE proposed by Lorenz (1955), in particular in relation to the omission or retention of terms A_4 to A_6 in the thermodynamic equation (3).

When the lower boundary is a geopotential surface, APE is given by

$$c_p g^{-1}(1+\kappa)^{-1} p_0^{-\kappa} \int_0^\infty (\overline{p^{1+\kappa}} - \bar{p}^{1+\kappa})\, d\theta \tag{9}$$

where the bars denote averages over an isentropic surface. Lorenz (1955) shows that this expression may be approximated by

$$\bar{A} = \frac{\kappa}{2} \frac{c_p}{g} p_0^{-\kappa} \int_0^{p_0} p^{\kappa-1} \left(-\frac{\partial\bar{\theta}}{\partial p}\right)^{-1} \overline{\theta_1^2}\, dp \tag{10}$$

in which the bars denote averages over an isobaric surface, and $\theta_1 = \theta - \bar{\theta}$. Neglecting time variations of $\bar{\theta}$, one obtains

$$\frac{\partial\bar{A}}{\partial t} \approx -\frac{R}{g}\int_0^{p_0} \frac{1}{p}\left(\frac{\partial\bar{\theta}}{\partial p}\right)^{-1}\left(\frac{p}{p_0}\right)^{\kappa} \times \frac{\partial}{\partial t}\left(\frac{\overline{\theta_1^2}}{2}\right) dp \tag{11}$$

In order to obtain an expression for the integrand in (11), we take the average, on an isobaric surface, of θ_1 times (4), obtaining (with direct transfer of relative Ro-orders),

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{\overline{\theta_1^2}}{2}\right)}_{A\ell(Ro^{-1})} + \underbrace{\overline{\mathbf{V}_x\cdot\nabla\left(\frac{\theta_1^2}{2}\right)}}_{A_5\,(1)} + \underbrace{\overline{\theta_1\omega}\frac{d\theta_s}{dp}}_{A_1, A_2\,(Ro^{-1})}$$
$$\underbrace{-\frac{\kappa}{p}\left[\overline{\theta_1\omega}(\bar{\theta}-\theta_s) + \overline{\omega\theta_1^2}\right]}_{A_4\,(1)}$$
$$\underbrace{+\overline{\theta_1\omega}\frac{\partial(\bar{\theta}-\theta_s)}{\partial p} + \overline{\omega\frac{\partial}{\partial p}\left(\frac{\theta_1^2}{2}\right)}}_{A_6\,(1)}$$
$$\underbrace{+\frac{\kappa}{p}\left[\overline{\theta_1\omega}(\bar{\theta}-\theta_s) + \overline{\omega\theta_1^2}\right]}_{A_6\,(1)} = 0 \tag{12}$$

Straightforward simplification of (12) gives

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{\overline{\theta_1^2}}{2}\right)}_{(Ro^{-1})} + \underbrace{\frac{\partial\bar{\theta}}{\partial p}\overline{\theta_1\omega}}_{(Ro^{-1})} + \underbrace{\overline{\mathbf{V}_x\cdot\nabla\left(\frac{\theta_1^2}{2}\right)}}_{(1)} + \underbrace{\overline{\omega\frac{\partial}{\partial p}\left(\frac{\theta_1^2}{2}\right)}}_{(1)} = 0 \tag{13}$$

Omitting the last two (small) terms of (13) and substituting into (11), we obtain Lorenz's result:

$$\frac{\partial\bar{A}}{\partial t} \approx \frac{1}{g}\int_0^{p_0} \frac{R}{p}\left(\frac{p}{p_0}\right)^{\kappa} \overline{\theta_1\omega}\, dp \approx \int \frac{R(T-T_s)}{p}\omega\, dM$$
$$= -\int \omega \frac{\partial}{\partial p}\{g(z-z_s)\}\, dM \tag{14}$$

which is in accordance with the expression in (6) for the general case. Approximation of $\partial\bar{\theta}/\partial p$ by $d\theta_s/dp$ in (11) and (12) would also lead to (14).

If the thermodynamic equation is simplified by omitting terms A_4 to A_6 as in (7) or (8), and $\partial\bar{\theta}/\partial p$ is replaced by $d\theta_s/dp$ in (11) and (12), the result once again is (14). If A_4 is retained and A_5 and A_6 are omitted, or vice versa, the dominant terms in (13) are not affected, so that (14) still emerges.

Thus, whether or not terms A_4 to A_6 are retained in the thermodynamic equation, we have approximate conservation of the sum of mean KE and approximate APE as defined in (10).

In each case, the chief contribution to $\partial\bar{A}/\partial t$ is made by terms originating from A_1 and A_2, whereas in the strict expression (6), the only term contributing to the rate of change of mean APE is A_4.

The way in which this comes about is that, in the evaluation of the actual rate of change of mean APE by integrating $c_p(p/p_0)^\kappa$ times (4), and of $\partial\bar{A}/\partial t$ by integrating $-Rp^{-1}(\partial\bar{\theta}/\partial p)^{-1}(p/p_0)^\kappa\ \theta_1$ times (4), it turns out that the "A_1, A_2" term in the latter integrand is approximately equal to the "A_4" term in the former, viz. $-\omega(R/p)(p/p_0)^\kappa(\theta-\theta_s)$.

We have thus demonstrated the analytical mechanism behind the usefulness of Lorenz's expression in the quasi-geostrophic formulation. What we have in fact shown is that the critical role of the perturbation part of the static stability term in the energy conversion is taken over, in the Lorenz formulation in quasi-geostrophic context, by the dominant part of that term.

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