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A study of the dynamics of four-dimensional data assimilation

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ABSTRACT

The problem of four-dimensional data assimilation is investigated on the inviscid linearized shallow-water equations. The conditions under which an assimilation of mass data, performed according to a simple updating scheme, will reconstruct the complete mass and velocity fields are studied. On the basis of a mathematical result proved elsewhere, it is shown that energy conservation ensures exact convergence towards any given solution, irrespective of geostrophy or lack of geostrophy, under the only condition that the available observations define a unique solution of the model equations. For most values of the relevant parameters, the divergent part of the wind field is reconstructed much more rapidly than the rotational part. The effect of a damping of gravity waves is considered, and shown to accelerate the reconstruction of the wind field only for scales which are large compared to the Rossby radius of deformation.

These results are generalized to the inviscid linearized multi-level primitive equations. The case of wind observations is also considered, and shown to lead to reconstitution of the complete mass field. Finally, comparison with numerical results obtained with non-linear equations shows that the main features deduced from the linearized theory are preserved, but also suggests that the non-linear advection can, at least in some cases, accelerate the process of reconstruction.

1. Introduction

The problem of specifying initial conditions for numerical weather prediction has not yet been satisfactorily solved. The difficulties encountered, which have been described in detail by several authors (see, e.g. Bengtsson, 1975), fundamentally arise from the impossibility of measuring all the relevant parameters at a given time and from the fact that many of the parameters cannot be measured with the necessary accuracy. We have to deal with measurements distributed in both space and time, and accept that they are incomplete at any given time. Moreover, among the available measurements, those which bring useful meteorological information are mostly measurements of pressure and temperature, which define the distribution of mass in the atmosphere, but do not bring any direct information on the wind. The problem of specifying complete initial conditions at a given moment from observations distributed in space and time has become generally known as the problem of *four-dimensional data assimilation.* It is substantially complicated by the fact that neither the observations nor the numerical model can be perfect, so that an exact adjustment of the model to the observations is in any circumstance impossible.

It has not yet been possible to define a method of four-dimensional data assimilation which takes full advantage of the available observations. The best techniques in operational use at present are basically statistical in nature (e.g. Lorenc et al., 1977), but there remain many unsolved questions relative to the dynamical aspects of the problem. In fact, even in the idealized case when the observations and the numerical model are assumed to be perfect, the theoretical and mathematical background of the problem is still rather obscure.

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A first, theoretical, question is the following: under which circumstances can initial conditions at a given instant be replaced by conditions distributed in time? In other words, which are the space-time distributions of observations which define a unique solution of the model equations, as do complete observations at a given time? It seems obvious that synchronous observations can be replaced by non-synchronous ones, provided these are numerous enough and adequately distributed. For linear equations, the theory of characteristics provides an appropriate theoretical framework for answering such questions, but this theory does not appear to be very tractable in many practical situations. See Bube (1978) for various aspects of the linear problem. For non-linear equations, the problem is substantially more difficult, and a complete solution is certainly far from being found. Ghil et al. (1977) however, have studied under which circumstances the knowledge of the mass field and of its time derivatives at a given instant would define uniquely the corresponding velocity field. They have shown that knowledge of the mass field together with its first and second time derivatives leads for the wind field to a second order non-linear partial differential equation which can be solved numerically, at least when it is elliptic.

A second question, closely linked to the previous one, but distinctly different, and closer to the practical problem, is the following. Supposing that observations distributed in space and time are available, and that we know, or at least have good reasons for assuming, that these observations define a unique solution of the basic equations, through which numerical procedure can we determine the corresponding complete initial conditions at a given time? Most procedures which have been proposed so far are variations of the natural approach of correcting the most recent forecast with the most recent observations. In these procedures, the numerical model is integrated over the time period where observations are available and, whenever the model time reaches an instant when observations are available, the state predicted by the model is corrected with the observations. The basic, heuristic, idea behind this procedure is that the successive corrections, provided they are frequent enough, will progressively induce the model flow towards the particular solution out of which the observations have been taken. A large variety of methods can be imagined as to the way the model state is "corrected" with new observations. Whatever the method used, a great number of numerical experiments (see, e.g. Bengtsson, 1975) have shown that this kind of approach is indeed successful, but only partially. It is successful in the sense that it leads to a gradual reconstruction of the complete flow, including the wind field when only mass observations are used. But the success is only partial in the sense that the reconstruction is always very slow, too slow in particular for the requirements of operational practice. Moreover, experiments of the "identical twin" type, in which "observations" are artificial values produced by the same model used by the assimilation, are definitely more successful than experiments carried out with real data. In the latter case, the difference between the model and the actual flow often stabilizes about a non-zero asymptotic value. The fact that a numerical model never reproduces exactly the real atmosphere has here a critical effect. For a more detailed review of the results obtained so far in four-dimensional data assimilation, and various aspects of the problem, see Bengtsson (1975), Miyakoda et al. (1976), Davies and Turner (1977) and Simmonds (1976, 1978).

Although the general approach just described is not very efficient, it appears that the simple repetition of updating the model with new data suffices for reconstructing the observed flow. This fact is by itself quite remarkable. Local perturbations caused by the introduction of observations could trigger instabilities which would in turn prevent any adjustment of the model state to the observed flow. It is generally considered that such instabilities are inhibited by the processes which maintain the model flow close to geostrophic balance. We will see that this view is incomplete and that more basic properties of the dynamical equations play a fundamental role in the assimilation.

A mathematical study of the assimilation process described above has been performed in the author's doctoral thesis (Talagrand, 1977), hereafter referred to as T77. This article presents a part of that study, more specifically devoted to the case of the linearized equations. Although some additional questions are also considered, this article deals mostly with the following question: will an assimilation performed on successive observations of the mass field reconstitute the complete state of the flow, i.e. the mass *and* velocity fields? It will be shown that, because of energy conservation, the assimilation will indeed reconstitute the complete state of the flow, provided the observations are numerous enough to define a unique solution of the model's equations. Moreover, it turns out that the observed slowness of the reconstitution of the wind field is limited to its rotational part. This is a direct consequence of the fact that the mass field appears only in the second time derivative of the vorticity.

Section 2 presents the basic mathematical formalism and a number of classical results to be used later. It is demonstrated in Section 3, on the basis of a result proved in T77, how energy conservation ensures the convergence of an assimilation performed with the linearized inviscid shallow-water equations. The influence of the relevant parameters on the efficiency of an assimilation is studied in Section 4. The results of Sections 3 and 4 are extended in Section 5 to the linearized, inviscid multi-level primitive equations. The influence of dissipation of gravity waves is studied in Section 6. The case of wind, or mixed wind/mass observations is considered in Section 7. Section 8 is devoted to a comparison between the previous theoretical results and numerical results obtained with the non-linear equations. The results are summarized, and a number of conclusions drawn, in Section 9.

2. The basic mathematical formalism

2.1. *The assimilation process*

We will assume that successive observations of the mass field, either complete or partial, have been carried out at successive times $t_1, t_2 \ldots$ and, as mentioned above, we will study under which conditions an assimilation performed on these observations is capable of reconstituting the complete mass and velocity fields at any given time. The following two hypotheses will be made about the assimilation process:

(i) Updating of the model with new observations is carried out in the simplest possible way. At each observation time t_i, the values predicted by the model for the observed parameters are simply replaced with the observed values, while the values predicted for the non-observed parameters are not altered in any way. This procedure calls for an important remark. In all systems of meteorological equations which use the pressure as independent vertical coordinate, the mass field appears in the horizontal momentum equation only through the gradient of the geopotential. As a consequence, the mass field appears in the first time derivative of the divergence, but only in the second time derivative of the vorticity. One can therefore expect that, in an assimilation of mass observations performed as just described (i.e. without modifying the wind field when the mass field is updated), the updating of mass observations will have a stronger effect on the divergent part of the wind field than on its rotational part. We will see later the important consequences which result from this fact.

(ii) Both the model and the observations are perfect. This is the standard "identical twin" hypothesis, which ignores the difficulties arising from observational and modelling errors. This amounts to assuming that the "observations" are exact values taken out of one particular model solution. This solution will be referred to in the following as the "observed solution".

The success or failure of an assimilation will be assessed according to the following criterion: an assimilation will be considered as successful if it is capable of reconstituting any model parameter (i.e. any mass or wind parameter) with any given accuracy, provided it is continued for a sufficiently long period. This means that for any model parameter, the difference between the model value and the corresponding value of the observed solution, tends to zero as the assimilation is continued over an infinite period of time. Now, the success of an assimilation, in that precise sense, may depend on the particular initial state from which the assimilation is started. It is preferable to define a criterion which is independent of the initial state. The assimilation procedure will be said to be *convergent* for a given set of observations if an assimilation carried out on these observations is successful in the above sense for any initial state. An obviously necessary condition for convergence in that precise sense is that only one solution of the model equations be compatible with the available observations. It is evident that, if several model solutions are compatible with the observations, and if an assimilation is started from a state which lies on a compatible solution other than the observed one, the model state will always follow the same solution as the assimilation will proceed, and will never tend to the observed solution.

For the above uniqueness condition to be satisfied, it is necessary that the number of observed data be at least equal to the number of parameters necessary to define a complete model state. For shallow-water equation models, which will mostly be considered here, the latter number is three times the number of parameters which define the mass field only. An assimilation can therefore converge only if the total number of data is at least three times the number of mass parameters in the model.

We will consider two different assimilation procedures. In the first one, which we will call *forward assimilation*, the model is always integrated forward in time, and new data are constantly fed into it as the assimilation proceeds. The study of the convergence of such a procedure is possible only if one assumes that an infinite number of observations are available. In the second procedure, which will be called *forward-backward assimilation*, only a finite number of observations, distributed over a finite period of time, are used. The model is integrated alternatively forward and backward in time over that period, the same data being introduced at each successive cycle.

2.2. *The shallow-water equations*

The problem of determining the velocity field from the history of the mass field makes sense only with models in which the mass and velocity fields are not related by a time-independent relationship. The simplest equations which satisfy the latter condition and whose solutions bear similarity to actual atmospheric flows are the shallow-water equations:

$$\frac{\partial \varphi}{\partial t} + \nabla \cdot (\varphi \mathbb{V}) = 0 \tag{2.1a}$$

$$\frac{\partial \mathbb{V}}{\partial t} + (\mathbb{V} \cdot \nabla)\mathbb{V} + \nabla \varphi + \mathbb{m} \times f \cdot \mathbb{V} = 0 \tag{2.1b}$$

where φ (geopotential) represents essentially a two-dimensional fluid density, and $\mathbb{V}$ is the horizontal velocity of the fluid, assumed to be independent of height; f is the Coriolis parameter, $\mathbb{m}$ a unit vertical vector positive upwards, and ∇ is the two-dimensional del operator.

We will assume that the flow covers a horizontal domain S, described by coordinates x and y, with no bottom topography and with periodic lateral boundary conditions, which ensure that the total mass and energy of the flow are conserved for any solution of (2.1).

The state defined by:

$$\varphi = \Phi_0 = \text{constant} \tag{2.2a}$$

$$\mathbb{V} = 0 \tag{2.2b}$$

is a stationary solution for eqs. (2.1). Linearized about this state, these equations become:

$$\frac{\partial \varphi}{\partial t} + \Phi_0 \nabla \cdot \mathbb{V} = 0 \tag{2.3a}$$

$$\frac{\partial \mathbb{V}}{\partial t} + \nabla \varphi + \mathbb{m} \times f \mathbb{V} = 0 \tag{2.3b}$$

where φ and $\mathbb{V}$ are now "perturbations" from the basic state (2.2). For any solution of the linearized equations (2.3), the following quadratic quantity:

$$E = \tfrac{1}{2} \int_S (\varphi^2 + \Phi_0 \mathbb{V}^2)\, dS \tag{2.4}$$

which represents the total energy of the "perturbed" flow, is conserved with time.

In the case when the domain S is plane and the Coriolis parameter f is constant, eqs. (2.3) can be written, when expressing the wind field by its divergence $D = \nabla \cdot \mathbb{V}$ and its vorticity $\zeta = \mathbb{m} \cdot \nabla \times \mathbb{V}$

$$\frac{\partial \varphi}{\partial t} + \Phi_0 D = 0$$

$$\frac{\partial D}{\partial t} + \nabla^2 \varphi - f\zeta = 0$$

$$\frac{\partial \zeta}{\partial t} + fD = 0.$$

Taking the Fourier transform of these equations, one obtains for wave vector $\mathbb{k}$:

$$\frac{d\varphi}{dt} + \Phi_0 D = 0 \tag{2.5a}$$

$$\frac{dD}{dt} - \mathbb{k}^2 \varphi - f\zeta = 0 \tag{2.5b}$$

$$\frac{d\zeta}{dt} + fD = 0 \tag{2.5c}$$

where φ, D and ζ are now components along the Fourier mode under consideration, and depend upon time only.

Equations (2.5) have three independent eigensolutions. One is associated with eigenvalue 0 and is therefore stationary. The corresponding flow is non-divergent and in geostropic balance. The other two solutions, associated with eigenvalues $\pm i\alpha$

$$\alpha^2 = f^2 + \mathbb{k}^2 \Phi_0 \tag{2.6}$$

are inertia-gravity wave solutions. Any solution to (2.5) is periodic, with period $2\pi/\alpha$. Given any state (φ, D, ζ), its component (φ_g, D_g, ζ_g) along the geostrophic eigensolution is

$$\varphi_g = \frac{1}{\alpha^2}(f^2\varphi - f\Phi_0\zeta) \tag{2.7a}$$

$$D_g = 0 \tag{2.7b}$$

$$\zeta_g = \frac{1}{\alpha^2}(-\mathbb{k}^2 f\varphi + \mathbb{k}^2\Phi_0\zeta) = -\frac{\mathbb{k}^2}{f}\varphi_g. \tag{2.7c}$$

The eqs. (2.5) depend on only one non-dimensional parameter, which can be defined as:

$$\gamma = \frac{f}{\sqrt{f^2 + \mathbb{k}^2\Phi_0}} = \frac{f}{\alpha}. \tag{2.8}$$

This expression will be more convenient for the sequel than the equivalent expression

$$\mathscr{A} = \frac{\mathbb{k}^2\Phi_0}{f^2} = \frac{1-\gamma^2}{\gamma^2}$$

ordinarily used. The value of γ is always comprised between 0 and 1. For values of γ close to 1 (i.e. for scales large compared to the Rossby radius of deformation $R = \sqrt{\Phi_0}/f$, the geostrophic component (2.7) is made up of the geopotential φ together with the vorticity $-(\mathbb{k}^2/f)\varphi$. For values of γ close to 0 (i.e. for scales small compared to R), the geostrophic component is made up of the vorticity ζ and the geopotential $-(f/\mathbb{k}^2)\zeta$. For given $\mathbb{k}$ and Φ_0, γ increases with latitude.

2.3. *Geostrophic adjustment*

One outstanding feature of the atmospheric flow is that it remains close to geostrophic balance, at least in middle and high latitudes. As this balance is permanently disrupted, either by the non-linear dynamic evolution of the flow, or by external causes (mountains, heating, etc.) there must be processes in the atmosphere, as well as in any realistic numerical model, which constantly maintain it. In the following, we will call a *geostrophic adjustment process* any process, either actually occurring in the atmosphere or artificially introduced in a numerical model, the effect of which is to maintain the flow in, or close to, geostrophic balance. As an example, it can be shown (Sadourny, 1972) that adding to the right-hand side of eqn. (2.3b) a term of the form $K\nabla(\nabla\cdot\mathbb{V})$, where K is a positive constant, has the effect of damping the inertia-gravity component of the flow, without altering the geostrophic component. Such a term represents an artificial geostrophic adjustment in the sense just defined.

As already mentioned, the fact that an assimilation is capable of reconstituting the complete flow is by itself quite remarkable, since it could very well be thought that instabilities would prevent such a reconstitution. The following heuristic explanation, which ascribes a basic role to geostrophic adjustment, is usually proposed: the introduction of data into the model disrupts the geostrophic balance, which is restored after some time by geostrophic adjustment. The repetition of data introductions, and of the subsequent restorations of geostrophic balance, will, in the long run, produce a state which will approximately be compatible with the data introduced, and which at the same time will be in approximate geostrophic balance. Indeed, Williamson and Dickinson (1972), considering the linearized shallow-water eqs. (2.5), and assuming the presence of a geostrophic adjustment process, have shown that an assimilation of successive observations of the complete mass field does reconstitute exactly the complete wind field (see also Talagrand, 1972; Morel and Talagrand, 1974; Blumen, 1975a, b). We will show that the presence of a geostrophic adjustment is actually not necessary for the reconstruction of the wind field with, in particular, the consequence that an assimilation is capable of reconstituting a non-zero divergence field.

3. The convergence of an assimilation for the linearized shallow-water equations

We will now proceed to the study of the convergence of an assimilation of mass observations performed with the linear eqs. (2.3), which contain no geostrophic adjustment and conserve energy. We will denote $\varphi_a(x, y, t), \mathbb{V}_a(x, y, t)$ the solution of eqs. (2.3) which is being observed. Each successive observation defines the exact instantaneous values of the mass field $\varphi_a(x, y, t)$ over either the entire domain S, or a part of S.

Let $\varphi_m(x, y, t)$ and $\mathbb{V}_m(x, y, t)$ denote the state of the model at any given time t. Since the model is assumed to be perfect, the time evolution of φ_m and $\mathbb{V}_m$ between two successive introductions of observations is also given by eqs. (2.3). Now, the relevant quantities to be considered are the differences

$$\varphi(x, y, t) = \varphi_m(x, y, t) - \varphi_a(x, y, t) \tag{3.1a}$$

$$\mathbb{V}(x, y, t) = \mathbb{V}_m(x, y, t) - \mathbb{V}_a(x, y, t) \tag{3.1b}$$

between the model state and the actual state to be reconstituted. As already said, an assimilation will be considered as successful if these differences tend to 0 everywhere on S when the assimilation is continued over an infinite length of time. By virtue of linearity, the time evolution of the difference fields (3.1) between two successive introductions of observations is also given by eqs (2.3). An introduction of new observations, as described in the previous section, results in setting the geopotential difference φ equal to 0 over the observational area, leaving it unmodified elsewhere. As for the wind difference $\mathbb{V}$, it is not modified at all.

The total energy E (eq. 2.4)) is an invariant for any solution of eqs. (2.3). If the constant Φ_0 is positive, as it must be for any realistic model, E is non-negative, and is equal to 0 if, and only if, φ and $\mathbb{V}$ are 0 everywhere on S. Between two consecutive introductions of observations, E remains constant. At an introduction time, when φ is set equal to 0 over a part or the whole of S, E can only decrease. When the number of observation introductions increases to infinity, E decreases to a limit $E_\infty \geqslant 0$. If $E_\infty = 0$, φ and $\mathbb{V}$ converge to 0 everywhere on S, and the assimilation is successful (the same argument has been developed independently, and in a slightly different context, by Bube (1978)). The assimilation procedure will be convergent in the sense defined in Section 2 if $E_\infty = 0$ for all initial values of φ and $\mathbb{V}$. The following result is proved in T77. In the case of a forward-backward assimilation, or in the case of a forward assimilation performed on observations which have a periodic time distribution, E_∞ is 0 for all initial values of φ and $\mathbb{V}$ if, and only if, the available observations define a unique solution of eqs. (2.3). As already said, the latter condition is always necessary for the convergence of an assimilation. It is sufficient in the present case.

That result is not specific to eqs. (2.3) but, as shown in T77, applies to any linear system defined by n parameters x_i ($i = 1, \ldots, n$) which conserves a positive definite quadratic form

$$\sum_{i=1}^{n} \alpha_i x_i^2 \qquad (\alpha_i > 0 \text{ for all } i\text{'s}),$$

provided the parameters observed at any observation time are among the x_i's. We shall later make use of this result.

It is noteworthy that, since the convergence of an assimilation depends only on the time evolution of the difference fields (3.1), it does not depend on any specific feature of the observed solution, and, in particular, on whether this solution is geostrophic or not.

4. The case of observations of the complete mass field

The foregoing developments show that the convergence of an assimilation is ensured provided that the observed solution is uniquely defined by the observations, but they do not bring any information as to the detailed features of that convergence and especially as to its rapidity. These features will depend to a large extent on the particular space-time distribution of the observations. A detailed study can be made rather easily for observations of the complete mass field, separated by a constant time interval. We will now proceed to that study, in the case of a plane domain S with a constant Coriolis parameter f. As the successive observations bear on the complete mass field, it is easy to study the assimilation for each Fourier mode independently. For the mode defined by the wave vector $\mathbb{k}$, eqs. (2.3) take the form (2.5).

The introduction of the observed complete mass field at a given time results in setting φ equal to 0, while D and ζ are not modified.

Now, any initial conditions ($\varphi(t)$, $D(t)$, $\zeta(t)$) at a given time t define a unique solution to eqs. (2.5). At time $t + \tau$, this solution assumes the values

$$\begin{pmatrix} \varphi(t+\tau) \\ D(t+\tau) \\ \zeta(t+\tau) \end{pmatrix} = R(\tau) \begin{pmatrix} \varphi(t) \\ D(t) \\ \zeta(t) \end{pmatrix} \tag{4.1}$$

where $R(\tau)$ is the following matrix:

$$R(\tau) = \begin{pmatrix} \gamma^2 + (1-\gamma^2)\cos\beta & -\dfrac{\Phi_0}{\alpha}\sin\beta & -\dfrac{f\Phi_0}{\alpha^2}(1-\cos\beta) \\ \dfrac{\mathbb{k}^2}{\alpha}\sin\beta & \cos\beta & \gamma\sin\beta \\ -\dfrac{f\mathbb{k}^2}{\alpha^2}(1-\cos\beta) & -\gamma\sin\beta & 1-\gamma^2(1-\cos\beta) \end{pmatrix} \tag{4.2}$$

In this expression, α and γ are defined by (2.6) and (2.8) respectively, and β by:

$$\beta = \alpha\tau \tag{4.3}$$

One sees that the matrix $R(\tau)$ is the unit matrix for $\beta = 2l\pi$, l integer, which corresponds to the fact that any solution of eqs. (2.5) is periodic, with period $2\pi/\alpha$.

It is now easy to describe the time evolution of the wind error (D, ζ) in the course of an assimilation. After the introduction of mass observations at time t has set the mass error φ equal to 0, the subsequent integration of (2.5) will start from initial conditions ($\varphi(t) = 0$, $D(t)$, $\zeta(t)$). At the next introduction time $t + \tau$ this integration will produce

(i) a mass error $\varphi(t + \tau)$ which will be set equal to 0 by the introduction of the new observations;

(ii) a wind error $(D(t + \tau), \zeta(t + \tau))$ which, according to (4.1) and (4.2) is equal to

$$\begin{pmatrix} D(t+\tau) \\ \zeta(t+\tau) \end{pmatrix} = B(\tau) \begin{pmatrix} D(t) \\ \zeta(t) \end{pmatrix}$$

where $B(\tau)$ is the 2×2 matrix obtained from the last two rows and two columns of $R(\tau)$, i.e.:

$$B(\tau) = \begin{pmatrix} \cos\beta & \gamma\sin\beta \\ -\gamma\sin\beta & 1-\gamma^2(1-\cos\beta) \end{pmatrix}. \tag{4.4}$$

It will be necessary at this stage to distinguish between the two assimilation procedures described in Section 2, i.e. forward assimilation and forward-backward assimilation.

4.1. *Forward assimilation*

At each assimilation step, the wind difference is multiplied by the matrix $B(\tau)$, which will be noted B. According to the definition given in Section 2, the assimilation procedure will be convergent if, for any initial value, the wind difference (D, ζ) tends to 0 as the assimilation is infinitely continued. This is equivalent to saying that the Nth power B^N tends to 0 as N tends to infinity. According to a basic result of matrix algebra (see e.g. Varga, 1962), the relevant parameter here is the *spectral radius* $\rho(B)$ of B, which is by definition the largest modulus of the eigenvalues of B. This result states that B^N tends to 0 as N tends to infinity if and only if $\rho(B)$ is strictly less than 1.

Fig. 1 shows the variations of $\rho(B)$ as a function of the two parameters β and γ. The spectral radius is everywhere $\leqslant 1$, as could be inferred *a priori*

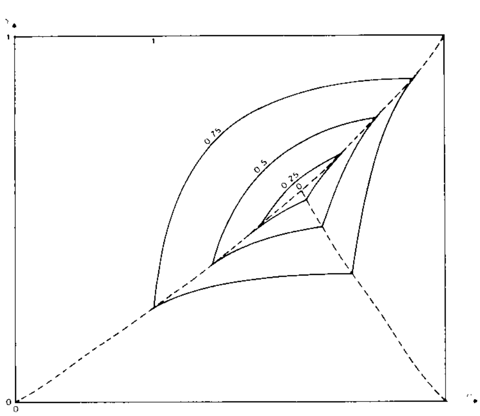

Fig. 1. Variations of the spectral radius of matrix B with respect to the parameters β and γ. Because of symmetries, only the range $0 \leqslant \beta \leqslant \pi$ is considered.

from the energy conservation (2.4). It assumes the value 1 for each of the four limiting values $\beta = 0$, $\beta = \pi$, $\gamma = 0$, $\gamma = 1$ and for these values only. It remains close to 1 over most of the variation range of β and γ, except in the vicinity of the point ($\beta = 2\pi/3$, $\gamma = 1/\sqrt{3}$). At this point, $\rho(B)$ is equal to 0.

The fact that $\rho(B)$ remains close to 1 over most of the variation range of β and γ means that the decrease of the wind error will be slow under most circumstances. From the expression (4.4) for B, it is possible to describe in detail the features of the convergence for any couple of values (β, γ). This is done in T77. However, it turns out to be sufficient for obtaining the most significant results, to consider only the four limiting cases for which $\rho(B)$ is equal to 1. We are going to see that these cases correspond to situations where the successive observations of the mass field do not define a unique solution to eqs. (2.5). This is in agreement with the result stated at the end of the previous section.

(i) When $\gamma = 0$, B assumes the following form

$$B = \begin{pmatrix} \cos\beta & 0 \\ 0 & 1 \end{pmatrix}.$$

At each step of the assimilation, the divergence difference is multiplied by $\cos\beta$, and will tend to 0 (if $\beta \neq l\pi$, l integer), while the vorticity difference is not modified. This situation, reached for $f = 0$, corresponds to the case when the rotational part of the flow is disconnected from the rest of the flow. The successive observations of φ do not bring any information on ζ, which remains undefined.

When γ is small, but different from 0, the assimilation will reconstruct the global wind field, since the spectral radius of B is now strictly less than 1. The divergence will however be reconstructed much more rapidly than the vorticity. It can be shown (see T77) that as the assimilation proceeds and D and ζ tend to 0, the ratio D/ζ tends to a limit which is less than 1, and which itself tends to 0 as γ tends to 0.

(ii) When $\gamma = 1$, B assumes the following form

$$B = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}.$$

B is a rotation matrix of angle $-\beta$. At each assimilation step, the error vector (D, ζ) will rotate by an angle $-\beta$, but its modulus will remain constant. D and ζ will vary periodically, without converging to any limit. This situation is reached for $\Phi_0 = 0$, which corresponds to a degenerate case of no physical interest. It is also reached for $\mathbb{k} = 0$, which means then an assimilation cannot reconstitute the spatial mean values of D and ζ. This is of no practical importance since these mean values are necessarily equal to 0 anyway. But it also means that an assimilation cannot reconstitute a pure inertia wave, i.e. a flow made up of a uniform φ and a uniform $\mathbb{V}$, the latter rotating in time with angular velocity f. This is so because, for a pure inertia wave, the time derivative of φ is zero, whatever is the value of $\mathbb{V}$, and therefore $\mathbb{V}$ cannot be determined from observations of φ. This implies that the wind field on an f-plane can be reconstructed at best up to an additive vector constant in space. But this does not apply on a sphere as pure inertia waves do not exist on a sphere.

When γ is close to 1, while strictly less than 1, an assimilation will reconstitute the complete wind field. It can be shown (see T77) that, contrary to what happens for small values of γ, neither the divergent nor the rotational part of the wind field will be reconstituted more rapidly than the other. As the assimilation proceeds, the ratio D/ζ will vary cyclically between $-\infty$ and $+\infty$, and will not tend to a limit.

(iii) When $\beta = 0$ (or, rather, $\beta = 2l\pi$, l integer), the observing period is equal to an integer times the intrinsic period $2\pi/\alpha$. The successive observations are redundant (in fact identical) and do not bring any information on the wind field. The matrix B correspondingly reduces to the identity matrix.

(iv) When $\beta = \pi$ (or $\beta = (2l + 1)\pi$, l integer), only the first two observations, which are separated by half an intrinsic period, are independent. From the third one on the observations are redundant, and cannot define the complete wind field. The matrix B assumes the form:

$$B = \begin{pmatrix} -1 & 0 \\ 0 & 1 - 2\gamma^2 \end{pmatrix}$$

which shows that the assimilation will in this case reconstitute the vorticity field only.

For a realistic atmospheric model, Φ_0 is to be taken equal to about 9×10^4 m^2 s^{-2}. For the largest wavelengths ($\lambda \simeq 6000$ km), one obtains

$$\alpha \simeq 3 \times 10^{-4}\ \mathrm{s}^{-1}$$

$$\gamma = \frac{f}{\alpha} \simeq .3$$

The parameter γ, which decreases as the wave number increases, will be small for all Fourier components. We can therefore expect the reconstitution of the divergence to be much more rapid than that of the vorticity, especially in the smallest scales, with the exception of the Fourier components such that

$$\beta = l\pi \qquad l \text{ integer} \tag{4.5}$$

for which, because of the "stroboscopic" effect described above, the divergence will not be reconstituted.

Numerical experiments have been performed and their results compared with the foregoing theoretical results. The numerical model used for these experiments is described in detail in Sadourny (1975). It is a spherical grid-point model, with a latitude-longitude staggered rectangular grid (of type C, in the terminology of Arakawa and Lamb, 1977). The spatial differencing schemes conserve total mass, energy and enstrophy, and the time differencing is performed according to the leapfrog scheme. For the experiments considered here, a rather low resolution (40 grid-points on one latitude circle, and 25 points on one longitude circle between both poles) was used. In accordance with eqs. (2.3), energy sources and sinks were removed.

The numerical experiments were actually not performed with the linear eqs. (2.3), but with the fully non-linear eqs. (2.1), the "observed" solution being the state of rest (2.2) (with $\Phi_0 = 8.13 \times 10^4$ m^2 s^{-2}). To first order, the time evolution of the difference between the model and the observed solution is then given by the linearized eqs. (2.3), which justifies comparing the numerical results with the theoretical results derived above.

The model being spherical, the coefficient $-\mathbb{k}^2$ in eqs. (2.5) and in all subsequent expressions must be replaced, for each spherical harmonic, by the corresponding eigenvalue of the Laplacian operator (the change from plane to spherical geometry also introduces additional terms in eqs. (2.5), which correspond to the latitudinal variation of f, and which can be ignored, as shown by the results to be presented below). On the non-discretized sphere, these eigenvalues are equal to $-(n(n+1)/a^2)$ ($n = 0, 1, \ldots, \infty$; a = radius of the sphere) where the integer n is essentially a latitudinal wave number. For the particular grid point model used here, it turns out (Basdevant, pers. com.) that the eigenvalues depend, not only on the latitudinal wave number n, but also on the longitudinal wave number m. However, the latter dependence is weak, with the consequence that, for a given n, the eigenvalues corresponding to the various m's all lie in the vicinity of the value $-(n(n+1)/a^2)$.

Fig. 2 shows the time variation of the global wind root-mean-square difference in the course of an assimilation where one complete mass field was updated every 4 hours. The wind difference first decreases, then, contrary to the previous theoretical results, begins to increase. Before considering this particular point, we note that, in the decreasing phase, the main features anticipated above are found. Figs. 3 and 4 show the spectra of the divergence and vorticity differences respectively at the beginning and after 9 days of assimilation. On the whole, the divergence difference decreases much more rapidly than the vorticity difference, especially for the highest harmonics, where the latter is hardly modified. The decrease of the divergence difference is much slower for abscissa values satisfying condition (4.5).

The increase of the wind difference after the assimilation has been continued for some time, is in contradiction with the energy conservation (2.4), which has been shown to prevent any growth of the difference. This increase is associated with a similar increase of the total energy of the

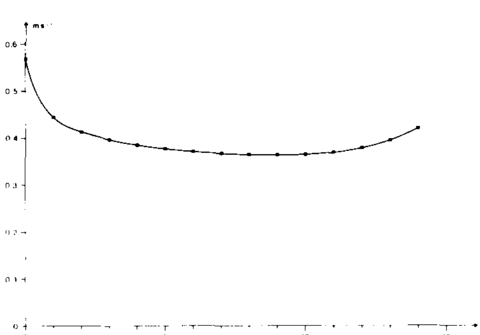

Fig. 2. Variation of the r.m.s. wind difference in a forward assimilation of mass observations.

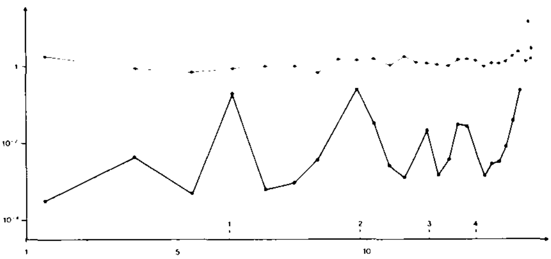

Fig. 3. Spectra of divergence difference at the beginning (upper curve) and after 9 days of assimilation (lower curve). The unit on the vertical axis is arbitrary, but the initial difference has been generated by a random perturbation with standard deviation 0.58 ms^{-1} on each wind component. The abscissa variable is the logarithm of the modulus of the Laplacian eigenvalue. It mostly depends on the latitudinal wave number of the corresponding eigenfunction. The arrows indicate abscissa values satisfying condition (4.5) together with the corresponding values of l.

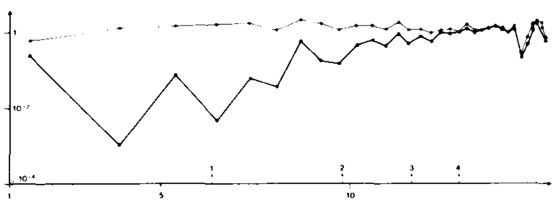

Fig. 4. Same as Fig. 3, for the vorticity difference.

flow. A first explanation may be that the experiment has been performed with the fully non-linear equation for which the expression of total energy contains additional terms beyond (2.4), and for which the results of Section 3 do not necessarily apply. This explanation, however, does not seem

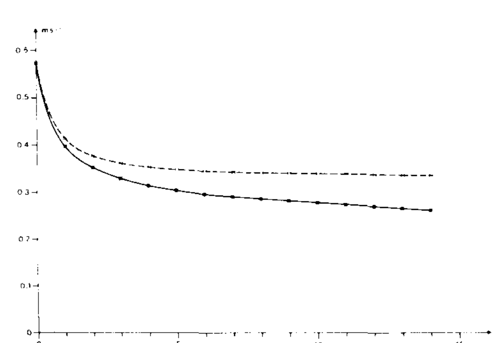

Fig. 5. Same as Fig. 2, in an assimilation with dissipation (solid curve), and in an integration without assimilation of data, but with the same amount of dissipation (dotted curve).

likely for at least two reasons. First, as already said, there exists some neighbourhood of the state of rest (2.2) in which the linear theory developed in Section 3 is valid, and the smaller is the wind difference, the more valid is the linear theory. The present explanation, however, would require that, although the wind difference decreases at the beginning, the linear approximation becomes less and less valid. Secondly, this explanation seems difficult to reconcile with other results, to be presented later in this paper, which suggest that non-linearity does in effect help the assimilation. A more likely explanation is the following. Due to the time discretization, the numerical model conserves the total energy (2.4) to *first order only* with respect to the time increment Δt. In the assimilation process, the highest order terms must accumulate in some way and determine in the long term a global increase of the energy.

Whatever is the reason for this somewhat unexpected energy increase, it can be avoided by adding a dissipative term to the model equations. This is shown by the lower curve of Fig. 5, which shows the wind difference variation in an assimilation with dissipation. The upper curve shows the variation of the same quantity in a run started from the same initial state, containing the same dissipation, but with no introduction of observations. The difference between the two curves therefore shows that part of the decrease which is due to the assimilation only.

4.2. *Forward-backward assimilation*

In a forward-backward assimilation carried out on $N + 1$ successive observations of the global mass field, separated by a constant time interval τ, the wind error (D, ζ) is multiplied N times by the matrix $B(-\tau)$ in the backward phase of an assimilation cycle, and N times by the matrix $B(\tau)$ in the forward phase. The wind error is therefore multiplied at each cycle by the following matrix

$$A = [B(\tau)]^N [B(-\tau)]^N.$$

As the number of cycles tends to infinity, the convergence of the assimilation will depend on the spectral radius $\rho(A)$ of A. We can tell from the energy conservation (2.4) that $\rho(A) \leqslant 1$. When $N > 1$ (if $N = 1$, only two observations per mass parameter are made, and the assimilation cannot converge; $\rho(A) = 1$ for all values of β and γ), the variations of $\rho(A)$ as a function of β and γ are quite similar to those of $\rho[B(\tau)]$ which have been described above: $\rho(A)$ varies between 0 at the point ($\beta = 2\pi/3$, $\gamma = 1/\sqrt{3}$) and 1 for the limit values $\gamma = 0$, $\gamma = 1$, $\beta = 0$, $\beta = \pi$. In the vicinity of these four limit values, the features of the convergence are similar to those already described for a forward assimilation, and the corresponding conclusions remain valid.

An interesting question is the following: for a given time distribution of observations, will the convergence be more rapid in a forward-backward assimilation, or in a purely forward assimilation? In one cycle of forward-backward assimilation, the wind difference is multiplied by the matrix A above. For the same length of model time, i.e. $2N\tau$, of a forward assimilation, the wind difference is multiplied by B^{2N}, where B again stands for $B(\tau)$. Comparing both convergence rates is therefore equivalent to comparing the spectral radii $\rho(A)$ and $\rho(B^{2N}) = |\rho(B)|^{2N}$. From (4.3) and (4.4) it is seen that the matrix $B(-\tau)$ is the transpose $\tilde{B}(\tau)$ of $B(\tau)$. Therefore $A = B^N \tilde{B}^N$. Now, according to a classical result of matrix algebra, the spectral radius of $A = C\tilde{C}$, where C is any matrix, is equal to the square of the Euclidean norm of matrix C (see e.g. Varga, 1962). This implies that

$$|\rho(C)|^2 \leqslant \rho(C\tilde{C})$$

which, for $C = B^N$, yields

$$\rho(B^{2N}) \leqslant \rho(A). \tag{4.6}$$

This means that a purely forward assimilation will converge at least as rapidly as a forward-backward assimilation. Fig. 6 shows the time variation of the global wind difference in a forward-backward assimilation, together with the same variation for a purely forward assimilation, already shown in Fig. 2. The initial decrease is slightly more rapid for the purely forward assimilation. This is in agreement with inequality (4.6) although, in view of the time truncation error described above, we cannot really be sure it is caused by inequality (4.6). The energy growth occurring later in the forward assimilation does not appear in the forward-backward assimilation. One can easily imagine that the energy increase which might occur in a forward integration would be dampened in the following backward integration, thereby preventing any rapid growth of the energy.

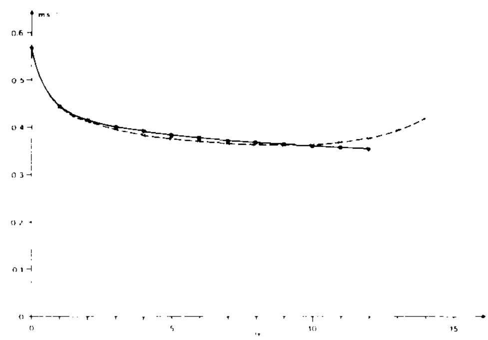

Fig. 6. Variation of the r.m.s. wind difference in a forward-backward assimilation of mass observations (solid curve). The dotted curve is the same as in Fig. 2, and corresponds to a forward assimilation.

4.3. *The effect of latitude on the assimilation*

The effect of the latitudinal variation of the Coriolis parameter can be seen in Fig. 7, which shows, in the case of a forward-backward assimilation, the time variations of the divergence and vorticity differences at high latitude and at the equator. The divergence difference decreases rapidly, at a rate which is independent of latitude. The vorticity difference decreases slowly at high latitude, and does not decrease at all at the equator. These facts agree with what could be *a priori* deduced from the fact that the Coriolis parameter does not appear in the interaction between φ and D, while it is only through f that the vorticity can be modified by an introduction of mass observation.

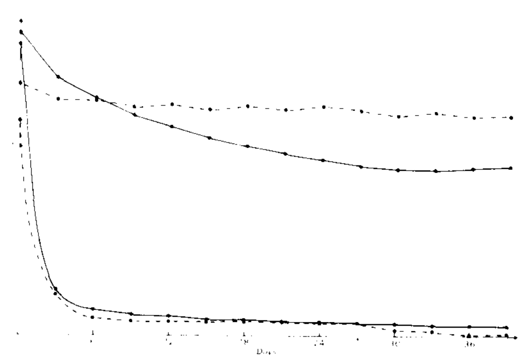

Fig. 7. Variations in a forward-backward assimilation of the r.m.s. of high latitude divergence difference (lower solid curve), equatorial divergence difference (lower dotted curve), high latitude vorticity difference (upper solid curve) and equatorial vorticity difference (upper dotted curve).

5. The case of the multi-level primitive equations

The results of the foregoing section, established for the shallow-water equations, can easily be extended to the multi-level, adiabatic, inviscid, primitive equations. We will consider a fluid occupying a horizontal domain S with, as before, periodic lateral boundary conditions. The pressure p being used as independent vertical coordinate, the state of the flow at a given time is completely defined by the surface pressure $p_S(x, y)$, the potential temperature $\theta(x, y, p)$ (which, here and in the following will be defined as $C_p T_p^{-\kappa}$) and the horizontal velocity $\mathbb{V}(x, y, p)$. The surface pressure p_S and the potential temperature θ define the distribution of mass, while $\mathbb{V}$ defines the distribution of velocity. In the case of no surface topography, the state defined by

$$p_S = p_0 = \text{constant} \tag{5.1a}$$

$$\theta = \theta_0(p) \tag{5.1b}$$

$$\mathbb{V} = 0 \tag{5.1c}$$

is a state of rest. It is convectively stable if $d\theta_0/dp < 0$. The primitive equations, linearized about this state, read:

$$\frac{\partial p_S}{\partial t} + \int_0^{p_0} \nabla \cdot \mathbb{V} dp = 0 \tag{5.2a}$$

$$\frac{\partial \theta}{\partial t} + \omega \frac{d\theta_0}{dp} = 0 \tag{5.2b}$$

$$\frac{\partial \mathbb{V}}{\partial t} + \nabla\varphi + \mathbb{n} \times f\mathbb{V} = 0 \tag{5.2c}$$

where p_s, θ and $\mathbb{V}$ now are "perturbations" from state (5.1). In (5.2), ω is the vertical velocity of the perturbed flow, defined at each level p by the continuity equation

$$\omega = -\int_0^p \nabla \cdot \mathbb{V} dp'$$

while φ is the geopotential "perturbation", defined at each level p by the linearized hydrostatic equation

$$\varphi = \int_{p^\kappa}^{p_0^\kappa} \theta(p')\, d(p'^\kappa) + \kappa p_0^{\kappa-1}\, \theta_0(p_0)\, p_S.$$

The other symbols have the same meaning as in eqs. (2.3).

It is easily verified that, for any solution of eqs. (5.2) the following quantity, which is quadratic with respect to p_S, θ and $\mathbb{V}$

$$E = \tfrac{1}{2}\int_S \int_0^{p_0} (\mathbb{V}^2 + \kappa p_0^{\kappa-2}\, \theta_0(p_0)\, p_S^2$$

$$- \kappa \frac{dp}{d\theta_0} p^{\kappa-1}\, \theta^2) dp\, dS \tag{5.3}$$

and which represents the total energy of the perturbed flow, is conserved with time. For a stable basic temperature profile ($d\theta_0/dp < 0$), E is positive definite, and the result stated at the end of Section 3 is applicable. An assimilation of observations of p_S and/or θ will therefore converge under the only condition that the observations uniquely define the observed solution.

In the case of observations of the complete mass field, it is possible to study quantitatively the convergence of the assimilation by separating eqs. (5.2) into their vertical modes. As it is well known, the time evolution of each of these modes is given by the linearized shallow-water eqs. (2.3), the quantity Φ_0 being replaced by the equivalent depth of the mode under consideration. This equivalent depth is always positive when the basic temperature profile is stable. The results obtained in Section 4 will apply to each vertical mode independently. For a meteorologically realistic model, the first mode (the external mode) has an equivalent depth of about 9×10^4 m^2 s^{-2}. The numerical results presented in Section 4 refer to this value. For higher order modes (the internal modes) the equivalent depth decreases and tends to 0 as the order tends to infinity. The decrease of equivalent depth corresponds to an increase of the parameter γ (eqn. (2.8)). We can therefore expect that, as the order will increase, the effect of the assimilation will become more equally distributed between divergence and vorticity. This is what can be seen in Figs. 8 and 9, relative to the (unique) internal mode of a two-level version of the model described in Section 4. The equivalent depth of the internal mode is $\Phi_0 = 1400$ m^2 s^{-2}. For the lowest harmonics, the divergence and the vorticity are now reconstituted at about the same rate (compare with Figs. 3 and 4). For highest harmonics, the divergence is still reconstituted faster than the vorticity, but the latter is reconstituted more rapidly than on the external mode. One can also note that all stroboscopic effects have disappeared. This is

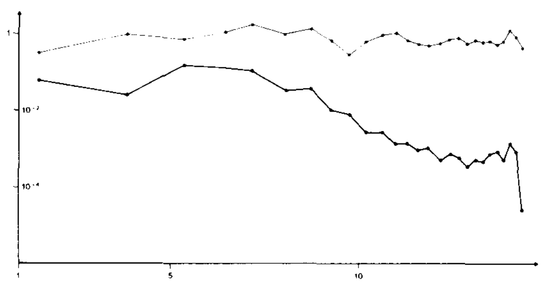

Fig. 8. Spectra of the divergence difference for the internal mode at the beginning (upper curve) and after 9 days of assimilation (lower curve). The units are the same as in Fig. 3.

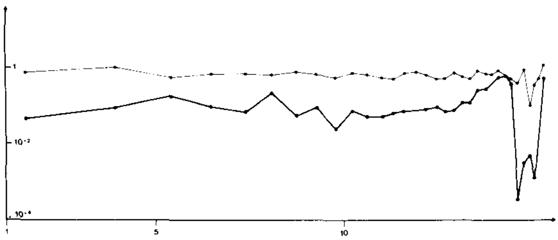

Fig. 9. Same as Fig. 8, for the vorticity difference.

so because the decrease of equivalent depth has moved the corresponding harmonics outside the range resolved by the model.

6. The effect of geostrophic adjustment

We have so far exclusively considered eqs. (2.3) and (5.2), which contain no geostrophic adjustment capable of restoring a disrupted geostrophic balance. It is interesting to study how the presence of a geostrophic adjustment will modify the results of the previous sections. We will consider again the linearized shallow-water eqs. (2.3), and will make the same hypotheses as Williamson and Dickinson (1972). First, the observed solution is exactly geostrophic, and known *a priori* to be so. Second, there exists in the assimilating model a geostrophic adjustment process which, after each introduction of observations, and before the next introduction, damps completely the gravity-wave component of the flow without altering its geostrophic component. It must be noted that these hypotheses change fundamentally the theoretical framework considered so far. Any model solution (and, in particular, the observed solution) is now exactly geostrophic, at least after a finite time of integration. The wind field is therefore a perfectly defined, time-independent, function of the mass field and the problem of determining the wind field from the history of the mass field becomes meaningless. The problem is now to determine the wind field which is in geostrophic balance with the observed mass field. But, of course, the assimilation can be considered as a possible procedure for solving this new problem. One important change is that an average of one datum only per mass parameter, instead of three, is now sufficient for defining completely the wind field.

Although the underlying basic problem now is of a nature different from the one previously considered, it is interesting to compare the corresponding rates of reconstitution of the wind field. At an introduction time, and as before, the difference φ of (3.1) is set equal to 0, while the difference $\mathbb{V}$ is not modified. But the gravity wave component of the difference flow is now completely damped between two introductions of observation, the geostrophic component being kept unchanged. This implies in particular that the energy (2.4) of the difference flow now decreases between two successive introductions of observations, as well as at the time of an introduction. Acting on the difference field ($\varphi = 0$, D, ζ) which results from the introduction of an observed complete mass field, the geostrophic adjustment will leave only the geostrophic component, which is stationary with time and defined by eqs. (2.7):

$$\varphi_g = -\frac{f\Phi_0}{a^2}\zeta$$

$$D_g = 0$$

$$\zeta_g = \frac{\mathbb{k}\,\Phi_0}{a^2}\zeta = (1-\gamma^2)\zeta.$$

The next introduction of observations will replace φ_g with 0, and we see that the wind difference (D, ζ) will be multiplied at each assimilation step by the following matrix G:

$$G = \begin{pmatrix} 0 & 0 \\ 0 & 1-\gamma^2 \end{pmatrix}$$

which replaces the matrix B of (4.4). The spectral radius of G is equal to $1 - \gamma^2$. It is always less than 1, except for $\gamma = 0$. It follows that the reconstitution of the wind field will be rapid for values of γ close to 1, and slow for values close to 0. These are, in a slightly different form, the results found by Williamson and Dickinson (1972).

G is independent of β, with the consequence that $\beta = 0$ and $\beta = \pi$ are no more particular values. This is so because only one datum per mass parameter is now sufficient to define entirely the observed solution, so that no indetermination can arise from a particular choice of the frequency with which the global mass field is observed.

A comparison between the spectral radii of B and G shows that the latter is smaller (i.e. the assimilation will converge more rapidly if a geostrophic adjustment is present) in the vicinity of the values $\gamma = 1$, $\beta = 0$, $\beta = \pi$, but is actually larger elsewhere, and in particular for small values of γ. The reconstitution of the wind field is now rapid when γ is close to 1 because the effect of geostrophic adjustment is to put the wind field in geostrophic balance with the mass field, without modifying the latter. Geostrophic adjustment can thus bring about what simple assimilation could not, namely modifying the wind field in a way consistent with the mass field.

For small values of γ, it is on the contrary the mass field which adjusts to the wind field. The information contained in the observations is lost in the adjustment, which is therefore of no use. Indeed the fact that the spectral radius of G is in this case larger than that of B means that geostrophic adjustment actually makes things worse.

7. The assimilation of wind observations

The previous sections have been devoted to the case of mass observations. It is obvious from eqns. (2.4) or (5.3) that the general result stated at the end of Section 3 also applies to wind observations or to mixed mass/wind observations. The convergence of an assimilation is therefore ensured for any kind of observations, provided the available observations uniquely define the observed solution. In the case of successive observations of the complete wind field, the reconstitution of the mass field can be studied through an approach similar to that presented in Section 4. The mathematics are in effect simpler, since convergence is now determined by a single parameter, which is the first entry

$$\gamma^2 - (1 - \gamma^2) \cos \beta \qquad (7.1)$$

of matrix (4.2). This leads to a slow reconstitution of the mass field for values of γ close to 1 while, for values close to 0, the reconstitution can be slow or rapid, depending on β. If a geostrophic adjustment is assumed to be present, coefficient (7.1) is replaced by γ^2, as can be found from eqs. (2.7). This leads to a slow reconstitution for values of γ close to 1, and a rapid one for values close to 0 (see also Williamson and Dickinson, 1972). It is of interest to mention at this point that Williamson and Kasahara (1971), in a similar study, did not find a more rapid reconstitution of the mass field at the equator, contrary to what one could expect from the presence of a geostrophic adjustment. The present results suggest that geostrophic adjustment was not in that case efficient enough to introduce a latitudinal variation of the reconstitution rate.

8. The case of the general non-linear equations

We have so far considered only the simplified and idealized case of the linear eqs. (2.3) or (5.2). The relevance of this simplified case to the practical assimilation problem is disputable, since it is well known that non-linearity plays an important role in the dynamical evolution of the atmosphere. We can first note that an extension of the above results to the non-linear equations cannot proceed from a simple generalization of the method used here. The reason why the linear eqs. (2.3) or (5.2) conserve a positive definite quadratic energy is simply that the corresponding states of rest (2.2) or (5.1) are states of minimal energy. The approach used here cannot therefore be extended to the case of the non-linear equations, where the particular solution to be reconstituted will not generally correspond to an energy minimum. It is nevertheless interesting at this point to compare the theoretical results obtained above with the numerical results obtained when using the fully non-linear equations.

Figs. 10 and 11 present the time variation of the divergence and vorticity difference spectra in an identical twin assimilation performed with a two-level non-linear primitive equation model. Figs. 10a and 11a refer to the external mode, Figs. 10b and 11b to the internal mode. The model used was the same as in Section 4. It additionally contained

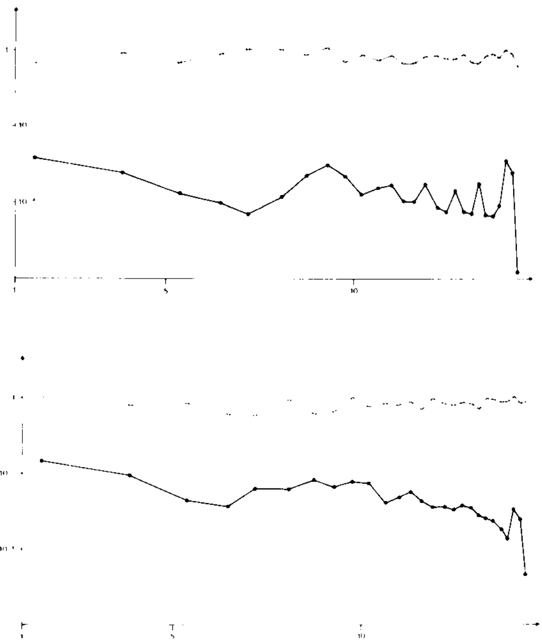

Fig. 10. Spectra of the divergence difference in an assimilation performed with non-linear equations at the beginning (upper curve) and after 9 days of assimilation (lower curve); (a) external mode, (b) internal mode.

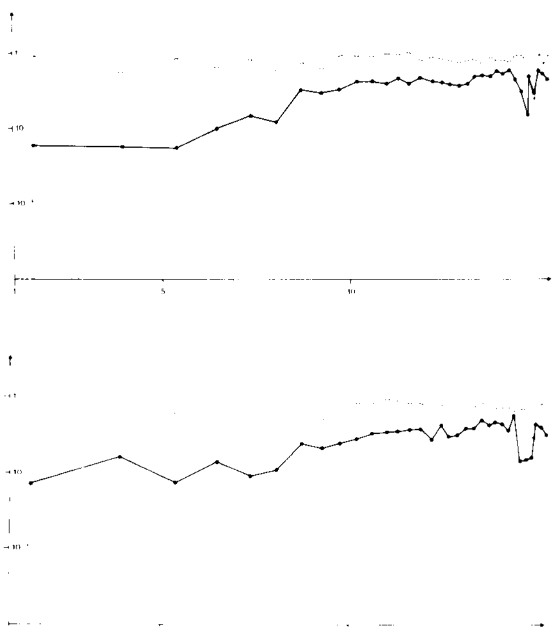

Fig. 11. Same as Fig. 10, for the vorticity difference; (a) external mode, (b) internal mode.

a representation of the basic "physical" processes, and a divergence damping, acting as geostrophic adjustment (see T77). The "observed" solution was taken as meteorologically realistic. In order to make easy comparisons with the results obtained in the linear case, the same space-time distribution of observations, was chosen, i.e. one complete mass field every 4 hours.

All four figures exhibit the same qualitative features obtained from the linearized equations (compare with Figs. 3, 4, 8, 9). Both the divergence and the vorticity are reconstructed for all harmonics, and on both the internal and external modes. Generally speaking, the divergence is reconstructed more rapidly than the vorticity, especially for high harmonics and for the external mode. It is noteworthy that all stroboscopic effects have disappeared, which, in the light of our previous results, can be attributed to the presence of a geostrophic adjustment and to the fact that non-linearity is likely to destroy any periodicity in the flow.

Comparison with results obtained in the linear case shows, however, a great difference. The reconstitution of the wind field is now more rapid, as can be seen in Fig. 12. This must be partly due to the presence of a geostrophic adjustment process which, as said above, accelerates the rate of convergence for certain values of the parameters β and γ. But geostrophic adjustment is certainly not the only cause of the more rapid reconstitution, since the latter is also observed for small values of the parameter γ, for which our conclusion was that the presence of a geostrophic adjustment would not accelerate the rate of convergence. In particular,

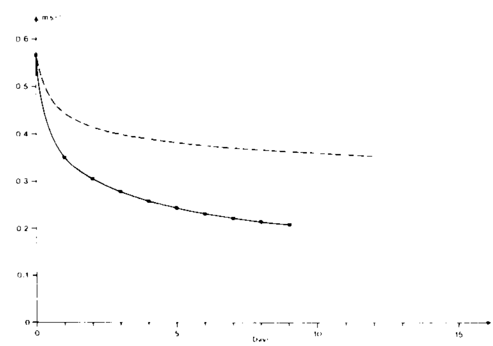

Fig. 12. Variations of the r.m.s. wind difference in two forward assimilations performed with linear (dotted curve) and non-linear (solid curve) equations respectively. The dotted line is the solid curve of Fig. 6.

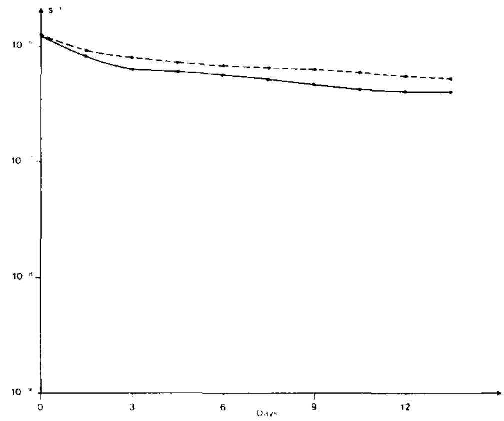

Fig. 13. Variation of the equatorial vorticity difference in an assimilation performed with non-linear equations, for the external (solid curve) and internal (dotted curve) modes respectively.

the high harmonics of the external mode vorticity are now reconstituted more rapidly, as can be seen by comparing Figs. 4 and 11. Also, Fig. 13 shows that equatorial vorticity is now reconstituted by the assimilation, while it was not in the linear case, as shown by the corresponding curve of Fig. 7. This suggests that the rate of reconstitution is accelerated by some non-linear advective interaction between divergence and vorticity, which adds up to the only interaction present in the linear equation, i.e. Coriolis acceleration.

9. Conclusion

On the basis of a mathematical result proved in T77, it has been shown that, because of energy conservation, any solution of the linearized shallow-water equations, or of the linearized primitive equations, can be completely reconstructed through repeated introductions of observations, provided the observations define a unique solution of the equations. No dissipative process, and in particular no geostrophic adjustment, is basically necessary for that. In the case of observations of the mass field, the divergent part of the wind field is reconstituted more rapidly than its rotational part. With the exception of the time discretization effect described in Section 4, these theoretical results are supported by numerical experiments.

The fact that an assimilation of mass observations is capable of reconstituting a non-zero divergence field, which it could not if geostrophic adjustment was the only process through which the wind field could be reconstructed, is of great potential interest. The divergent component of the wind field is small and inaccessible to direct measurements, but nevertheless dynamically important. The assimilation provides a theoretical way to extract that otherwise inaccessible component from the time evolution of the mass field. For instance, a surface pressure tendency of 1 mb hr^{-1} corresponds to a vertically averaged divergence of $3 \times 10^{-7}\ s^{-1}$, several orders of magnitude smaller than the accuracy of direct measurements. The assimilation offers a possibility for reconstructing that divergence in a way coherent with the other available data. And this possibility is not limited to the use of surface pressure tendencies, but applies to all observations which bring information on the time evolution of the mass field.

Numerical results also show that non-linearity can accelerate the convergence. This is of particular interest since non-linearity makes the basic equations unstable, in the sense that two initially close solutions separate rapidly. Obviously, this latter fact is by itself highly unfavourable for the convergence of an assimilation. It would be of great interest to try and determine under which precise conditions non-linearity is favourable or unfavourable to the convergence of an assimilation.

Finally it is clear that if the assimilation as described above provides a theoretical possibility for using the observed history of the mass field, it is certainly not the most efficient approach to that end. The simple updating procedure considered in this paper, in which predicted values are simply replaced by observed values, while the non-observed parameters are not modified, belongs more to brute force than to carefully considered strategy. Numerous more sophisticated methods, some of which are in operational use, have indeed been defined, but they have not been intended at extracting the information contained in the time evolution of the observed fields. The results presented in this paper show that the reappraisal of these methods in the light of this specific objective and, possibly, the definition of new methods, are potentially of great interest.

10. Acknowledgements

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НЕКОТОРЫЕ АСПЕКТЫ ЧЕТЫРЕХМЕРНОГО АНАЛИЗА

Проблема четырехмерного анализа и усвоения данных исследуется с помощью линеаризованных уравнений мелкой воды без вязкости. Изучаются условия, при которых усвоение данных о поле массы, выполненное по простой процедуре инициализации, будет восстанавливать полные поля массы и скорости. На основе математического результата, доказанного в другом месте, показано, что сохранение энергии обеспечивает точную сходимость к любому заданному решению, независимо от того, выполняется геострофическое соотношение или нет, при единственном условии, что доступные данные наблюдений определяют единственное решение уравнений модели. Для большинства значений соответствующих параметров сопеноидапьная комлонента поля ветра восстанавливается значительно более быстро, чем вихревая. Рассмотрен эффект подавления гравитационных волн и показано, что он ускоряет реконструкцию поля ветра только для масштабов, больших по сравнению с радиусом деформации Россби.

Эти результаты обобщаются на случай невязких линеаризованных многоуровенных примитивных уравнений. Рассмотрен также случай с данными наблюдений поля ветра и показано, что и здесь возможно восстановление полного поля массы. Наконец, сравнение с численными результатами, полученными по нелинейным уравнениям, показывает, что основные особенности, полученные из линеаризованных уравнений, сохраняются, но оно также дает основания полагать, что нелинейная адвекция может, по крайней мере, в некоторых случаях ускорять процесс реконструкции.