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On inertial instability of the equatorial undercurrent

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ABSTRACT

Hughes (1979, *Tellus 31*, 447–455) has noted that given a particular distribution of potential vorticity with stream function, it may theoretically be possible to have two forms for the equatorial undercurrent. As the vertical thickness of the layer, through which the undercurrent is flowing, decreases the undercurrents approach each other in form until a critical thickness is reached at which the undercurrent forms coalesce. The present study considers the behaviour of a slowly moving, long wavelength, small amplitude symmetric wave disturbance in the presence of an undercurrent of near coalescent form. It is shown that the narrower of the undercurrents is inertially unstable and subcritical to a Kelvin-like disturbance. The wider of the undercurrents is inertially stable and supercritical to such a disturbance. High mode Rossby waves are only slightly affected by the equatorially confined undercurrent.

1. Introduction

The study of instabilities in the equatorial ocean is extremely important to our understanding of the mechanics and predictability of that ocean. Philander (1976), in a pioneering work, has given a good review of the studies relevant to the stability of equatorial currents in an oceanographic context. Philander (1976, 1978) has concluded that the westward surface flow in the Pacific and Atlantic Oceans may be unstable to westward propagating waves with a period of about 30 days and a wavelength of 1100 km. He has found strong observational evidence to support his ideas. Philander (1978) has cited wave motions observed in the numerical model of Cane (1979) as further evidence of this instability. Philander has also concluded that the equatorial undercurrent in the Pacific Ocean may be unstable in March and April. He found that such an instability would have a wavelength of about 900 km and a period of approximately 40 days. He suggested that such an instability may explain the measurements of Taft et al. (1974).

Hughes (1979) (referred to as I) has noted that for a given distribution of potential vorticity with stream function it is often theoretically possible to have two equatorial undercurrent forms. The presence of two solutions suggests that possibly one solution is stable and the other unstable to long wave disturbances. It is of interest in the study of the undercurrent to establish the stability characteristics of these multiple solutions. In this work the stability of these solutions to slow moving symmetric disturbances is considered. However, it should be noted that most earlier studies have suggested that an anti-symmetric disturbance rather than a symmetric disturbance is likely to be the most unstable mode for a symmetric current system (see Philander, 1976).

2. The model

The equatorial ocean is represented here by a simple three-layer model. Each of the layers is of constant density. The upper layer corresponds to the well-mixed surface layer of the ocean. This layer is of the order of 10^2 m in thickness. It is sufficiently slowly moving that advective accelerations do not contribute significantly to the dynamical balance of the layer. The centre layer is associated with the region of the ocean occupied

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by the thermocline. This layer is also of the order of 10^2 m in thickness. However, it is not sufficiently slowly moving that advective accelerations may be neglected everywhere in the dynamical balance. This layer contains the equatorial undercurrent. The lower layer represents the region below the thermocline. This layer is taken to be infinitely deep and passive. Fig. 1 gives a diagrammatic representation of the vertical structure of the model. The use of this three-layer model is a gross simplification of the observed structure of the equatorial oceans. However, the model may be used to give valuable insight into the expected stability of the equatorial undercurrent.

The undercurrent, which is confined to the centre layer, is assumed to be equatorially symmetric in this study. Field observations (for example those of Rotschi et al., 1972) indicate that this assumption is often reasonable. Philander (1973) has discussed this assumption in detail. He has concluded that systematic variations in the structure of the undercurrent from symmetry are not discernible with the data available. However there is strong evidence to suggest that equatorially symmetry may be disturbed by the presence of a local north–south wind stress. The undercurrent considered here is confined to latitudes, y, less than some value, y_e, which is of the order of the first baroclinic mode equatorial radius. Field observations show that the equatorial undercurrent is often meridionally bounded. In the Pacific Ocean it is flanked on both sides by the westward-flowing south equatorial current. This current is neglected in the model presented here. The undercurrent half width, y_e, is observed to be about 1.5 deg. of latitude.

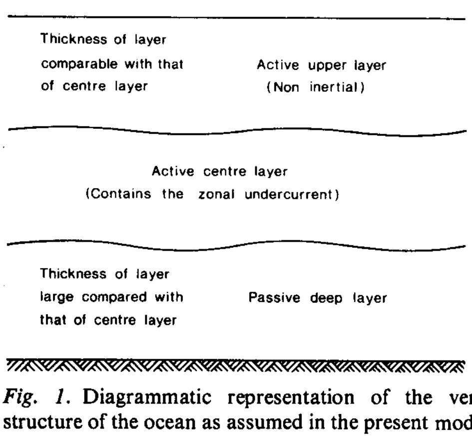

Fig. 1. Diagrammatic representation of the vertical structure of the ocean as assumed in the present model.

In order to study the stability of the undercurrent it is necessary to introduce a disturbance to the undercurrent. It is assumed here that the undercurrent is disturbed by a small amplitude periodic wave. This wave disturbance is taken to be of long wavelength. This requires the wavelength to be much greater than both an equatorial radius of deformation and the inertial-geostrophic distance scale. The inertial-geostrophic distance scale is defined by $\sqrt{U/\beta}$ where U is a characteristic zonal velocity in the undercurrent and β is the meridional gradient of the Coriolis parameter. This condition on the wavelength of the disturbance is generally satisfied for a disturbance of wavelength greater than 15 deg. of longitude. It is also necessary to apply an upper bound to the wavelength of the disturbance, so that the disturbance may be regarded as propagating through a zonally homogeneous environment. This restriction is generally met for a disturbance with a wavelength less than the basin width. In this study the phase velocity, C_s, is restricted so that $|C_s^2| \ll |UU_w|$ where U_w is a characteristic zonal velocity amplitude induced by the small disturbance. This restriction is not as limiting as it first appears because it will be shown that for the instability mechanism given here, neutral stability of the undercurrent is accompanied by a zero phase velocity of the disturbance.

Before proceeding with the analysis of the stability of the undercurrent, it is necessary to note that in the model just described the behaviour of the centre layer can be studied separately from that of the upper and lower layers. This can be understood by noting that the zonal and meridional momentum equations are independent of the behaviour of the velocity and pressure in the other layers. Thus the only coupling between the layers is via the continuity equation through the dependence of the layer thickness on the pressure field in the other layers. If vertical accelerations are very much smaller than the reduced gravitational accelerations between the layers, hydrostatic balance may be used to obtain the depths of the various layers in terms of their pressures. However, the pressure in the low layer must be constant because the layer is passive. Also a simple order of magnitude argument based on the properties of the model shows that for a slowly moving disturbance the pressure variation in the slowly moving upper layer are small compared with the pressure variations in the centre layer. This enables the centre layer thickness, H, to be written as

$$H = H_\infty + \frac{p}{\rho} \cdot \left(\frac{1}{g_{12}} + \frac{1}{g_{23}} \right) \tag{1}$$

where g_{12} and g_{23} are the reduced gravitational accelerations between the upper and centre layers and the centre and lower layers, H_∞ is the thickness of the centre layer at the edge of the undercurrent at $y = \pm y_e$, p is the pressure in the centre layer and ρ is the density of that layer. Hence the depth variations predicted by the continuity equation for the centre layer can be used to estimate the pressure variations in that layer. The set of equations governing the behaviour of the centre layer is thus

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \beta y v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \beta y u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{3}$$

$$\frac{\partial H}{\partial t} + \frac{\partial Hu}{\partial x} + \frac{\partial Hv}{\partial y} = 0 \tag{4}$$

together with eq. (1), where u and v are the zonal and meridional velocities in the centre layer. These four equations form a closed set of equations which are independent of the behaviour of the other layers. For the remainder of this study attention is restricted to a consideration of the centre layer.

3. The wave in the region of the undercurrent

I used relationships between the potential vorticity and the Bernoulli head. The reason for the choice of the Bernoulli head rather than the more usual stream function was to facilitate the inclusion of baroclinic effects in the formulation of the problem. To utilize the results of this earlier study it is necessary to continue with this procedure. If standard stability calculations were to be carried out on the currents used as examples by I, the stability calculation would lack the generality which is sought here.

In the absence of unsteady terms both the potential vorticity, A, and the Bernoulli head are conserved following a particle of fluid. However in the case of unsteady motion the Bernoulli head is not conserved. Using eqs. (2) and (3) it is possible to find a modified Bernoulli head, B, which is conserved. This is given by

$$B = \tfrac{1}{2}(u^2 + v^2) + \frac{p}{\rho} - \zeta \tag{5}$$

where ζ is given by

$$\frac{D\zeta}{Dt} = \frac{\partial p/\rho}{\partial t} \tag{6}$$

where D/Dt is the Lagrangian time derivative

$$\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

Before proceeding it is necessary to solve eq. (6) for ζ. Adding the zero quantity $f_c(A)(DA/Dt)$, where $f_c(A)$ is an unspecified function of the potential vorticity, to the left-hand side of eq. (2) and using the expression $\partial/\partial t = -C_s(\partial/\partial x)$ gives eq. (6) where ζ is given by

$$\zeta = C_s(-\tfrac{1}{2}\beta y^2 + u) + C_s \int f_c(A)\, dA \tag{7}$$

In the absence of waves it is desirable that the modified Bernoulli head should reduce to the Bernoulli head. hence ζ is zero in the absence of waves. Using the definition of the potential vorticity, this requires the function f_c to be given by

$$f_c(A_0) = \frac{H_0 A_0}{dA_0/dy} \tag{8}$$

where $(\ )_0$ denotes the value of $(\ )$ in the absence of waves. Equations (7) and (8) completely determine ζ and so the modified Bernoulli head given by eq. (5) is known.

Because attention has been restricted to long waves, it is possible to simplify eq. (3) to give an equation for zonal geostrophic balance,

$$\beta y u = -\frac{\partial p/\rho}{\partial y} \tag{9}$$

This simplification is essential to the theory given here. The theory is therefore not valid when the basic zonal flow in the absence of waves is not in zonal geostrophic balance. Such a situation may occur in the western Pacific Ocean, see Philander (1973). However in the eastern Pacific Ocean, where the stability of the undercurrent has received considerable attention (as discussed at the Workshop on GATE Equatorial Experiment, Miami, 1977), the undercurrent appears to be geostrophic. Two other consequences of the long wave assumption need to be noted here. These are

$$|u| \gg |v| \quad \text{and} \quad \left|\frac{\partial}{\partial y}\right| \gg \left|\frac{\partial}{\partial x}\right| \tag{10}$$

Again for the theory to be valid they must apply to the basic flow in the absence of waves. Except in the immediate neighbourhood of boundaries both of these requirements are met.

Now because the modified Bernoulli head and the potential vorticity are both conserved following a particle of fluid, if they are initially related by a function f_B they must remain related by the same function f_B for all time. Hence using eqs. (9) and (10) this can be written as

$$\frac{1}{\beta}\frac{\partial}{\partial y}\frac{1}{y}\frac{\partial p/\rho}{\partial y} + \beta y = Hf_B\left(\frac{1}{2\beta^2}\frac{1}{y^2}\left(\frac{\partial p/\rho}{\partial y}\right)^2 + \frac{p}{\rho} - \zeta\right) \tag{11}$$

It should be noted that H is related to p by eq. (1). Because ζ is not directly determined by p, eq. (11) cannot be integrated directly to obtain p on any longitudinal section. However by taking $\zeta \equiv 0$, eq. (11) may be integrated to obtain the pressure field in the absence of waves. The influence of ζ is discussed later.

It is important to note the boundary conditions on eq. (11). In the absence of any waves the velocity and pressure are both zero at the edge of the undercurrent as is ζ. Thus the streamline marking the edge of the undercurrent is characterized by a zero modified Bernoulli head. That is

$$\frac{1}{2\beta^2}\cdot\frac{1}{y^2}\cdot\left(\frac{\partial p/\rho}{\partial y}\right)^2 + \frac{p}{\rho} - \zeta = 0 \tag{12}$$

at the edge of the undercurrent both in the presence and absence of a wave. The modified Bernoulli head is continuous across the edge of the undercurrent. However the potential vorticity need not be continuous. From eq. (11) it can be seen that the potential vorticity, A, takes on a value of $f_B(0)$ on the inside of the undercurrent edge. The value of the potential vorticity on the outside of the undercurrent edge is $\beta y_e/H$.

It is very constructive to consider the influence of wave-induced variations in ζ and

$$\tfrac{1}{2}u^2\left(= \frac{1}{2\beta^2}\frac{1}{y^2}\cdot\left(\frac{\partial p}{\partial y}\right)^2\right)$$

on the solution of eq. (11). At the edge of the undercurrent, the undercurrent velocity is that of the wave only. Hence at the edge of the undercurrent ζ dominates the velocity head, $\tfrac{1}{2}u^2$; the former being of first order in wave amplitude and the latter being of second order. However for lower latitudes where the undercurrent is stronger the variations with time in $\tfrac{1}{2}u^2$ dominate those in ζ because $|U| \gg |C_s|$ and $\zeta \propto C_s$ from eq. (7). ζ is therefore only significant in the small region near the edge of the undercurrent.

The width of the region where ζ is significant is of order $|C_s y_e/U|$. However using eq. (4) the outer streamline of the undercurrent shows meridional variations of order $|U_w y_e/C_s|$. Because the phase velocity of the waves being considered here has been restricted such that $|U_w U| \gg |C_s^2|$, the meridional variations in the position of the outer streamline are much greater than the width of the region for which ζ is significant. Hence the influence of ζ on the determination of the position and pressure at the outer streamline is insignificant. ζ may thus be neglected when integrating eq. (11) even if a wave disturbance is present.

If eq. (11) is perturbed about the solution in the absence of waves, which has $p = 0$ at the outer boundary, an equatorially anti-symmetric equation for the pressure perturbation results. From the asymmetry of this equation it follows that small amplitude disturbances may be either pure symmetric or pure asymmetric. Other disturbances do not show the same amplitude in the disturbance in the pressure and meridional displacement at the edge of the undercurrent on the two sides of the equator. This symmetry or asymmetry is fundamental for matching the wave motion in the undercurrent to the wave motion outside of the undercurrent. Here only symmetric waves are considered.

4. The wave near the edge of the undercurrent

At the edge of the undercurrent the velocity and pressure are those of the wave alone. There is no contribution to the pressure and velocity from the undercurrent here. Hence to first order in wave amplitude eq. (12) requires that $\zeta = p/\rho$ at the undercurrent edge. Combining this with eq. (7) gives

$$u = \frac{p}{\rho C_s} + \tfrac{1}{2}\beta y^2 - \int f_c(A)\, dA \tag{13}$$

at the edge of the undercurrent. This expression gives the zonal velocity at the undercurrent edge on the equatorial side.

A similar expression for the zonal velocity at the edge of the undercurrent, applicable on the outer side, can be obtained from time integrating the linearized form of eq. (2) which is applicable outside of the undercurrent. The resulting expression for u is

$$u = \frac{p}{\rho C_s} + \tfrac{1}{2}\beta y^2 + \text{constant} \tag{14}$$

Comparing eqs. (13) and (14) and using eq. (8) to evaluate the constant in eq. (14) gives it as $-\int^{y_{e_0}} H_0 A_0\, dy$. This shows that there is no discontinuity in the zonal velocity across the edge of the undercurrent provided p is continuous across the edge.

It is thus only necessary to match the pressure and meridional velocity across the edge of the undercurrent. Matching this behaviour will give the dispersion relation for the long wave disturbance. The dispersion relation can then be used to study the stability of the undercurrent to long wave disturbances.

Suppose that having integrated eq. (11), the pressure at the edge of the undercurrent is not $p = 0$ (as given by eq. (1) when $H = H_\infty$ in the absence of a wave) but $p = p_w|_{\text{edge}}$. Also suppose that the associated change in the meridional position of the edge of the undercurrent from the equator is $y_{e_w} - y_{e_0}$. Defining Ξ by $p_w|_{\text{edge}}/(y_{e_w} - y_{e_0})\rho$, it follows that

$$\rho \frac{v_w|_{\text{edge}}}{\dfrac{\partial p_w|_{\text{edge}}}{\partial t}} = \frac{1}{\Xi} \tag{15}$$

where

$$v_w|_{\text{edge}} = \frac{\partial y_{e_w} - y_{e_0}}{\partial t}$$

is the meridional velocity at the edge of the undercurrent. This equation serves as a boundary condition on the region outside of the undercurrent.

5. The wave in the region outside of the undercurrent

In the region outside of the undercurrent the velocity is that induced by the disturbance. Therefore as the wave amplitude is small the linearized forms of eqs. (1), (2), (3) and (4) are relevant to this region. These equations must be solved subject to the matching condition, eq. (15), on p and v at the edge of the undercurrent. At large $|y|$ the amplitude of the wave disturbance must decay to zero.

Eliminating u and p from the linearized form of eqs. (1), (2), (3) and (4) gives

$$\frac{\partial^2 v}{\partial y^2} + \left(-\frac{\beta}{C_s} - \frac{\beta^2}{c^2} y^2\right) v = 0 \tag{16}$$

in the long wave limit, where

$$c^2 = H_\infty \bigg/ \left(\frac{1}{g_{12}} + \frac{1}{g_{23}}\right)$$

The solution of this equation can be written in terms of parabolic cylinder functions (see Abramowitz and Stegun, 1970). The solution which decays for large $|y|$ is of the form

$$v = e_1 W_1\left(\frac{c}{2C_s}, y\sqrt{\frac{2\beta}{c}}\right) \tag{17}$$

where e_1 is oscillatory in time and zonal distance with a phase speed of C_s and W_1 is the decaying parabolic cylinder function. Using eqs. (2) and (3) it follows that

$$\frac{1}{\rho}\frac{\partial p}{\partial t} = -e_1 \frac{\left(-\dfrac{1}{C_s}\beta y W_1\left(\dfrac{c}{2C_s}, y\sqrt{\dfrac{2\beta}{c}}\right) + \dfrac{d}{dy}\left(W_1\left(\dfrac{c}{2C_s}, y\sqrt{\dfrac{2\beta}{c}}\right)\right)\right)}{\dfrac{1}{c^2} - \dfrac{1}{C_s^2}} \tag{18}$$

Combining eqs. (15), (17) and (18) implies that the dispersion relation for the disturbance is

$$\frac{W_1^1(T_1, T_2)}{W_1(T_1, T_2)} = T_3(1 - 4T_1^2) + T_2 T_1 \tag{19}$$

where

$$T_1 = \frac{c}{2C_s}, \quad T_2 = y_e\sqrt{\frac{2\beta}{c}} \quad \text{and} \quad T_3 = -\Xi\frac{1}{\sqrt{2\beta c^3}}$$

This relationship can be modified using an expansion for the derivative of a parabolic cylinder function in terms of parabolic cylinder functions to give

$$\frac{W_1(T_1 - 1, T_2)}{W_1(T_1, T_2)} = -T_3 \cdot (1 - 4T_1^2) + \tfrac{1}{2}T_2 - T_2 T_1 \tag{20}$$

This form of the dispersion relation will be useful later.

It should be noted that although the solution ($T_1 \to -\frac{1}{2}$, T_2 arbitrary, $T_3 \to \pm\infty$) is a valid solution of eq. (20), it is not associated with $u, p \to 0$ as $|y| \to \infty$ even though $v \to 0$. This example (even though it is spurious because it does not satisfy $C_s \ll c$) casts doubt on the completeness of the dispersion relation. A parameter range is apparently associated with solutions which do not have a decaying zonal velocity and pressure at large distances. The determination of the parameter range for which meridional decay does not occur is important in understanding the influence of the undercurrent on waves.

This problem can be easily understood by the following argument. Suppose that $v \to 0$ as $|y| \to \infty$ but that u and p do not. Then for sufficiently large $|y|$ eqs. (1), (3) and (4) imply that $p \propto \exp(-\frac{1}{2}\beta y^2 C_s/c^2)$. Hence it immediately follows that for p not to decay $R_e(C_s) < 0$ where R_e denotes the real part. Also because u grows exponentially like p while yv can grow at most linearly, it follows from eqs. (2) and (3) that the only case where the dispersion relation breaks down is where

$$C_s = -c \tag{21}$$

This is the case of $T_1 = -\frac{1}{2}$. When eq. (21) is not satisfied identically the solutions of eq. (20) do indeed represent true solutions although the wave to which they correspond may not decay until large $|y|$.

6. The undercurrent form

I has integrated eq. (11) with $\zeta \equiv 0$ for several specific functions, f_B. That study showed that it is often possible to have two possible solutions for the same distribution of potential vorticity with Bernoulli head (or alternatively stream function). One of these solutions was that of an undercurrent which widened as the layer thickness decreased, while the other solution was of an undercurrent which narrowed. For a more detailed description of these undercurrents the reader is referred to the original article.

As discussed two paragraphs after eq. (12) of this study the influence of ζ on the form of the wave disturbance to the undercurrent is negligible. The results of I with $\zeta \equiv 0$, therefore seem relevant to the problem here. However because that study was concerned with symmetric undercurrents the application of those results to this study limits the present study to equatorially symmetric waves as noted earlier. Fig. 2 is typical of the results obtained by I.

Fig. 2 shows the loci in the plane of the meridian at which termination of the integration occurred and the pressure at which this termination occurred, generated by varying the pressure at the equator. Termination of the integration can occur because either the outer streamline of the undercurrent (for which the Bernoulli head is zero) is encountered or because a reversal in the direction of the undercurrent develops. The former

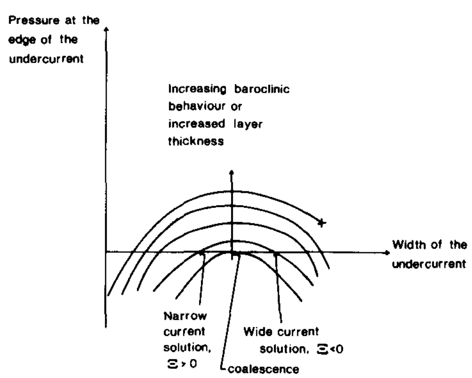

Fig. 2. Characteristic pattern as given by I of the relationship between the width of the undercurrent and the pressure at the undercurrent edge. A solution for the undercurrent occurs when the pressure at the edge of the undercurrent is zero.

occurs below the axis $p = 0$, while the latter occurs above the axis. A proof of this is given by I. The intersection of a locus with the axis $p = 0$ represents a solution to the undercurrent when unperturbed by a disturbance. The gradient of the locus just below such an intersection is a measure of Ξ which was defined to be $p_w|_{\text{edge}}/(y_{e_w} - y_{e_0})p$. The gradient above such an intersection is of no relevance here because the outer streamline is not encountered before termination of the integration of eq. (11) occurs.

When there are two solutions present, the solution represented by the intersection of the locus with $p = 0$ at small $|y|$ is generally associated with Ξ being positive. The other solution at large y is generally associated with Ξ being negative. As the layer depth decreases the two solutions coalesce. At the coalescence point Ξ is zero. It is constructive to consider the behaviour of a wave disturbance to an undercurrent in the neighbourhood of the coalescence point. The depth of the layer containing the undercurrent is then marginally larger than that required for coalescence. Ξ is then of small magnitude and negative for one solution and small and positive for the other.

Fig. 3 shows the pressure field in two actual solutions of eq. (11) with $\zeta \equiv 0$. The corresponding velocity fields are shown in Fig. 4. It can be seen that the wider of the two solutions has the higher pressure and thus distorts the vertical stratification more. However, it is associated with the slower undercurrent and has less meridional shear. Thus Ξ is in some sense a measure of the shear in the undercurrent. It should be noted that for the undercurrents of interest here the local shear gradient may exceed β. This introduces the possibility of instability in the current according to the Rayleigh criterion. See Boyd (1976) for when the gradient of the shear is weak.

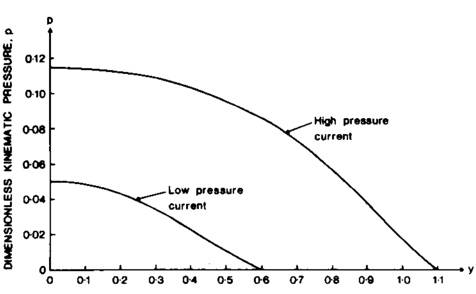

Fig. 3. An example of the pressure field in two conjugate currents as noted by I. Non-dimensionalization is achieved by taking the distance scale as ζ (arbitrary) and the time scale as $1/\beta\zeta$.

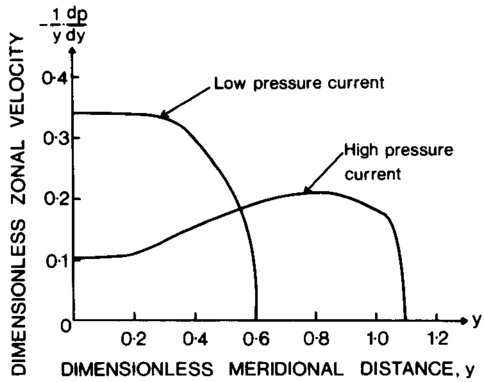

Fig. 4. The non-dimensional zonal velocity corresponding to the conjugate currents of Fig. 3.

7. Instability near the point of coalescence

Because attention has been restricted to small wave speeds, the parameter T_1 which was introduced in eq. (19) is very large in magnitude. Hence for finite T_2 the left-hand side of eq. (20) may be approximated by $T_1^{1/2}$ provided $|\arg(T_1)| < \pi$. The dispersion relation can then be expressed in the simpler form

$$T_3 = \frac{-T_1^{1/2} + \frac{1}{2}T_2 - T_1 T_2}{1 - 4T_1^2} \tag{22}$$

Because the magnitude of T_1 is large the theory presented here is only valid when the magnitude of T_3 is small. This corresponds to a small value of Ξ which is what is observed in the neighbourhood of the coalescence point (described in the previous section).

Observations of the equatorial undercurrent in the Pacific, Atlantic and Indian Oceans all suggest that the parameter T_2 is of order unity. Hence using eq. (22) and the definitions of T_1 and T_3, following eq. (19), it follows that

$$C_s \approx -\Xi \sqrt{\frac{2}{\beta c}} \left(1 - 2\left(\frac{-\Xi}{\sqrt{2\beta c^3}}\right)^{1/2}\right) \tag{23}$$

It can be seen that at the coalescence point where $\Xi = 0$, C_s is zero and the disturbance does not grow or propagate. The undercurrent is in neutral equilibrium. When Ξ is small in magnitude but negative the disturbance does not grow but may propagate eastward with a speed of $-\Xi\sqrt{2/\beta c}$. This speed increases as the magnitude of Ξ increases. When Ξ is small but positive the disturbance propagates westward but may grow with an e-folding time of the wavelength/$\Xi^{1/2}$ $10(1/\beta c^3)^{1/4}$. This decreases (and so the instability becomes more rapid) as Ξ increases in magnitude.

As described in the last section Ξ is generally negative for the solution to the undercurrent with large meridional width. Disturbance therefore propagate away in such a solution. However, Ξ is generally positive for the solution to the undercurrent with a small meridional width. Disturbances therefore grow and the solution for the undercurrent is weakly unstable. It is in agreement with intuition that the undercurrent which is narrow and fast should be unstable while the wide undercurrent which is flowing slowly should be stable.

It should be noticed that the stability analysis has been restricted to a consideration of long wavelength disturbances in an undercurrent with a depth close to that which causes coalescence. No consideration has been given to undercurrents in a layer of much greater depth. However, the instability predicted here in narrow undercurrents becomes more rapid as the layer depth increases. This suggests that all solutions with a width less than the width of the undercurrent at coalescence are unstable to long wave disturbances. This hypothesis needs further investigation.

The instability is more rapid for short wavelength disturbances than long wavelength ones.

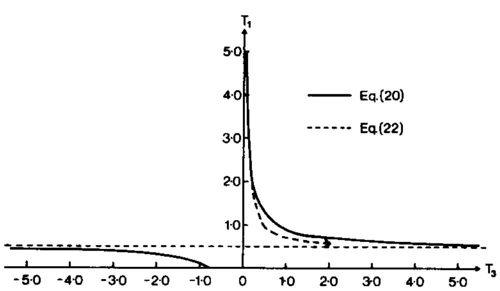

Fig. 5. Plot of T_1 against the stiffness parameter T_3 for the case of $T_2 = 1$. Note the asymptotic behaviour towards $T_1 = 0.5$ for large $|T_3|$. This limit corresponds to a Kelvin wave.

However, the theory given here is only valid for wavelengths much greater than both the equatorial radius, $\sqrt{c/\beta}$, and the inertial-geostrophic distance, $\sqrt{U/\beta}$. Using values for U and c of 1 m/s (typical for the Pacific Ocean), values of order 10^1 days for the e-folding time are valid before the theory fails for waves of suitable amplitude.

It is interesting to ask, what type of wave takes on the dispersion relation eq. (22) when an undercurrent is present such that $|T_1| \to \infty$? To answer this question we note that the assumption of large $|T_1|$ was needed only to obtain a value of Ξ. Provided that a value of Ξ is obtained from the basic definition of Ξ rather than from the gradient of plots similar to those of Fig. 2, the dispersion relation eq. (20) is quite general. The parameter T_3 is then a measure of the rigidity of the undercurrent. Fig. 5 shows how T_1 varies with T_3 in the case of $T_2 = 1$ and $\arg(T_1) = 0$. It is easy to see that as $|T_3| \to \infty$, eq. (20) implies

$$T_1 \to \tfrac{1}{2} \tag{24}$$

This describes the speed of a Kelvin wave in the absence of an undercurrent. The limit $|T_3| \to \infty$ reflects the absence of meridional motion in such a wave. Hence eq. (22) is the dispersion relation for a Kelvin wave after modification by the undercurrent.

8. High-mode Rossby waves

Consider now the solution of eq. (20) for what $\arg(T_1) = \pi$. Three cases need to be considered. These are

$$\text{(i)}\ 4T_1T_3 - T_2 < 0 \tag{25}$$

$$\text{(ii)}\ 4T_1T_3 - T_2 = 0 \tag{26}$$

$$\text{(iii)}\ 4T_1T_3 - T_2 > 0 \tag{27}$$

Suppose that case (i) is pertaining. Then for large negative T_1 and T_2 of order unity as required here, the right-hand side of eq. (20) is positive and large. This requires

$$T_1 = -2n - \tfrac{3}{2} \tag{28}$$

That is

$$C_s = -c/(4n+3) = -c/(4(n+1)-1) \tag{29}$$

where n is a positive integer. As $|T_1|$ must be large n must also be large. This restricts attention to the high-mode number waves. Suppose now that T_3 is varied. As case (ii) is approached, the right-hand side of eq. (20) decreases rapidly until it is zero at some value of T_3 near that required by eq. (26). This requires

$$T_1 = -2n - \tfrac{1}{2} \tag{30}$$

That is

$$C_s = -c/(4n+1) = -c/(4(n+\tfrac{1}{2}) - 1) \tag{31}$$

As $4T_1T_3 - T_2$ increases further the right-hand side of eq. (20) rapidly becomes large and negative. Thus case (iii) requires

$$T_1 = -2n + \tfrac{1}{2} \tag{32}$$

That is

$$C_s = -c/(4n-1) \tag{33}$$

This is the usual speed of a long symmetric Rossby wave of mode $2n$ in the absence of an undercurrent.

It can be seen that the dispersion relation for high-mode Rossby waves (note that n is large) remains undisturbed by the presence of an undercurrent. This is to be expected because the meridional width of such waves is much greater than that of the undercurrent. It is interesting that a modal jump occurs when $4T_1T_3 - T_2$ changes sign as T_3 varies. The sharpness of the transition depends on the magnitude of T_1. For very large $|T_1|$ this transition is extremely sharp. As $(-T_1)$ and T_2 are both positive, a transition can only occur if T_3 is negative. This requires $\Xi > 0$ and such a current has been found to be unstable to Kelvin-wave-like disturbances. No instability is generated by long symmetric Rossby waves.

No information is available on the lower mode waves as they do not appear to move sufficiently slowly for the theory presented here to be valid except in terms of the more general interpretation of T_3 as discussed in the previous section. As explained earlier the present study is restricted to equatorially symmetric waves and hence does not give any information on odd-mode Rossby waves.

9. Discussion

A general analysis of slowly varying symmetric long disturbances has been carried out in the neighbourhood of the coalescence point of the undercurrent solutions as discovered by I. In this neighbourhood the wider of the two undercurrents shows an eastward slowly propagating symmetric long Kelvin-like waves. The narrower undercurrent has a slow westward-propagating wave in the neighbourhood of the coalescence point. This behaviour is reminiscent of subcritical and supercritical flow behaviour. (See Streeter, 1971.)

The gradient of absolute vorticity of both undercurrent forms obtained by I changes sign near the boundary of the undercurrent. Both undercurrents may therefore be unstable. However there is a long wavelength slow instability only in the case of the narrower of the undercurrents. This undercurrent has a mean velocity (and shear) which is higher than that of the other undercurrent solution. It is therefore not surprising that it should be associated with the long wavelength inertial instability. The unstable solution is characterized by a widening as the undercurrent flows into a region of decreased layer thickness. It should be noted that the extent of deformation of the density structure by the undercurrent is not a measure of the stability of the undercurrent. The wider undercurrent, which is stable to the above disturbances, deforms the density structure of the thermocline more than the narrower unstable undercurrent. Both the phase velocity of waves in the case of the wider undercurrent and the growth rate and velocity of the instability in the case of the narrower undercurrent decrease as the undercurrent flows into a region of decreased layer thickness. At the layer thickness at which coalescence of the two solutions occurs, a disturbance can neither grow nor propagate away.

The above study has placed the strict requirement on a stable undercurrent. The undercurrent must decrease in width as the layer in which it is flowing decreases in thickness. The undercurrent must therefore flow with a higher characteristic velocity when the layer is thinner. Philander (1979) has argued that critical layer absorption will dampen slowly eastward-propagating disturbances. Thus if such a disturbance is observed it must be supplied with energy from an instability of the undercurrent. Although this does not detract from the present model, it is in contradiction with the results of the present model which does not have a critical layer because of its simple vertical structure.

The instability studied here has been associated with a Kelvin wave. There have been several other studies of Kelvin waves in an oceanic rather than atmospheric environment. McPhaden and Knox (1979) studied a Kelvin wave with a speed very much greater than that of the undercurrent. They found that long Kelvin waves were only slightly affected by the presence of the undercurrent. The main effects of the undercurrent were to introduce a small Doppler shift and a small meridional velocity. Philander (1979) obtained similar results with a different basic undercurrent form. Although the present study is not directly applicable to these studies because it requires a meridionally bounded undercurrent, it would appear from Fig. 5 that the cores of the undercurrents used were rather inflexible ($|T_3|$ high). This is not surprising because vorticity distributions used are similar to those of current 3 studied by I. This current showed a general inflexibility compared with the two other currents studied by I.

The present study has illustrated the complexity of the general study of equatorial wave–current interaction in the ocean. The calculation of the dispersion relation for waves in particular equatorial currents is likely to remain the most profitable method of understanding this problem. However, care needs to be shown in applying such results universally.

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ОБ ИНЕРЦИОННОЙ НЕУСТОЙЧИВОСТИ ЭКВАТОРИАЛЬНОГО ПРОТИВОТЕЧЕНИЯ

Хьюз (1979) отметил, что для некоторой заданной формы зависимости потенциального вихря от функции тока теоретически возможно иметь два вида экваториальных противотечений. По мере того, как толщина слоя, через который проходит противотечение, уменьшается, оба течения приближаются друг к другу по форме и при достижении критической толщины слоя обе формы противотечений сливаются. В данной работе рассматривается поведение медленно движущегося симметричного волнового возмущения малой амплитуды и большой длины в присутствие противотечения почти сливающейся формы. Показано, что более узкое противотечение инерционно неустойчиво и докритично к возмущению типа волны Кельвина. Более широкое противотечение, наоборот, инерционно устойчиво и неустойчиво по отношению к подобному возмущению. Волны Россби высоких мод испытывают лишь слабое влияние экваториального противотечения.