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On the theory of the surface-stress induced entrainment at a buoyancy interface (toward interpretation of KP and KPA experiments)

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ABSTRACT

An attempt is made to explain KP (Kato and Phillips, 1969) and KPA (Kantha, Phillips and Azad, 1977) laboratory experiments on shear generated turbulent entrainment in stratified fluid. The approach is based on the results of recent computations by Garnich and the author (Garnich and Kitaigorodskii, 1977, 1978; see also Kitaigorodskii, 1979), where it was shown that the effective surface-stress induced entrainment is due to locally generated turbulence in the interfacial shear region. If this is so, then the assumption that the entrainment rate is independent of the character of density stratification below the buoyancy interface leads to reasonable agreement between KP and KPA data.

1. Introduction

In the first paper on surface-stress induced entrainment, Kato and Phillips (1969) presented the experimental data on entrainment velocity $u_e = \partial D/\partial t$ in the form

$$u_e v_*^{-1} = E^{KP}(\Pi_1) \tag{1}$$

where, $v_* = \sqrt{\tau_0/\rho}$, τ_0 is the surface stress, ρ is the reference density and $\Pi_1 = \Delta b D/v_*^2$ is the so-called bulk Richardson number based on the time-dependent buoyancy jump across the interface $\Delta b(t)$ and on the depth of well-mixed turbulent layer $D(t)$; $E^{KP}(\Pi_1)$ is some nondimensional function to be determined either theoretically or experimentally. In recent laboratory experiments on surface-stress induced entrainment, Kantha, Phillips and Azad (1977) discovered a surprising feature: for the same values of bulk Richardson number, there is a definite 100% difference between the values of nondimensional entrainment rate in two-layer fluid (KPA case) and initially linearly stratified fluid (KP case). Phillips (1976) attributed this to the possible influence of the density stratification below the buoyancy interface, which can reduce the entrainment rate in KP case (compare KPA case) due to the internal wave radiation effects (for a balance of momentum and turbulent energy). Without entirely disregarding this idea, we will try to show here that the discrepancy between KP and KPA data can be also explained from the opposite point of view, i.e. with the assumption that the character of density stratification below the buoyancy interface has no *direct effect* on the entrainment rate. This idea in implicit form was first suggested by Garnich and Kitaigorodskii (1977). One purpose of this paper is to clarify those aspects of the problem of shear-induced entrainment which were not discussed in previous publications (Garnich and Kitaigorodskii, 1977, 1978; Kitaigorodskii, 1979).

2. General similarity arguments

It is first useful to emphasize that the presentation of experimental data on entrainment rate in the form (1) for both KP and KPA experiments is not a direct result of dimensional analysis and thus $E^{KP}(\Pi_1)$ in (1) is not necessarily a universal function for both KP and KPA data. To show this, let us consider the general case of the deepening of a turbulent mixed layer in stratified nonrotating fluid with arbitrary but nonzero initial conditions

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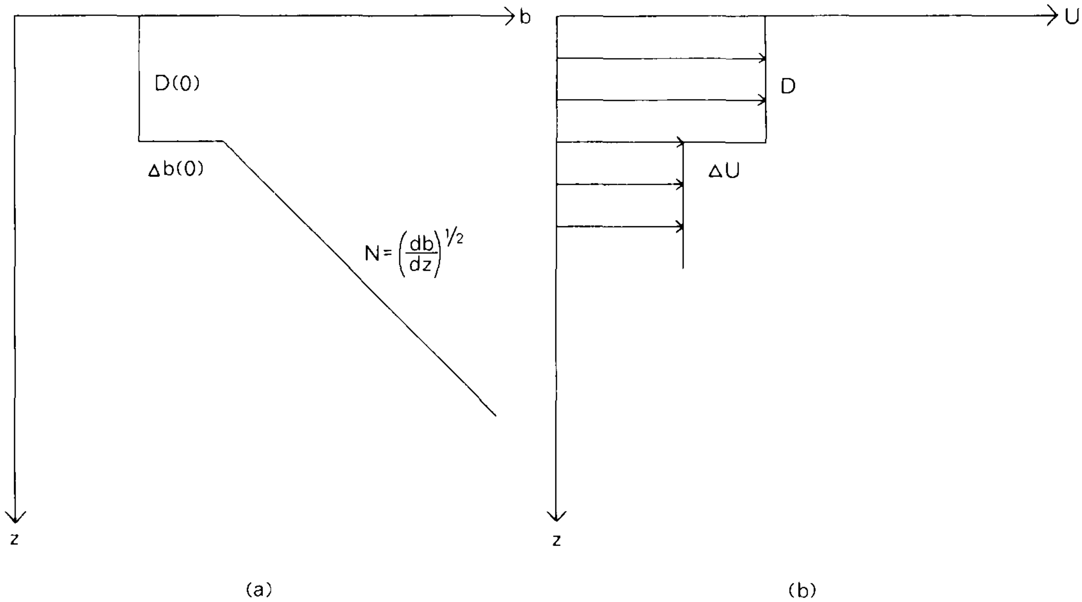


Fig. 1 (a and b). Sketch of mean density and velocity distribution in pure slab approximation.

for Δb and D (Fig. 1a). Our choice of finite initial values for Δb and D is dictated by the desire to bring together the KP and KPA data*. For this reason it is better to have from the beginning the complete set of governing parameters for the deepening of turbulent layer both in linearly stratified and two-layer fluids.

For the simplest picture of the time evolution of mixed layer, when density stratification below it is characterized by buoyancy jump Δb and constant values of Brunt–Vaisala frequency N (Fig. 1a) and by the absence of the surface buoyancy flux, the conservation of buoyancy implies that

$$\Delta b D - \frac{N^2 D^2}{2} = \Delta b(0) D(0) - \frac{N^2 D^2(0)}{2} \tag{2}$$

where $f(0) = f(t = 0)$ characterizes the initial conditions. Equation (2), of course, is exact when the interfacial region is modeled by the discontinuity in buoyancy distribution.

For the shear-induced entrainment we must also consider mean velocity distribution. If it is characterized by the time-dependent velocity jump across the interface ΔU with constant velocity below the interface (Fig. 1b) (we will refer to such velocity distribution as *pure slab* model), then the conservation of mean momentum in case of constant surface stress $\tau_0(\partial \tau_0/\partial t \equiv 0)$ simply implies that

$$\Delta U D - \Delta U(0) D(0) = v_*^2 t \tag{3}$$

In deriving (2) and (3), the following conditions at the nonstationary density interface were used:

$$\Delta q = \Delta b u_e \tag{4}$$

$$\Delta \tau = \Delta U u_e \tag{5}$$

where Δq and $\Delta \tau$ are the differences in buoyancy (q) and momentum (τ) fluxes on both sides of the interface. Then to get (2) and (3) it is necessary to assume that independent of the character of density stratification *below* the interface, the turbulent fluxes of buoyancy and momentum below the interface are zero. This assumption gives the well-known formulas for entrainment buoyancy flux q_e and entrainment stress τ_e

$$q_e = \Delta b u_e \tag{6}$$

$$\tau_e = \Delta U u_e \tag{7}$$

* It must be remembered in this connection that KP experiments were performed with zero initial conditions for Δb and D, whereas KPA experiments correspond to nonzero initial values of Δb and D.

When the initial velocity jump across the interface $\Delta U(0)$ is zero formula (3) reduces to the simplest form of momentum balance for a slab mixed layer.

$$\Delta UD = v_*^2 t \tag{8}$$

From now on let us consider just this simplest case of *pure slab* model of mixed layer (The generalization of this approach will be discussed in Section 5.) The general statement for the time evolution of $D(t)$ for conditions described above will be

$$\frac{\partial D}{\partial t} = u_e[D(0), \Delta b(0), v_*, t, N] \tag{9}$$

Making use of (2), we can write it also as

$$u_e = u_e[D(0), D, \Delta b, v_*, t, N] \tag{10}$$

Finally, we can eliminate $D(0)$ in (10) by solving (10) with the initial condition

$$t = 0 \qquad D = D(0) \tag{11}$$

Thus without any loss of generality, we can write that

$$u_e = u_e(D, \Delta b, v_*, t, N) \tag{12}$$

Let us now formulate our general hypothesis. We assume for any given values of D, Δb, t, v_* that the entrainment rate u_e is independent of the character of the density stratification below the interface, i.e. independent of the exact value of N. Introducing the nondimensional variables

$$\Pi_2 = \frac{D}{v_* t}; \qquad \tilde{u}_e = u_e v_*^{-1} \tag{13}$$

we find that our assumption leads to the following general entrainment law

$$\tilde{u}_e = E(\Pi_1, \Pi_2) \tag{14}$$

where it is seen also from (8) and (13) that in pure slab approximation

$$\Pi_2 = \frac{v_*}{\Delta U} \tag{15}$$

so Π_2 becomes the drag coefficient. It is useful to note now that the nondimensional parameter Π_2 in (14) cannot be interpreted either in terms of nondimensional time or nondimensional depth D. Neither can we keep D constant when we change the time or vice versa, when we consider the entrainment process in any particular experiment. The advantage of using the parameter Π_2 instead of any other [based, for example, on initial values of $D(0)$ or $\Delta b(0)$] is that it does not contain initial values of Δb and D, or N, and thus can be used to describe both KP and KPA data. In addition to (15), the other possible physical interpretation of Π_2 can be done by taking into account that surface-stress induced entrainment in a *homogeneous* fluid produces a turbulent layer $D_{\mathrm{hom}} \propto v_* t$ (Phillips, 1977). Thus Π_2 in the case of surface-stress induced entrainment is the ratio of the characteristic depths of turbulent layers in stratified and homogeneous fluids. Because of this we must expect (at least asymptotically) that $\Pi_2 = \ll 1$, which allows us to consider a single limiting procedure, i.e. $\Pi_2 \to 0 (\Delta U \to \infty$ for fixed v_*). We can then expect that as $\Pi_2 \to 0$, entrainment will *still depend on the exact values* of the drag coefficient *because* ΔU is the only source of turbulence in a pure slab model (Fig. 1b). This latter consideration suggests that for $\Pi_2 \to 0$ $E(\Pi_1, \Pi_2)$ (14) must behave as

$$E(\Pi_1, \Pi_2) = \Pi_2^n \Psi_n(\Pi_1) \tag{16}$$

It is interesting to note that $\Psi_n(\Pi_1)$ has different physical meanings for different values of the power exponent n. For example, for positive values of n using (7, 8) we get for $n = 1$

$$\Psi_1(\Pi_1) = \frac{\tau_e}{v_*^2} \tag{17}$$

and for $n = 2$

$$\Psi_2(\Pi_1) = \frac{2P_h}{v_*^3} \tag{18}$$

$$P_h = -\int_D^{D+h} \tau \frac{\partial U}{\partial z} dz = u_e \frac{(\Delta U)^2}{2} \tag{19}$$

So, P_h is the shear production of turbulent energy in an interfacial region of thickness h in the case $h/D \to 0$ (see Appendix). Thus, $\Psi_1(\Pi_1) = \tilde{u}_e \Pi_2^{-1}$ characterizes the ratio of entrainment stress to the surface stress, Ψ_2 is the normalized integral shear production in pure slab approximation. On Fig. 2a,b, the results of the determination of $\Psi_n(\Pi_1)$ for $n = 1, 2$ use some of the KP and KPA data present. From these figures it is seen that there is still a big scatter in KPA points on Ψ_1, Ψ_2 for the same values of Π_1. This shows that the hypothesis

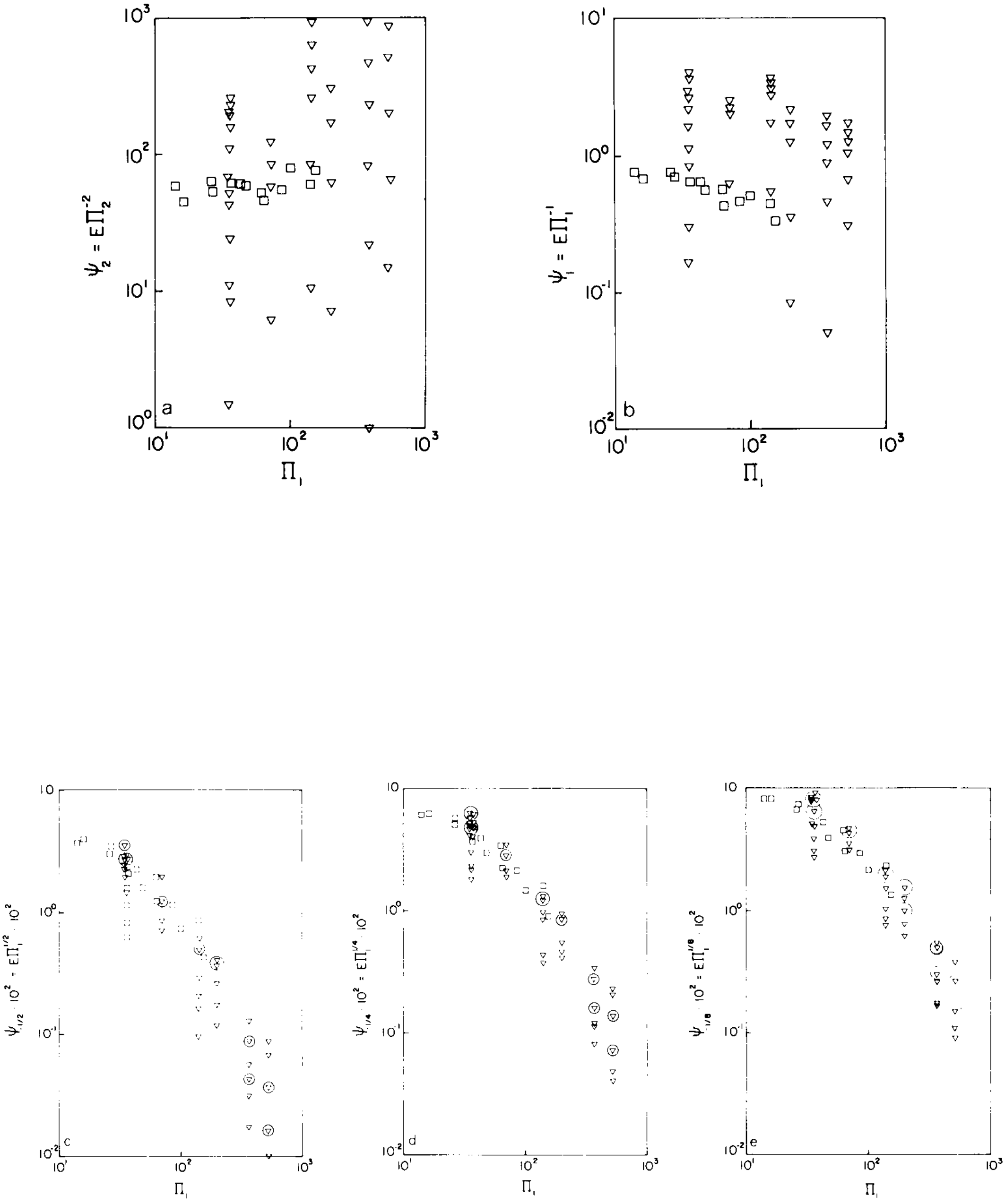


Fig. 2. (*a–e*) The relationships between ψ_n and bulk Richardson Π_1 for KP and KPA data. (a) ψ_2, (b) ψ_1, (c) $\psi_{-1/2}$, (d) $\psi_{-1/4}$, (c) $\psi_{-1/8}$. (□) KP data, (▽) KPA data. In Figs. 2c,d,e big circles indicate those of KPA experimental points which have Π_2 values close to Π_2^{KP} for the same values of bulk Richardson number Π_1 (see also Fig. 4).

(17, 18) in a two-layer case (KPA) is not generally valid. Because with the increase of ΔU as the only production source of turbulence for fixed values of v_* and Π_1, it seems more natural that the entrainment rate will *increase* with a *decrease* of drag. This means that we can expect n to be negative in (16). Even if the entrainment rate is not very sensitive to changes in the drag coefficient it will still be dependent on Π_2 (or ΔU). In Fig. 2c,d,e, we present some results of the determination of Ψ_n for $n = -1/2, -1/4, -1/8$ as an attempt to bring KP and KPA data together using the particular form (16) of the general entrainment law (14). The *additional hypothesis* about the asymptotic behavior of $E(\Pi_2, \Pi_1)$ (14) can be based on the idea that for very big values of ΔU ($\Pi_2 \to 0$) and of bulk Richardson number Π_1 ($\Pi_1 \to \infty$) the entrainment rate does not depend on exact values of the thickness of turbulent layer D (but still depends on ΔU and Δb). This hypothesis for $\Pi_2 \to 0$, $\Pi_1 \to \infty$ leads to the prediction

$$\Psi_n \propto \Pi_1^{-n} \tag{20}$$

which gives

$$E \propto \Pi_3^{-n}; \qquad \Pi_3 = \Pi_2^{-1}\,\Pi_1 \tag{21}$$

The function $E = E(\Pi_3)$ is shown on Fig. 3. From this figure it is seen that *some*, but not all points from both KP and KPA experiments can be described using (21) when $n = 1/2$. But the case $n > 0$ contradicts our intuitive expectation of the asymptotic behavior of $E(\Pi_1, \Pi_2)$ for $\Pi_2 \to 0$. Thus we can see that not all *a priori assumptions* about asymptotic behavior of the universal entrainment function $E(\Pi_1, \Pi_2)$ (14) permit us to explain the discrepancy between KPA and KP results in terms of the relationship between entrainment velocity and bulk Richardson number. Fortunately, there is another way to establish relationships between entrainment velocities in KP and KPA cases. Before doing this, let us reemphasize the meaning of the general entrainment formula (14). It shows that the nondimensional entrainment rate $\tilde{u}_e$ is a universal function of two variables Π_1 and Π_2 for all values of N including, of course, $N \equiv 0$. Now when we introduce nondimensional time $t_N = Nt$

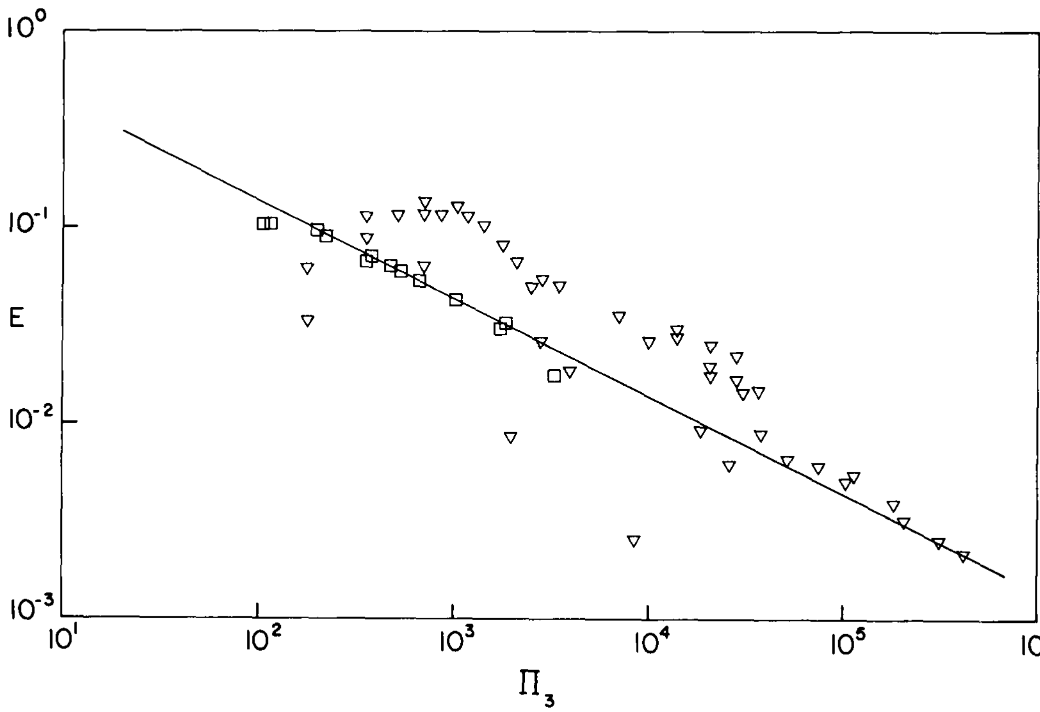


Fig. 3. The nondimensional function $E(\Pi_3)$ (21) according to KP and KPA data. (□) KP data, (▽) KPA data.

and depth of mixed layer $D_N = DN/v_*$, we can rewrite (14) as

$$\tilde{u}_e = \frac{\partial D_N}{\partial t_N} = E(\Pi_1, \Pi_2)$$

Because for a two-layer fluid, when $E = E^{\mathrm{KPA}}$ it follows from (2) that $\partial \Pi_1/\partial t_N = 0$, and if we assume that not only entrainment, but *also* drag are independent of exact values of N, we have for the KPA case

$$\frac{\partial \Pi_1}{\partial t_N} = \frac{\partial \Pi_2}{\partial t_N} \equiv 0$$

With latter valid for the KPA case, we can integrate (14), using the initial condition (11). We then get

$$\Pi_2 \left(1 - \frac{D(0)}{D}\right) = E^{\mathrm{KPA}}(\Pi_1, \Pi_2) \tag{22}$$

In principle this formula permits us to construct the function $E^{\mathrm{KPA}}(\Pi_1, \Pi_2)$ using only the values of Π_2, $D(0)/D$ and Π_1. These are the directly measurable parameters for KPA (as well as KP) experiments. E^{KPA} and E^{KP} were previously estimated by *differentiating* experimental curves $D(t)$. Of course, the accuracy of such estimates of entrainment is much less than that for estimating $D(t)$ itself.

According to (22) as time goes by ($D(0)/D \to 0$) asymptotically

$$\frac{\Pi_2}{E^{\mathrm{KPA}}} \to 1 \tag{23}$$

In a two-layer (KPA) case when $\Pi_2 = \Pi_2^{\mathrm{KPA}}$ in pure slab approximation, it simply means that $\Psi_1 \to 1$, i.e. asymptotically ($D(0)/D \to 0$) all surface stress is used for entrainment. In initially linearly stratified fluid (KP case) we must put $D(0) = 0$ in (22) and $\Pi_2 = \Pi_2^{\mathrm{KP}}$. Then according to (2)

$$\Pi_1 = \Pi_N; \qquad \Pi_N = \frac{D_N^2}{2} \tag{24}$$

and from (22) we get

$$\Pi_2^{\mathrm{KP}} = E^{\mathrm{KPA}}(\Pi_1, \Pi_2^{\mathrm{KP}})|_{\Pi_1 = \Pi_N}. \tag{25}$$

3. Basic relationships for surface-stress induced entrainment in KPA case

To proceed further, i.e. to find some relationship between E^{KP} and E^{KPA} using (25), we must consider the two-layer case in more detail. If we apply the same arguments as (9–12), we will find that for $N \equiv 0$

$$\tilde{u}_e = \tilde{u}_e^{\mathrm{KPA}} = E^{\mathrm{KPA}}(\Pi_1, \Pi_2) \tag{26}$$

where index KPA corresponds to the two-layer experiment on shear induced entrainment (Kantha, Phillips and Azad, 1977). If we consider the situation when

$$\frac{\partial E^{\mathrm{KPA}}}{\partial t} = 0 \tag{27}$$

we can see from (2, 26, 27) that in this case, also,

$$\frac{\partial \Pi_2^{\mathrm{KPA}}}{\partial t} = 0 \tag{28}$$

or vice versa. This means that if we also hypothesize, for example, that the drag coefficient is a universal function of the bulk Richardson number, i.e.

$$\Pi_2 = \xi(\Pi_1) \tag{29}$$

that we will satisfy the conditions (27, 28) for the two-layer case. If (29) is valid, then for $\Pi_1 = \Pi_1^{\mathrm{KPA}} = \mathrm{const}$

$$E^{\mathrm{KPA}} = \xi(\Pi_1) \tag{30}$$

The hypothesis of marginal stability first suggested by Pollard et al. (1973),

$$\mathrm{Ri}_v = \frac{\Delta b D}{(\Delta U)^2} = \mathrm{const} = A^2 \infty 1 \tag{31}$$

is just one particular case of the more general hypothesis (29) because in a pure slab approximation formula (31) implies that in (29)

$$\xi(\Pi_1) = \frac{A}{\Pi_1^{1/2}} \tag{32}$$

The validity of the assumption (31) (the subject of long discussions in many recent papers on ocean modeling) remains to be proved. It certainly does not work well for the initial period of the deepening of the mixed layer in the KPA case when $D(0)/D$ is not small enough. Figs. 2a,b illustrate this fact. It is useful in this context (following Garnich and Kitaigorodskii, 1977, 1978) to consider the simplified form of turbulent energy balance due to surface-stress induced entrainment. In pure slab approximation when the only source for mixing is due to shear-production in the interfacial layer, the balance of turbulent energy reduces to the form

$$W_e - \phi_2 P_h = 0 \tag{33}$$

where

$$W_e = \frac{q_e D}{2} = \frac{u_e \Delta b D}{2}$$

is the rate of change of potential energy due to entrainment. Here, ϕ_2 is the mixing efficiency of interfacial shear-induced entrainment (Kitaigorodskii, 1979) which shows what the *a priori* unknown fraction of P_h is used for deepening the mixed layer. Equation (33) can be written also as

$$\Pi_2 = \left(\frac{\phi_2}{\Pi_1}\right)^{1/2} \tag{34}$$

Thus, we can see that the assumption used by Garnich and Kitaigorodskii (1977, 1978)

$$\phi_2 = \phi_2(\Pi_1) \tag{35}$$

is exactly equivalent to (29) and leads in the two-layer case to (27, 28, 29). If we prefer instead of (29) or (35) to make a "weaker" assumption like

$$\frac{\tau_e}{v_*^2} = \zeta(\Pi_1) \tag{36}$$

where $\zeta(\Pi_1)$ is a universal function of the bulk Richardson number, then we get in pure slab approximation

$$\frac{\tilde{u}_e}{\Pi_2} = \zeta(\Pi_1) \tag{37}$$

For the two-layer case after integrating (37) with respect to time, we find that (37) can be satisfied only for

$$\zeta \equiv 1 \tag{38}$$

which means that assumption (36) can be valid in KPA case only when all surface stress is used for entrainment. In such a case, we again get from (26, 29) that $E^{\mathrm{KPA}} = \Pi_2^{\mathrm{KPA}} = \xi(\Pi_1)$. Thus we have been able to establish the nontrivial fact that *closure assumptions* of different types, like (29), (35) or (36), lead to the same universal dependence of E^{KPA} on bulk Richardson number Π_1. According to general formula (26) they all must be considered as asymptotic approximations. However, we can use them if we limit ourselves to considering pure slab approximation.

The entrainment in a stratified *two-layer fluid* in pure slab approximation occurs only when $t \gg D(0)/v_*$ or when $\Pi_2 \ll (D/D(0))$. Thus we can reasonably guess that for $\Pi_2^{\mathrm{KPA}} \ll (D/D(0))$

$$E^{\mathrm{KPA}}(\Pi_1, \Pi_2) \to \Pi_2^{\mathrm{KPA}} = \xi(\Pi_1) = E_\infty^{\mathrm{KPA}}(\Pi_1) \tag{39}$$

It follows from Fig. 2b that this asymptotic regime was not characteristic for the majority of the KPA data, partly because of the finite and not small values of $(D(0)/D)$. Of course, it must be remembered that the conclusions drawn above were essentially based on the assumed validity of *pure-slab approximation*. But even in this very idealized approach, it is possible to find a reasonable explanation for the difference in the KPA and KP entrainment rates.

4. The relationship between KP and KPA entrainment velocities

The surprising result of (22) and (39) is that some range of the values of $\Pi_2 \left[\Pi_2 \ll \left(\frac{D}{D(0)}\right)^{\mathrm{KPA}}\right]$ must exist when for $D(0)/D \ll 1$

$$\Pi_2 = E_\infty^{\mathrm{KPA}}(\Pi_1) \tag{40}$$

We can see now that in a stratified fluid ($N \neq 0$) as $D(0)/D \to 0$, not nondimensional entrainment velocity $\tilde{u}_e = E(\Pi_1, \Pi_2)$, but $\Pi_2 = D/D_{\mathrm{hom}}$, tend to be a universal function of the bulk Richardson number. According to (22, 25, 39) we can find immediately that for $D(0) \equiv 0$ ($\Pi_2 = \Pi_2^{\mathrm{KP}}$ in this case) and

$$\Pi_2^{\mathrm{KP}} \ll \frac{D}{D(0)}^{\mathrm{KPA}}$$

$$\Pi_2^{\mathrm{KP}} = E_\infty^{\mathrm{KPA}}(\Pi_1)|_{\Pi_1 = \Pi_N} \tag{41}$$

This equation relates the entrainment velocities in KP and KPA case. Indeed, by rewriting (41) as

$$D^{\mathrm{KP}} = v_* t E_\infty^{\mathrm{KPA}}(\Pi_1)|_{\Pi_1 = \Pi_N}$$

and differentiating (42) with respect to time taking into account (24) we get

$$E^{KP} = \frac{E_{\infty}^{KPA}(\Pi_1)}{1 - 2\dfrac{\partial \ln E_{\infty}^{KPA}}{\partial \Pi_1}\Pi_1} \tag{43}$$

So if, for example,

$$E_{\infty}^{KPA} \propto \Pi_1^{-m}; \qquad m = \text{const} > 0 \tag{44}$$

then

$$E^{KP} = \frac{E_{\infty}^{KPA}(\Pi_1)}{1 + 2m} \tag{45}$$

This is, from our point of view, the simplest explanation of the difference between the E^{KP} and E^{KPA} values for the same values of bulk Richardson number Π_1. The ratio E^{KP}/E_{∞}^{KPA} in (45) is simply a number, which depends on our choice of m. Garnich and Kitaigorodskii (1977) agree with Kantha, Phillips and Azad (1977) that $E_{\infty}^{KPA} \propto \Pi_1^{-1}$, which leads according to (45) $E^{KP}/E_{\infty}^{KPA} = \frac{1}{3}$(!). The other hypothesis (31) according to (32, 39) leads to $E_{\infty}^{KPA} \propto \Pi_1^{-1/2}$ and thus $E^{KP}/E_{\infty}^{KPA} = \frac{1}{2}$(!). This last choice was also made recently by Kantha (1978). It is worthwhile noticing that before choosing between the values of m in (44), it may be better to consider the arbitrary form of $E_{\infty}^{KPA}(\Pi_1)$. In this case, we can expect that the KPA data on entrainment rate corresponding to $D(0)/D \ll 1$ and $\Pi_2 \ll |D/D(0)|^{KPA}$ and KP data on E^{KP} must be related universally as

$$\frac{E^{KP}}{E_{\infty}^{KPA}} = \Theta(\Pi_1) \tag{46}$$

We have attempted to construct this universal function using only those KPA experimental points for which $\Pi_2^{KPA} \simeq \Pi_2^{KP}$ for the same values of Π_1. Probably the easiest way to choose such KPA experimental points is to present Π_2 as a function of Π_1 for both KP and KPA data. The results are shown in Fig. 4. As it was expected, only those of KPA data which are characterized by the small values of the ratio

$$\frac{D(0)}{D}\left(\frac{D(0)}{D} \lesssim 0.24\right)$$

are close to the corresponding values of the Π_2 in KP case. In Fig. 4 such points are indicated as open circles, whereas the remaining part of KPA

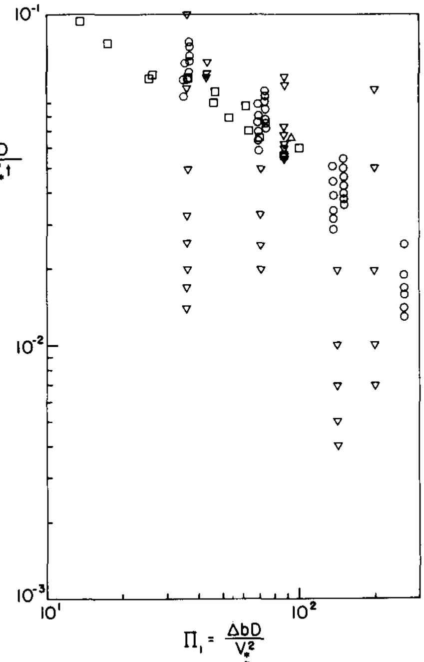


Fig. 4. The relationship between Π_2 and Π_1 for KP and KPA data. (□) KP data, (▽) KPA data in cases when $D(0) = 5.7$ cm (these points correspond to the situations where side-wall effects were negligible, but the ratio of $D(0)/D$ was usually 0.5), (○) KPA data in cases when $D(0) = 2.7$ cm (these points correspond to the situations where side-wall effects were negligible and the ratio $D(0)/D$ was less than 0.24).

data (open triangles) being relatively free from side-wall effects has usually the values of $D(0)/D \gtrsim 0.5$. Thus we were able to use only a very small portion of KPA data for construction $\Theta(\Pi_1)$ in (46). The results of such determination of the ratio E^{KP}/E_{∞}^{KPA} based on the average values of E^{KP} E_{∞}^{KPA} for different Richardson numbers Π_1 are presented on Fig. 5. We can see from this figure that the ratio E^{KP}/E_{∞}^{KPA} does not vary with Π_1 and its value is close to 0.65.

The explanation of KP and KPA data given above is based on the following statement concerning our universal entrainment law (14)

$$\lim_{\Pi_2 \to E_{\infty}^{KPA}(\Pi_1)} E(\Pi_1, \Pi_2) = \begin{cases} E^{KP}(\Pi_1) & \text{for } N \neq 0 \ \dfrac{D}{D(0)} \to \infty \\ E_{\infty}^{KPA}(\Pi_1) & \text{for } N = 0 \ \dfrac{D}{D(0)} \to \infty \end{cases} \tag{47}$$

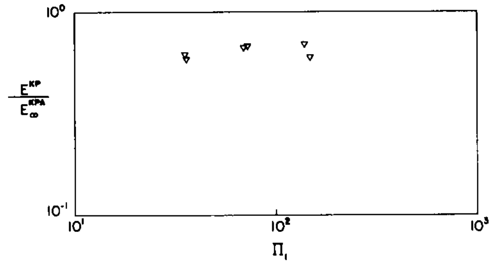


Fig. 5. The universal function E^{KP}/E_{∞}^{KPA} (46) (only those points in KPA experiments were used which have Π_2 values close to Π_2^{KP} for the same values of bulk Richardson number Π_1, see Fig. 4).

However, we do not find this result in any way surprising. More to the point, we think that it is probably the only generalized way to understand physically the difference between the KP and the KPA data. The reason for this is because in an asymptotic regime corresponding to $t \to \infty (D(0)/D \to 0)$, $\tilde{u}_e^{KP}$ and $\tilde{u}_e^{KPA}$ must behave differently. In the first case the buoyancy jump across the interface Δb *increases* (as well as Π_1), whereas in the two-layer case Δb will decrease with time (Π_1 being constant). This means, of course, that in the case $N \neq 0$ as time goes by $(D(0)/D \to 0)$, the interfacial region will become stratified more and more strongly, which will necessarily prevent entrainment. In the two-layer case, the opposite situation must occur. Equation (47) reflects this fact.

Because we have drawn our conclusions here using pure slab approximation, we shall discuss some generalizations of the pure slab approximation to show that the relationships between E^{KP} and E_{∞}^{KPA} are not very sensitive to the method of the parametrization of the vertical structure of mean velocity field.

5. The simple generalization of pure slab approximation for surface-stress induced entrainment

Let us consider now the more general vertical structure of mean velocity field which contains not only ΔU but also a velocity jump in a thin constant stress layer (Fig. 6a). For simplicity we assume also that this velocity jump ΔU_0 is developed practically instantaneously and that the ratio $\Delta U_0/v_*$ remains constant with time. Then the general statement about entrainment rate (14) will be modified in the following way

$$\tilde{u}_e = E(\Pi_0, \Pi_1, \Pi_2) \tag{48}$$

where

$$\Pi_0 = \frac{v_*}{\Delta U_0} = \text{const} \tag{49}$$

According to available experimental data (Kitaigorodskii, 1979), $\Pi_0 \approx 0.12 \div 0.06$. Introducing the additional parameter Π_0, we must realize that the existence of a *surface shear zone from the very start of the action of surface stress* permits turbulence generated by shear in a thin surface layer to produce entrainment. This can occur even *before* the velocity jump ΔU exists at the density interface. This stage of entrainment due to the self-diffusion of turbulence from the surface shear zone can be identified with the entrainment process in the case of grid-generated turbulence (Garnich and Kitaigorodskii, 1978). To show how this can influence the relationship between KP and KPA entrainment velocities derived in Section 4, it is convenient to consider the balance of a turbulent energy more general than that in (33), when

$$W_e - \phi_2 P_h = \phi_1 P_\delta \tag{50}$$

where

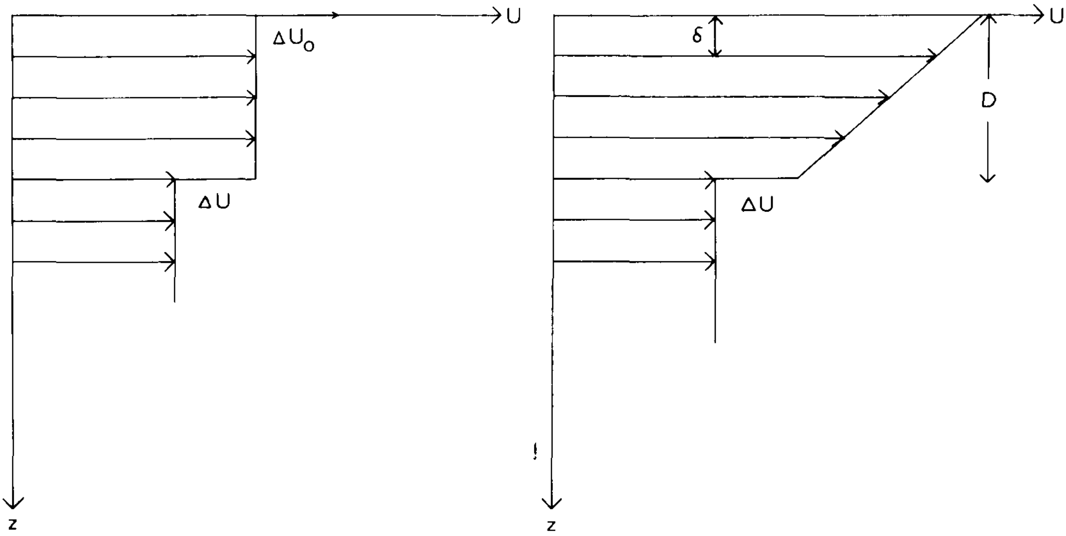
$$P_\delta = -\int_0^\delta \tau \frac{\partial U}{\partial z} dz = \tau_0 \Delta U_0 = \text{const} \tag{51}$$

is the shear production in the surface constant stress layer of the thickness $\delta(\delta/D \to 0$ as in the case of Fig. 6b). Here, ϕ_1 is the mixing efficiency of the entrainment due to the vertical self-diffusion of turbulence (Garnich and Kitaigorodskii, 1977, 1978). In nondimensional form, (5) can be written as

$$\tilde{u}_e \left[\Pi_1 - \frac{\phi_2}{\Pi_2^2}\right] = \frac{2\phi_1}{\Pi_0} \tag{52}$$

Now let us again consider the *two-layer fluid* (*KPA case**). Because $\Pi_0 = \text{const}$, the assumptions $\Pi_2 = \xi(\Pi_1)$ (29) or $\tau_e/\tau_0 = \zeta(\Pi_1)$ (36) will once again

* It must be noted that, in the KPA experiments, the real flux of turbulent energy from the thin surface shear region can be $G = P_\delta + M_0$, where M_0 is due to the wave roughness of the screen, whose direct action on water produces not only ΔU_0 but also turbulence.



(a) (b)

Fig. 6 (a and b). The scheme of mean velocity distribution in well-mixed layer. (a) Two velocity jumps (at the surface and at the buoyancy interface). (b) Linear velocity distribution in mixed layer.

determine the universal entrainment rate E_∞^{KPA} (39). Using (52) we are now able to establish more definitely the conditions under which (39, 41) are valid. Indeed, if we now consider the regime when $\tau_e \to \tau_0$, i.e. when all surface stress is used for entrainment in the KPA case (*real surface* stress induced entrainment), then because $\tilde{u}_e \to \Pi_2 \to E^{\mathrm{KPA}}$ (36, 38) we derive from (52) that

$$E^{\mathrm{KPA}} = \frac{\phi_1}{\Pi_1 \Pi_0}(1 + S) \tag{53}$$

where

$$S = \left(1 + \frac{\phi_2 \Pi_0^2 \Pi_1}{\phi_1^2}\right)^{1/2} \tag{54}$$

In the case when $\phi_2 \Pi_0^2 \Pi_1 / \phi_1^2 \ll 1 S = 1$ and we get from (53)

$$E^{\mathrm{KPA}} = E_0^{\mathrm{KPA}} = \frac{2\phi_1}{\Pi_0 \Pi_1} \tag{55}$$

Because E_0^{KPA} is related to the deepening of the mixed layer at the self-diffusion stage, we must assume (Garnich and Kitaigorodskii, 1977, 1978) that

$$E_0^{\mathrm{KPA}} \propto \Pi_1^{-3/2} \tag{56}$$

and thus

$$\phi_1 \propto \Pi_1^{-1/2} \tag{57}$$

In another asymptotic regime when $S \gg 1$ and

$$\left[\frac{\phi_2 \Pi_0^2 \Pi_1}{\phi_1^2}\right]^{1/2} \gg 1 \tag{58}$$

we get from (34, 39, 53, 54)

$$E^{\mathrm{KPA}} = \left(\frac{\phi_2}{\Pi_1}\right)^{1/2} = E_\infty^{\mathrm{KPA}}(\Pi_1) \tag{59}$$

Now, the condition (58) can also be written in the form

$$\left(\frac{E_\infty^{\mathrm{KPA}}}{E_0^{\mathrm{KPA}}}\right)^{1/2} \gg \tfrac{1}{2} \tag{60}$$

Because $E_\infty^{\mathrm{KPA}} \propto \Pi_1^{-m}$ $(m = 1 \div \frac{1}{2})$ and $E_0^{\mathrm{KPA}} \propto \Pi_1^{-3/2}$, (60) leads to some limitations on the bulk Richardson number Π_1 of the type

$$\Pi_1^{(3-2m/4)} \gg 1 \tag{61}$$

Garnich and Kitaigorodskii (1978) showed that with reasonable assumptions for the values of E_∞^{KPA} and E_0^{KPA} the conditions (60, 61) were satisfied for practically the whole range of bulk Richardson numbers Π_1 used in KPA experiments.

The equation (50) is valid now for

$$t \gg \frac{D(0)}{(P_\delta)^{1/3}} \tag{62}$$

which makes the restriction on Π_2 even weaker than in Sections 3 and 4 because (62) means that

$$\Pi_2 \ll \left(\frac{D}{D(0)}\right) \Pi_0^{-1/3} \tag{63}$$

Thus because $\Pi_0 \ll 1$, the relationship between E^{KP} and E_∞^{KPA} (43) can exist for smaller values of Π_1 (61), and even when Π_2 is not much smaller than $|D/D(0)|$.

6. Conclusions

We were able to demonstrate what kind of differences between KP and KPA entrainment rates (E^{KP} and E_∞^{KPA}) must exist. But we have not discussed how to determine the functional form of $E_\infty^{\mathrm{KPA}}(\Pi)$ (or $E^{\mathrm{KP}}(\Pi_1)$). The closure assumptions (31, 32) permit us to find $E_\infty^{\mathrm{KPA}}(\Pi_1)$ as well as $E^{\mathrm{KP}}(\Pi_1)$, but they do not clarify much, because the *critical Richardson number arguments* can be formulated also in other, and maybe more phvsical ways. For example, they can be formulated in terms of the gradient Richardson number $\mathrm{Ri} = (db/dz)(dU/dz)^2$, as well as the flux Richardson number $R_f = q/\tau(dU/dz)$ *for turbulence in the interfacial layer of finite thickness h*. In this case, we must speculate about the form of such relationships as

$$\frac{h}{D} = \frac{h}{D}(\Pi_1, \Pi_2) \tag{64}$$

or more simply

$$\frac{h}{D} = \frac{h}{D}(\Pi_1) \tag{65}$$

The experience with grid-generated turbulence, where we can expect that for $\Pi_1 \to \infty$ $h/D \to$ const (Crapper and Linden, 1974), is not of direct help here, because we already have shown that the crucial role at the main stages of surface-stress induced entrainment belong to the local shear instability in the interfacial layer (P_h). The discussion of turbulence structure and the physics of turbulent entrainment in such situations is outside the scope of the present paper.

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8. Appendix

The derivation of shear production of turbulence during surface-stress induced entrainment

To derive eqs. (50) and (52), we must calculate the shear production of the turbulent energy in a stratified boundary layer with density interface and an arbitrary mean velocity distribution. To do this, let us first use the following general rule based on the mean momentum equation $\partial U/\partial t = -\partial\tau/\partial z$

$$\int_{a_1(t)}^{a_2(t)} \tau \frac{\partial U}{\partial z} dz = \tau U \Big|_{a_1}^{a_2} + \frac{1}{2}\int_{a_1}^{a_2} U^2 dz - \frac{1}{2}\left\{\frac{\partial a_2}{\partial t} U^2(a_2, t) - \frac{\partial a_1}{\partial t} U^2(a_1, t)\right\} \tag{A.1}$$

Let us then calculate $P_h = -\int_D^{D+h} \tau(\partial U/\partial z)dz$. If in (A.1) $a_2 = D + h$, $a_1 = D$, and if we assume $\tau_{D+h} = 0$ (7), we get

$$P_h = \tau_D U_D - \frac{1}{2}\frac{\partial}{\partial t}(h\tilde{U}^2) + \frac{1}{2}\left\{U_{D+h}^2 \frac{\partial}{\partial t}(D+h) - U_D^2 \frac{\partial D}{\partial t}\right\} \quad (A.2)$$

where

$$\tilde{f} = \frac{1}{h}\int_D^{D+h} f(z,t)\,dz \quad (A.3)$$

Now if we assume that according to (7)

$$\tau_D = \tau_e = \frac{\partial D}{\partial t}[U_{D+h} - U_D] = \frac{\partial D}{\partial t}\Delta U,$$

and if we neglect in (A.2) all terms containing h, we get from (A.2)

$$P_h = \frac{1}{2}\frac{\partial D}{\partial t}(\Delta U)^2 \quad (A.4)$$

To calculate the shear production in mixed layer P_D, let us first calculate the shear production in the constant stress sublayer $0 \leq z \leq \delta$ (Fig. 6b). We then get

$$P_\delta = -\int_0^\delta \tau \frac{\partial U}{\partial z}\,dz = \tau_0(U_0 - U_\delta) \quad (A.5)$$

The remaining part of the shear production in the mixed layer $P_{D-\delta}$ using (A.1) when $a_1 = \delta$ and $a_2 = D$ can be written as

$$P_{D-\delta} = -\int_\delta^D \tau \frac{\partial U}{\partial z}\,dz = \tau_0 U_\delta - \tau_D U_D - \frac{1}{2}\frac{\partial}{\partial t}\int_\delta^D U^2\,dz + \frac{1}{2}\left(\frac{\partial D}{\partial t}U_D^2 - \frac{\partial \delta}{\partial t}U_\delta^2\right) \quad (A.6)$$

Let us now introduce the quantity $P'_{D-\delta}$ according to formula

$$P'_{D-\delta} = \frac{1}{2}\frac{\partial}{\partial t}\int_\delta^D (U - \bar{U})^2\,dz \quad (A.7)$$

where

$$\bar{f} = \frac{1}{D}\int_0^D f(z,t)\,dz \quad (A.8)$$

Then $P'_{D-\delta}$ will characterize the influence of shear production in the main body of mixed layer on the total shear production in the mixed layer P_D. The expression (A.6) with $\delta/D \to 0$ can be written in the form

$$P_{D-\delta} = \tau_0 U_\delta - \tau_D U_D - P'_{D-\delta} - \bar{U}\frac{\partial}{\partial \tau}(\bar{U}D) + \frac{1}{2}\frac{\partial D}{\partial t}(\bar{U}^2 + U_D^2). \quad (A.9)$$

From the momentum equation $\partial U/\partial t = -\partial\tau/\partial z$ with $\tau_D = \Delta U(\partial D/\partial t)$, we have

$$\frac{\partial}{\partial t}(\bar{U}D) = \tau_0 - \frac{\partial D}{\partial t}\Delta U + \frac{\partial D}{\partial t}U_D \quad (A.10)$$

Substituting (A.10) in (A.9) we get

$$P_{D-\delta} = \tau_0(U_\delta - \bar{U}) - P'_{D-\delta} - \frac{\partial D}{\partial t}\Delta U U_D - \frac{\partial D}{\partial t}\bar{U}U_{D+h} + \frac{1}{2}\frac{\partial D}{\partial t}(\bar{U}^2 + U_D^2) \quad (A.11)$$

Now the total turbulent shear production $\sum P_i$ using (A.4) and (A.5) can be presented in the form

$$\sum P_i = P_\delta + P_{D-\delta} + P_h = \tau_0(U_0 - \bar{U}) + \frac{1}{2}\frac{\partial D}{\partial t}(\bar{U} - U_{D+h})^2 - P'_{D-\delta} \quad (A.12)$$

If we then neglect the contribution of the term $P'_{D-\delta}$ and put $U_{D+h} \equiv 0$, we have the eq. (50) where now in the expressions for P_δ (51) and P_h (19) instead of slab velocity U we can use the average velocity in mixed layer $\bar{U}$. From (A.10) it is seen that all the conclusions of the theory developed in this paper remain *unchanged* except when the definition of Π_2 (or drag coefficient) is given in terms of average velocity $\bar{U}$, so that $\Pi_2 = v_*/\bar{U}$ (instead of $\Pi_2 = v_*/U$ in pure slab approximation). To estimate the contribution of the term $P'_{D-\delta}$ to the right part of (A.12), it is of course necessary to know the profile of the mean velocity $U(z)$ in the mixed layer. This is still quite controversial, and $U(z)$ has not been adequately explored at least in natural conditions. According to some sources (Niller, 1975; Mellor and Durbin, 1975), $\Delta U \sim$ 10–20 cm s^{-1} and is even larger than the velocity difference in the major part of mixed layer $\delta U = U_\delta - U_D$. Laboratory experiments by Moore and Long (1971) have shown that $\delta U/\Delta U$ can decrease slightly with an increase in density jump Δb. For a very preliminary estimate, we can take

$$\delta U = U_\delta - U_D \lesssim \Delta U = U_D - U_{D+h} = U_D \quad \text{(A.13)}$$

so that

$$1 \lesssim \frac{U_\delta}{U_D} \lesssim 2 \quad \text{(A.14)}$$

The last condition is enough to show that if, for example, the velocity distribution in a mixed layer is linear, $P'_{D-\delta}$ in (A.12) will be much smaller than the second term on the right side of (A.12). If this is so (if $P'_{D-\delta}$ can be neglected in (A.12)), then the conclusions of this theory are not sensitive to the vertical distribution of mean velocity and instead of the approximation used in Section 5, it is possible to use an even more general one (for example, the velocity in a mixed layer $0 \leq z \leq D$ can be presented as

$$\frac{U_0 - U(z)}{V_*} = \Psi\left(\frac{z}{\delta}, \frac{z}{D}\right)).$$

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К ТЕОРИИ ВОВЛЕЧЕНИЯ В СТРАТИФИЦИРОВАННОЙ ЖИДКОСТИ, ВЫЗВАННОГО ДЕЙСТВИЕМ ПОВЕРХНОСТНОГО КАСАТЕЛЬНОГО НАПРЯЖЕНИЯ

Предпринята попытка объяснить различие в скоростях вовлечения в двухслойной и непрерывно стратифицированной жидкости в опытах Като-Филлипса и Канта-Филлипса. Используемый подход основан на результатах вычислений Гарнича и автора (Гарнич, Н. Г., Китайгородский, С. А., 1977, 1978, см также Китайгородский, 1979), показавших, что процесс вовлечения, вызванного действием поверхностного касательного напряжения, становится наиболее эффективным, когда сдвиг скорости проникает в область скачка плотности. Если такая стадия наступает, то предположение о том, что скорость вовлечения не зависит от характера плотностной стратификации под поверхностью раздела, приводит к разумным выводам о различии в скоростях вовлечения в опытах Като-Филлипса и Канта-Филлипса.