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Balanced motion of a stratified, rotating fluid induced by bottom topography

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ABSTRACT

The topographically bound, balanced motion of an incompressible, inviscid, and stratified fluid on the f-plane is studied. The potential vorticity of the fluid is specified as a positive function of its density.

For an infinite, straight bottom ridge, it is shown that a solution with constant density along the entire bottom surface is possible only when the crest height is less than a critical value, which depends on the shape of the ridge profile and on its width in comparison with the deformation radius. If the crest height exceeds the critical value, the upper part of the ridge must protrude into less dense layers above, disrupting the lowest isopycnic surfaces. The critical crest height is calculated for certain simple ridge profiles.

The situation seems to be qualitatively similar for a circular bottom topography. In this case, the critical summit height must be determined from the non-linear gradient wind equation, since the geostrophic approximation fails when the critical height is approached. Although no explicit expression for the critical height is offered, it seems plausible that its value will be smaller than for a ridge of similar profile.

1. Introduction

Stratified flow past obstacles on the rotating earth presents an important problem both in meteorology and oceanography. Leaving out of consideration the various kinds of mountain waves, the effect of the obstacle is to force the current to bend partly over, partly around it. The characteristics of the flow depend upon the shape and height of the obstacle and the velocity and stratification of the oncoming current. The problem may be studied either for a gas where the specific entropy is conserved for every particle, or for an incompressible fluid where the density is conserved. In either case, the potential vorticity is also conserved for every particle.

In recent years there has been much discussion about to what extent the fluid flows "around", respectively "over" the obstacle. The meaning of these expressions is not quite clear, however, and it may be useful to consider the following classification of flows, assuming that the lowest isentropic (or, for an incompressible fluid, isopycnic) surface coincides with the bottom boundary in the far upstream region.

First we classify the flows according to the continuity of the isentropic surfaces: the obstacle surface may either coincide with the lowest isentropic surface, or its summit may protrude into potentially warmer (less dense) fluid. We shall refer to these two contingencies as "*isentropic obstacle*" and "*warm obstacle*", respectively. The two cases are topologically different; in the "warm obstacle" case, the lower isentropic surfaces form doubly connected regions.

Another classification concerns the non-occurrence or occurrence of closed streamlines without stagnation points. We shall refer to these two cases as "*open streamlines*" and "*closed streamlines*".

The two classifications above are independent, so that there are four combined contingencies, all of which may occur: "isentropic-open", "isentropic-closed", "warm-open" and "warm-closed".

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If the flow is steady, then every streamline is confined to an isentropic surface, and the lowest fluid layers must clearly flow *around* the obstacle in the "warm" cases. But also in the "isentropic-closed" case the flow may be said to go *around*; only in the "isentropic-open" case is there a clear-cut flow *over* the obstacle.

In the steady-state problem, information about the entropy and potential vorticity is carried along the streamline characteristics from a far upstream inflow boundary where these properties are prescribed. In the closed streamline case, the region of closed streamlines without stagnation points does not receive this information, so that the problem is indeterminate, as has been pointed out by Merkine and Kálnay-Rivas (1976) and others. But also in the open streamline case, the steady-state problem may not have a unique solution, since one does not know in advance whether the obstacle will be "isentropic" or "warm", and in the latter case, how warm. However, we cannot be sure that the steady flow problem is not unique until it has been demonstrated that two or more solutions exist with the same far-field properties.

Steady flow past a circular obstacle with a uniform far-field current has been studied by Huppert (1975), Merkine and Kálnay-Rivas (1976) and Buzzi and Speranza (1979). They all make use of the geostrophic approximation in some form, thus excluding gravity waves. In the first and the last of the papers mentioned, the quasi-geostrophic equations in Charney's formulations are used (e.g. Charney, 1973), whereas Merkine and Kálnay-Rivas make use of the geostrophic momentum approximation. To make the problem determinate, they all *assume* that the obstacle is "isentropic".

However, Buzzi and Speranza also treat one case (a vertical disk) where the obstacle may be more or less "warm", and obtain in this case an infinite number of solutions, indicating non-uniqueness of the steady flow problem. The vertical disk is a rather special type of obstacle, however, and it is perhaps premature to conclude that the steady flow problem is ambiguous for other obstacle shapes.

When the obstacle is assumed to be "isentropic", the results of the papers referred to are in good agreement as to the general features of the steady flow. Compressed isentropic (or isopycnic) layers and anticyclonic vorticity are found over the obstacle, whereas vertically expanded isentropic layers and cyclonic vorticity appear in the surroundings. The motion may be viewed as a uniform current with a superimposed topographically bound anticyclonic vortex over the obstacle. From Charney's quasi-geostrophic equations it follows that the topographically bound vortex is independent of the far-field current velocity, and may be studied in the absence of such a current.

When the maximum velocity of the topographically bound vortex exceeds the far-field current velocity, closed streamlines appear; Huppert (1975) has studied the conditions for this to happen.

Huppert and Bryan (1976) have performed numerical integrations for a temporally variable stratified current past a circular obstacle. Starting from rest, they studied the evolution of the flow as the impressed current grew to a constant value. Thus the obstacle was initially "warm", and they found that it would remain "warm" provided the final current velocity did not exceed a certain critical value; but if the current velocity grew stronger, the warm fluid would blow off the obstacle, which would become "isentropic". Thus it seems that the question of whether the obstacle is "isentropic" or "warm" (and how "warm") can possibly be solved by resorting to the initial value problem.

There is a difficulty, however. Suppose, in an integration experiment with a supercritical current velocity and an "isentropic" obstacle, that the current velocity is again reduced to a subcritical value; would then the warm obstacle reappear? It would clearly not be possible by a mere advection of entropy along the lower boundary, nor by diffusion along the boundary. Thus it is not quite clear to what extent the final state, for a given current velocity, depends on the previous history. If it does, then there would be more than one solution to the steady flow problem.

The present paper deals with the topographically bound motion without a general current, assuming the density and potential vorticity for every particle to be the same as in a hypothetical state of rest over a level bottom surface. This is an adjustment problem, where the state of rest over the level bottom is the initial state, and we ask for a steady state after the bottom topography has grown to its final shape, and transient wave motions have dispersed.

For a topography of arbitrary shape, a possible steady flow will be three-dimensional, and the problem is quite involved. We shall here consider only the special case when the bottom contours are either straight and parallel, or concentric circles. In these cases a steady state is possible where the motion is strictly horizontal, following the bottom contours. Such a motion is horizontally non-divergent, and therefore determined by its vorticity. Moreover, the flow is in strict hydrostatic and geostrophic (or gradient wind) balance, so that the theory is exact.

For an infinite ridge, the theory shows that a solution with an "isentropic" crest is possible only when the crest height does not exceed a critical value which depends on the ridge profile and the properties of the undisturbed fluid layer. Higher ridges must necessarily be "warm". The solution is clearly not unique if the ridge profile contains minor valleys, where pockets of heavy fluid may or may not be trapped. However, for smooth profiles like the Gaussian, it seems likely that the solution is unique whether the ridge height is subcritical or supercritical, although no proof of uniqueness is offered.

The situation seems to be similar for a circular topography, although the analysis is complicated by the non-linearity of the gradient wind equation.

The case of a bottom mount growing in an initially resting fluid is admittedly different from flow past an obstacle, but may still not be entirely irrelevant. For an obstacle of finite extent (e.g. circular), it seems likely that the criterion for whether the obstacle is "isentropic" or "warm" in the absence of a general current should be approximately valid also in the presence of a current, provided the current is weak compared to the topographically bound flow.

The matter is different for an infinite ridge, however. In the absence of a cross-ridge current, a supercritical ridge will cleave the lower isentropical surfaces into two separate parts; and with this configuration no steady cross-ridge flow is possible, no matter how weak. Thus the problem of the infinite ridge with a current across seems rather different from that of a ridge without such a current. The infinite ridge is also physically unrealistic; it is dealt with in this paper solely for reasons of simplicity.

Flow across an infinite ridge has been treated by Merkine (1975). Using a verison of the geostrophic approximation, he found that his solution would break down when the ridge is too steep. This break-down may correspond to the transition from an "isentropic" to a "warm" ridge in the absence of a cross-ridge current.

2. Fluid model

We consider an incompressible, inviscid, and stratified fluid and its motion relative to a Cartesian coordinate frame xyz, which rotates in the absolute sense around the z-axis with the constant angular velocity $f/2$. The field of apparent gravity is uniform, with potential gz (g constant).

In such a fluid there are two conservative properties which remain unchanged for every fluid particle, viz. the specific volume α (inverse density) and the potential vorticity q, defined as

$$q = g^{-1}(f\nabla z + \nabla \times \boldsymbol{v})\cdot\alpha\nabla\alpha \tag{2.1}$$

Here $\boldsymbol{v}$ denotes fluid velocity and ∇ the three-dimensional gradient operator. The constant factor g^{-1} is attached for convenience. In the quasi-static approximation, $\boldsymbol{v}$ is to be interpreted as the horizontal velocity.

We assume that the fluid is stably stratified. Thus α increases monotonically with height, and we may use α as vertical coordinate instead of z. The dependent variables: q, z, pressure p, and horizontal velocity components u, v may all be regarded as functions of x, y, α, and time t. Assuming further that the motion may be considered as hydrostatic, the potential vorticity (2.1) may be expressed as

$$q = \left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Big/\left(-\frac{\partial p}{\partial \alpha}\right) \tag{2.2}$$

where $\partial/\partial x$ and $\partial/\partial y$ are taken at constant α. Moreover, the horizontal pressure gradient force may be written $(-\partial m/\partial x, -\partial m/\partial y)$, where

$$m = gz + \alpha p \tag{2.3}$$

is the Montgomery potential, and

$$\partial m/\partial\alpha = p \tag{2.4}$$

expresses hydrostatic equilibrium.

We shall restrict our consideration to cases where the fields of α and q are *homotropic*, i.e. where q is constant in every isopycnic surface.

Then in all points and at all times, q and α satisfy a relation

$$q = Q(\alpha) \tag{2.5}$$

As a consequence, there is no advection of potential vorticity in isopycnic surfaces. As a further consequence, a possible state of the fluid is the state of rest, which we shall take as reference state.

Since the fluid layer is assumed to be statically stable, a necessary requirement is $Q > 0$.

3. The reference state of rest

In the state of rest, the isobaric and isopycnic surfaces are level, and hence, p, z and m are functions of α only; these will be denoted by capital letters: $P(\alpha)$, $Z(\alpha)$, $M(\alpha)$. The specific volume is assumed to vary from α_B at the bottom of the layer to α_T at the top. The fluid layer rests upon a level surface $z = 0$ so that

$$gZ(\alpha_B) = M(\alpha_B) - \alpha_B P(\alpha_B) = 0 \tag{3.1}$$

The top surface is supposed to be a free surface at constant pressure p_T:

$$P(\alpha_T) = p_T = \text{constant}. \tag{3.2}$$

Equation (2.2) takes the form:

$$-\frac{d^2 M}{d\alpha^2} = -\frac{dP}{d\alpha} = \frac{f}{Q} \tag{3.3}$$

When $Q(\alpha)$ and the constants α_B, α_T, and p_T are specified, the functions $P(\alpha)$, $M(\alpha)$, and $Z(\alpha)$ can be obtained from the above equations, and the reference state is thus completely defined.

For later reference, we note the relation

$$Q/f = (N\alpha/g)^2 \tag{3.4}$$

where N is the buoyancy frequency $(gd \ln \alpha/dz)^{1/2}$.

As specific examples, we now consider two fluid layer models.

I. *Fluid layer of finite depth with constant potential vorticity*

With Q = constant, $N \sim \alpha^{-1}$, we find from the preceding equations:

$$P(\alpha) = p_T + \frac{f}{Q}(\alpha_T - \alpha) \tag{3.5}$$

$$M(\alpha) = \left(p_T + \frac{f}{Q}\alpha_T\right)\alpha - \frac{f}{2Q}(\alpha^2 + \alpha_B^2) \tag{3.6}$$

The depth of the layer is

$$Z_T = \frac{f}{2gQ}(\alpha_T^2 - \alpha_B^2) \tag{3.7}$$

II. *Fluid later of infinite depth with constant buoyancy frequency*

N = constant, $Q \sim \alpha^2$, $\alpha_T = \infty$, $p_T = 0$. In this case:

$$P(\alpha) = gH/\alpha \tag{3.8}$$

$$M(\alpha) = gH[1 + \ln(\alpha/\alpha_B)] \tag{3.9}$$

where H is the density scale height

$$H = g/N^2 \tag{3.10}$$

4. Motion induced by an infinite bottom ridge

Consider a fluid layer initially in the reference state of rest, and suppose that the bottom boundary rises and assumes the shape

$$z_B = hF(x) = h\int_{-\infty}^{+\infty} F_k \exp(ikx)\,dk \tag{4.1}$$

In principle, $F(x)$ may be any function, but we shall here assume that the function represents a symmetric ridge, centred at $x = 0$, and with $F(0) = 1$ so that h represents the crest height. If the fluid layer is confined within walls at $x = \pm a$, say, then the integral should be replaced by a Fourier sum.

We seek a stationary motion with $u = 0$ and where the other variables v, z, and p, and m are functions of x and α. Since there is no pressure gradient in the y-direction, the velocities in this direction are produced solely by the Coriolis forces:

$$v(x, \alpha) = -f\xi(x, \alpha) \tag{4.2}$$

where ξ is the particle displacements, such that a particle which in the final state is located at x was initially at $x - \xi(x, \alpha)$. The final motion is strictly geostrophic and hydrostatic:

$$v = f^{-1}\,\partial m/\partial x \tag{4.3}$$

$$p = \partial m/\partial \alpha \tag{4.4}$$

When these expressions are entered into (2.2) we find, on account of (2.5)

$$\frac{\partial^2 m}{\partial x^2}+fQ\frac{\partial^2 m}{\partial \alpha^2}=-f^2 \tag{4.5}$$

This equation is the two-dimensional and incompressible version of the expression for geostrophic potential vorticity for a compressible atmosphere in isentropic coordinates, presented by Charney and Phillips (1953). They proposed a method of numerical integration based on this equation, together with geostrophic advection of potential vorticity. Their method was not tested until 20 years later, when Bleck (1973, 1974) showed results of numerical integration experiments, using essentially the same method.

In the present problem, (4.5) is exact. The boundary condition at the top is

$$p(x, \alpha_T)=(\partial m/\partial \alpha)_{\alpha_T}=p_T \tag{4.6}$$

The bottom boundary condition is, under the *assumption* that the ridge is "isentropic", i.e. the isopycnic surface $\alpha=\alpha_B$ is still continuous and in contact with the bottom boundary over its entire extent,

$$gz(x, \alpha_B)=(m-\alpha\partial m/\partial\alpha)_{\alpha_B}=ghF(x) \tag{4.7}$$

Equations (4.5–4.7) is a a linear boundary value problem which defines $m(x, \alpha)$ uniquely when $F(x)$ is prescribed. In particular, for $F(x)\equiv 0$ the solution is the reference field $m=M(\alpha)$.

It is convenient to express m as a deviation from $M(\alpha)$:

$$\begin{aligned} m(x, \alpha) &= M(\alpha)+gh\mu(x, \alpha) \\ &= M(\alpha)+gh\textstyle\int_{-\infty}^{+\infty} v_k(\alpha)F_k\exp(ikx)\,dk \end{aligned} \tag{4.8}$$

Since $M(\alpha)$ satisfies (4.5), $\mu(x,\alpha)$ satisfies the corresponding homogeneous equation, and $v_k(\alpha)$ satisfies

$$v_k''-\frac{k^2}{fQ}v_k=0 \tag{4.9}$$

where primes denote differentiation with respect to α. The boundary conditions (4.6) and (4.7) become

$$v_k'(\alpha_T)=0 \tag{4.10}$$

$$v_k(\alpha_B)-\alpha_B v_k'(\alpha_B)=1 \tag{4.11}$$

When $Q(\alpha)$ is a specified, positive function, the last three questions define $v_k(\alpha)$ uniquely. It can be seen directly from these equations that $v_k(\alpha)$ must behave as shown in Fig. 1. For constant k, it decreases monotonically with height, and its tangent in the lowest point (α_B) intersects the line $\alpha=0$ in the point $v_k=1$.

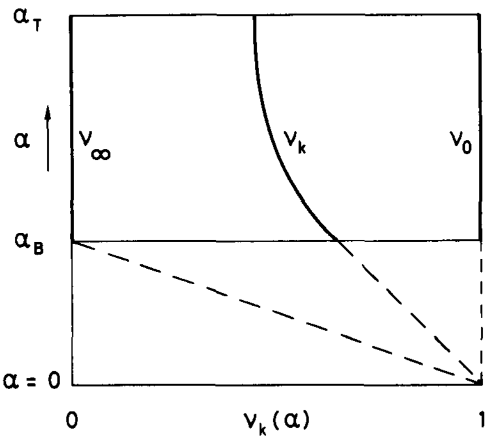

Fig. 1. The function $v_k(\alpha)$.

For constant α, v_k is a monotonically decreasing function of $|k|$. For small values of $|k|$,

$$v_k=1-O(k^2), \quad k\to 0 \tag{4.12}$$

For large $|k|$, we have the asymptotic formula

$$v_k(\alpha)\approx f^{1/2}\alpha_B^{-1}Q(\alpha_B)^{1/4}Q(\alpha)^{1/4}|k|^{-1} \times \exp\left[-|k|\textstyle\int_{\alpha_B}^{\alpha}(fQ)^{-1/2}\,d\alpha\right] \tag{4.13}$$

valid for $\alpha_B\le\alpha<\alpha_T$. (The author is indebted to Dr Eyvind Riis for this expression.) In particular, the value at the bottom is

$$v_k(\alpha_B)\approx\alpha_B^{-1}[fQ(\alpha_B)]^{1/2}|k|^{-1}, \quad |k|\to\infty \tag{4.14}$$

For the two fluid layer models defined in Section 3, eqs. (4.9)–(4.11) are readily solved. For model I ($Q=$ const.) with

$$\kappa=k(fQ)^{-1/2}=gk/(fN\alpha) \tag{4.15}$$

the solution is

$$v_k(\alpha)=v_k(\alpha_B)\frac{\cosh\kappa(\alpha_T-\alpha)}{\cosh\kappa(\alpha_T-\alpha_B)} \tag{4.16}$$

with the bottom value

$$v_k(\alpha_B)=[1+\kappa\alpha_B\tanh\kappa(\alpha_T-\alpha_B)]^{-1} \tag{4.17}$$

For model II we write

$$\lambda=+\left[\left(\frac{gk}{Nf}\right)^2+\frac{1}{4}\right]^{1/2} \tag{4.18}$$

The solution is

$$v_k(\alpha) = v_k(\alpha_B)(\alpha_B/\alpha)^{\lambda - 1/2} \quad (4.19)$$

where

$$v_k(\alpha_B) = (\lambda + 1/2)^{-1} = (Nf/gk)^2(\lambda - 1/2) \quad (4.20)$$

Note that (4.17) and (4.20) are both in agreement with (4.12) and (4.14).

The resulting field of $m(x, \alpha)$ is obtained by entering the solution for $v_k(\alpha)$ into the Fourier integral (4.8). The corresponding distributions of v and p are then obtained from (4.3) and (4.4), and z from $gz = m - \alpha\partial m/\partial\alpha$.

Note that

$$\int_{-\infty}^{+\infty} [p(x, \alpha) - P(\alpha)]\, dx = ghv_0'(\alpha) F_0 = 0 \quad (4.21)$$

on account of (4.12). This equation expresses that the total mass above any isopycnic surface is the same as in the reference state, and is an obvious consequence of mass continuity. It also follows that

$$\int_{-\infty}^{+\infty} [z(x, \alpha) - Z(\alpha)]\, dx = h[v_0(\alpha) - \alpha v_0'(\alpha)] F_0$$
$$= hF_0 = \int_{-\infty}^{+\infty} z_B\, dx \quad (4.22)$$

Thus, for any isopycnic surface, the displaced area in the xz-plane equals the area of the bottom profile.

5. The critical crest height

In order that the solution of the preceding section shall be physically meaningful, we must require that α increases, and p decreases with height everywhere. It suffices to require $\partial^2 m/\partial\alpha^2 < 0$; on account of (2.2)–(2.5), we then have

$$-\frac{\partial p}{\partial\alpha} = \frac{f + \partial v/\partial x}{Q} = \frac{g}{\alpha}\frac{\partial z}{\partial\alpha} = -\frac{\partial^2 m}{\partial\alpha^2} > 0 \quad (5.1)$$

From (2.4) and (4.8) we find, on account of (3.3) and (4.9):

$$-fQ\, \partial p/\partial\alpha = f^2 - gh \int_{-\infty}^{+\infty} k^2 v_k(\alpha) F_k \exp(ikx)\, dk \quad (5.2)$$

Therefore, in order to satisfy (5.1), h must be less than a critical height h_c, defined by

$$h_c^{-1} = \max\{gf^{-2} \int_{-\infty}^{+\infty} k^2 v_k(\alpha) F_k \exp(ikx)\, dk\} \quad (5.3)$$

For many bell-shaped ridge profiles, this expression will assume its maximum value at $x = 0$ and $\alpha = \alpha_B$. In particular, this is the case for profiles whose Fourier transform (F_k) is positive for all k, as e.g. the Gaussian profile, because $v_k(\alpha)$ is positive and largest at $\alpha = \alpha_B$. For such profiles,

$$h_c^{-1} = gf^{-2} \int_{-\infty}^{+\infty} k^2 v_k(\alpha_B) F_k\, dk \quad (5.4)$$

On account of (4.14),

$$k^2 v_k(\alpha_B) \sim |k|, \quad |k| \to \infty \quad (5.5)$$

so that the integral (5.4) converges provided

$$F_k = o(k^{-2}), \quad |k| \to \infty \quad (5.6)$$

It follows from the Riemann–Lebesque lemma that this condition is fulfilled for a ridge profile $F(x)$ whose second derivative exists in all points. Hence, for such bottom profiles, h_c has a finite positive value.

However, if we consider a ridge with a sharp edge at the crest, as e.g. $F(x) = \exp(-|x|/a)$, then $F_k \sim k^{-2}$ for large $|k|$, and the integral (5.4) does not converge. Consequently, $h_c = 0$ such ridges, and the lowest isopycnic surfaces will break for all $h > 0$.

If the crest height h approaches h_c, (5.2) shows that $-\partial p/\partial\alpha$ becomes small over the crest, i.e. isopycnic layers become very thin (Fig. 2a). If $h = h_c$, $\partial z/\partial\alpha$ is zero just at the crest, and when $h > h_c$, the solution of Section 4 gives $\partial z/\partial\alpha < 0$ in a region above the crest, as shown in Fig. 2b. This, of course, is not physically possible. The only way to obtain a realistic solution in this case is to drop the assumption that the isopycnic surface $\alpha = \alpha_B$ is continuous. Instead, we must assume that the crest is "warm", i.e. it has broken through some of the lowest isopycnic layer, as shown in Fig. 2c. This means that the lower boundary condition (4.7, 4.11) must be reformulated. This will be dealt with in Section 6.

The critical crest height is easily calculated for a cosine bottom profile:

$$F(x) = \cos Kx \quad (5.7)$$

In this case, we could place lateral boundaries at $x = \pm\pi/K$. Equation (5.4) now becomes

$$h_c = f^2[gK^2 v_K(\alpha_B)]^{-1} \quad (5.8)$$

In the model I, $Q(\alpha)$ = const., this becomes on account of (4.17),

$$h_c/Z_T = 2\gamma(KL)^{-2} + (1 - \gamma)(KL)^{-1} \tanh(KL) \quad (5.9)$$

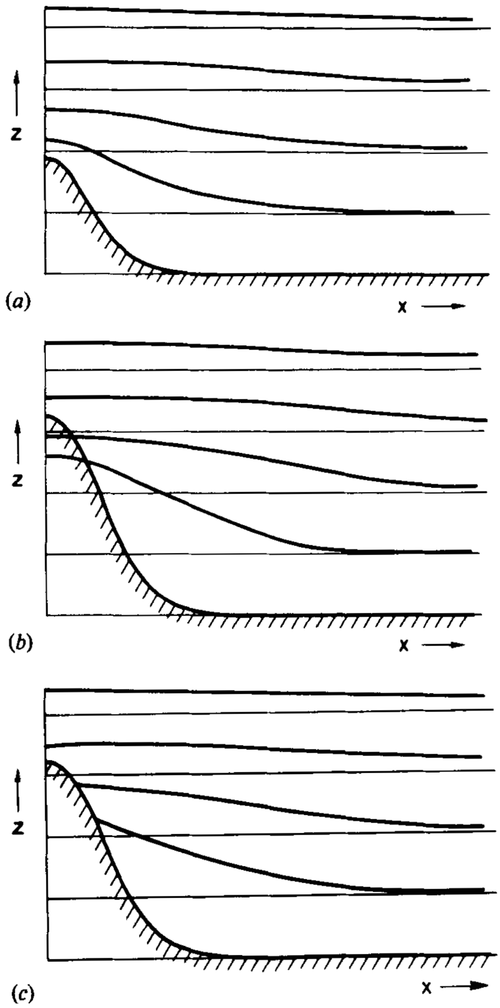

Fig. 2. Isopycnics and ridge profile in the xz-plane. a: $h < h_c$. b: $h > h_c$, erroneous boundary condition. c: $h > h_c$, correct boundary condition.

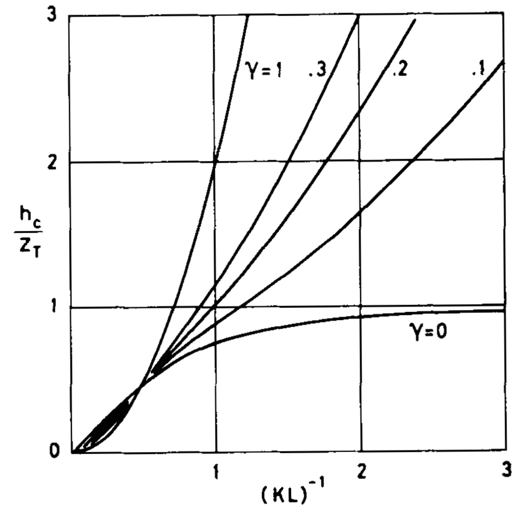

Fig. 3. Critical height of a cosine ridge in fluid layer model I (constant potential vorticity), as a function of wave length.

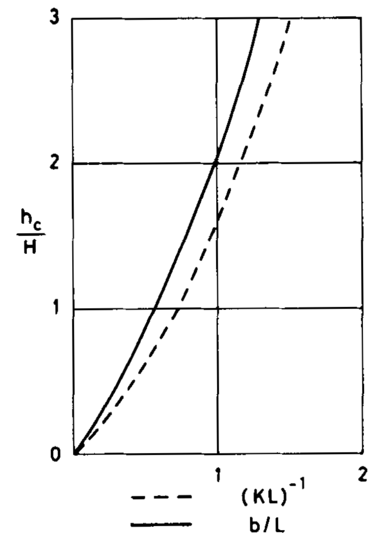

Fig. 4. Critical height of a ridge in fluid layer model II (constant buoyancy frequency), as a function of the ridge half-width. Dashed line: cosine ridge. Solid line: Gaussian ridge.

where Z_T (eq. (3.7)) is the depth of the undisturbed fluid layer,

$$L = \frac{\alpha}{\frac{1}{2}(\alpha_B + \alpha_T)} \frac{N}{f} Z_T = \frac{\bar{N}}{f} Z_T \tag{5.10}$$

is the Rossby deformation radius for a stratified fluid layer, and

$$\gamma = (\alpha_T - \alpha_B)/(\alpha_T + \alpha_B), \quad 0 < \gamma < 1 \tag{5.11}$$

measures the heterogeneity of the fluid layer.

Fig. 3 shows h_c/Z_T as a function of $(KL)^{-1}$ according to (5.9) for some values of γ.

The dashed curve in Fig. 4 shows h_c/H as a function of $(KL)^{-1}$ for the model II with $N =$ constant, according to the formula

$$H/h_c = (K^2 L^2 + 1/4)^{1/2} - 1/2 \tag{5.12}$$

which follows from (5.8) and (4.20). H is the scale height (3.10) and $L = NH/f = g/(fN)$.

Calculation of the critical crest height for other ridge profiles gives results which are qualitatively similar. As an example, the fully drawn curve of Fig. 4 gives the result of a calculation from (5.4) for a Gaussian ridge $F(x) = \exp(-x^2/2b^2)$ with fluid model II (N = constant).

The curves show tnat a narrow ridge will break through the lowest isopycnic surfaces more easily than a broad ridge. For instance, a Gaussian ridge of half width $b = 0.2L$ will become "warm" when $h > 0.22H$, whereas a ridge of half width $b = L$ must grow to a height $h < 1.6H$ in order to acquire a warm crest.

In his treatment of stratified flow over an infinite ridge, Merkine (1975), for reasons of consistency of his approximations, imposed the condition that the ridge slope should not be steeper than $O(f/N)$. When this condition ($\delta \leq 0(1)$ in his notation) was violated, he noted that the isentropic stream surfaces would collapse over the crest of the ridge, in much the same way as found in the present paper. Merkine also found that this collapse would be accompanied by an indefinite increase of the vertical shear just over the crest, and pointed out that the inviscid model would be unrealistic in this case. The situation is not quite the same in the present case without a cross-ridge flow, since the velocity is always zero over the crest itself.

6. The supercritical ridge

Fig. 5 gives a schematic picture of the isopycnic surfaces for a symmetric ridge with $h > h_c$. The highest part of the ridge ($|x| < a$, say) has broken through the isopycnic $\alpha = \alpha_B$ into higher α-values.

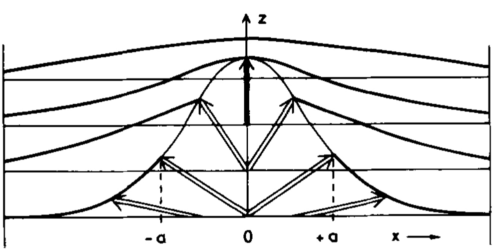

Fig. 5. Isopycnics and displacements for a supercritical ridge.

Accordingly, along the lower boundary α assumes values which may be written

$$\alpha(z = hF(x)) = \begin{cases} \alpha_b(x) > \alpha_B & \text{when } |x| < a \\ \alpha_B & \text{when } |x| \geq a \end{cases} \tag{6.1}$$

In terms of $\mu(x, \alpha)$ defined by (4.8), the lower boundary condition becomes:

$$[\mu - \alpha\,\partial\mu/\partial\alpha + Z(\alpha)/h]_{\alpha=\alpha_b(x)} = F(x), \quad |x| < a$$

$$(\mu - \alpha\,\partial\mu/\partial\alpha)_{\alpha=\alpha_B} = F(x), \quad |x| \geq a \tag{6.2}$$

Here we encounter the difficulty that the function $\alpha_b(x)$ is unknown, and so is the value of a.

The displacement vectors, connecting the initial and final position in the xz-plane, are shown in Fig. 5 for some particles at the lower boundary. Particles with $|x| \geq a$ originate from the level bottom surface $z = 0$, whereas particles with $|x| \leq a$ originate from positions above the level bottom surface; for reasons of symmetry their initial positions were on the line $x = 0$. It follows that the particles with $|x| \leq a$ have had displacements $\xi = x$; hence from (4.2), (4.3), and (4.8),

$$v(x, \alpha_b(x)) = \frac{gh}{f}\left(\frac{\partial\mu}{\partial x}\right)_{\alpha=\alpha_b} = -fx, \quad |x| \leq a \tag{6.3}$$

This gives an additional boundary condition which makes up for the lack of knowledge of the function $\alpha_b(x)$.

The author has not been able to solve analytically the problem for a supercritical ridge, not even for the simplest case of a cosine ridge. However, it is possible to turn the problem around and ask for the particular bottom profile $F_1(x)$ which corresponds to

$$\mu(x, \alpha) = v_K(\alpha)\cos Kx \tag{6.4}$$

for a particular K, when $h > h_c$. Equation (6.3) then gives

$$v_K(\alpha_b(x))\sin Kx/Kx = f^2/ghK^2 = v_K(\alpha_B)\,h_c/h \tag{6.5}$$

The last expression follows from (5.8).

Since $\alpha_b(a) = \alpha_B$, a is defined by

$$\sin Ka/Ka = h_c/h, \quad Ka < \pi \tag{6.6}$$

When $v_K(\alpha)$ is known, (6.5) determines $\alpha_b(x)$, and it can be readily shown that $-\partial p/\partial\alpha > 0$ when

$\alpha > \alpha_b(x)$, so that the solution is physically meaningful. The corresponding bottom profile $F_1(x)$ can be obtained from (6.2). Clearly we have

$$F_1(x) \begin{array}{ll} \neq \cos Kx, & |x| < a \\ = \cos Kx, & |x| \geq a \end{array} \tag{6.7}$$

For fluid layer model II with ν_k given by (4.19), eq. (6.5) gives

$$\frac{\alpha_b(x)}{\alpha_B} = \left(\frac{\sin Kx/Kx}{h_c/h}\right)^{1/(\lambda-\frac{1}{2})} \tag{6.8}$$

Fig. 6 shows the isopycnic surfaces and the modified bottom profile according to eqs. (6.4) and (4.19), with $KL = \pi$ and $h/h_c = 2$.

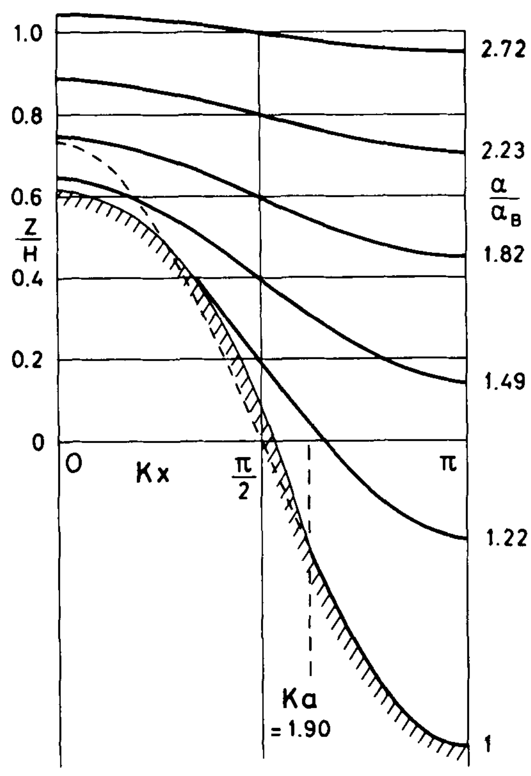

Fig. 6. Isopycnic surfaces in the *xz*-plane according to eq. (6.4) with fluid layer model II (constant *N*), and with parameter values $KL = \pi$, $h_c/H = 0.368$, $h/h_c = 2$. Heavy drawn curves: isopycnics. Thin horizontal lines: reference state isopycnics. Dashed curve: original supercritical ridge. Thin curve: modified topography.

7. The topographically bound vortex for a circular topography

Finally we consider a circular bottom topography of the form

$$z_B = hF(r), \quad r = (x^2 + y^2)^{1/2} \tag{7.1}$$

assuming that $\max\{F(r)\} = F(0) = 1$. The final motion is a stationary circular vortex. Let the particle velocity be

$$u = f\omega r \tag{7.2}$$

reckoned positive when the motion is cyclonic. Then ω is a local Rossby number, and $f\omega$ the angular velocity. With (r, α) as coordinates, the potential vorticity is

$$Q(\alpha) = f\left[1 + \frac{1}{r}\frac{\partial}{\partial r}(r^2\omega)\right]\Big/\left(-\frac{\partial p}{\partial \alpha}\right) \tag{7.3}$$

As before, we write

$$m(r, \alpha) = M(\alpha) + gh\mu(r, \alpha) \tag{7.4}$$

The hydrostatic equation holds rigorously:

$$p(r, \alpha) = \partial m/\partial \alpha = P(\alpha) + gh\, \partial\mu/\partial\alpha \tag{7.5}$$

and (7.3) becomes, on account of (3.3),

$$\frac{1}{r}\frac{\partial}{\partial r}(r^2\omega) + \frac{Qgh}{f}\frac{\partial^2\mu}{\partial\alpha^2} = 0 \tag{7.6}$$

As in the case of an infinite ridge, we must require

$$-\partial p/\partial\alpha = f/Q - gh\, \partial^2\mu/\partial\alpha^2 > 0 \tag{7.7}$$

in order that the solution shall be physically possible.

An approximate equation in μ may be obtained by substituting for ω its geostrophic value ω_g, defined by

$$fu_g = f^2\omega_g r = gh\, \partial\mu/\partial r \tag{7.8}$$

This gives the geostrophic potential vorticity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\mu}{\partial r}\right) + fQ\frac{\partial^2\mu}{\partial\alpha^2} = 0 \tag{7.9}$$

with boundary conditions (for an isentropic summit)

$$\partial\mu/\partial\alpha = 0, \quad \alpha = \alpha_T \tag{7.10}$$

$$\mu - \alpha\,\partial\mu/\partial\alpha = F(r), \quad \alpha = \alpha_B \tag{7.11}$$

$$\frac{\partial \mu}{\partial r} = 0, \quad r = 0 \tag{7.12}$$

In addition, an external boundary condition is required. For an infinite domain, the radial displacement, and hence, the geostrophic circulation should tend to zero for large r; this gives

$$r\,\partial\mu/\partial r \to 0, \quad r \to \infty \tag{7.13}$$

The geostrophic boundary value problem (7.9–7.13) is linear and may be solved by applying the Hankel transform:

$$F(r) = \int_0^\infty \hat{F}_k J_0(kr)\,k\,dk \tag{7.14}$$

$$\mu(r, \alpha) = \int_0^\infty \nu_k(\alpha)\,\hat{F}_k J_0(kr) k\,dk \tag{7.15}$$

Since

$$\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} J_0(kr) = -k^2 J_0(kr) \tag{7.16}$$

it follows that $\nu_k(\alpha)$ must satisfy (4.9–4.11) and is thus identically the same as the $\nu_k(\alpha)$ appearing in the Fourier integral (4.8). As in the case of the infinite ridge, the condition (7.7) gives an expression for the maximum obstacle height h_c compatible with an isentropic boundary condition, i.e. with the assumption $\alpha = \alpha_B$ over the entire bottom surface. In particular, if the topographic profile is simply $F(r) = J_0(kr)$, then h_c determined from the geostrophic approximation will be just the same as for an infinite cosine ridge (eqs. (5.8), (5.9) and (5.12)).

It appears, however, that the geostrophic approximation is not a valid basis for calculation of h_c for a circular topography. The gradient wind equation, expressing balance of forces in radial direction, is

$$\omega_g = \omega + \omega^2 \tag{7.17}$$

As h approaches h_c, then $(-\partial p/\partial\alpha)$ tends to zero at $r = 0$, $\alpha = \alpha_B$. Since the potential vorticity (7.3) is conserved, it follows that the absolute vorticity tends to zero also; therefore, in this point

$$\omega \to -1/2, \quad \omega_g \to -1/4 \tag{7.18}$$

and the geostrophic approximation fails; this was noted also by Merkine and Kalnay-Rivas (1976). It follows that the full gradient wind equation (7.17) must be used in order to determine the critical height.

If $\omega = \omega_g - \omega^2$ is substituted into (7.6) and using (7.2), we obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\mu}{\partial r}\right) + fQ\frac{\partial^2\mu}{\partial\alpha^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{u^2}{gh}\right) \tag{7.19}$$

This exact equation differs from the approximate equation (7.9) in the appearance of the right-hand member. This term is unknown, but must necessarily be positive near the axis. This makes it plausible that $\partial^2\mu/\partial\alpha^2$ is actually larger near the axis than the geostrophic estimate (obtained from (7.9)), and hence, by (7.7), that the true h_c must be *smaller* than the geostrophc estimate for a circular topography.

No solutions to the non-linear problem can be offered in this paper. However, eq. (7.19) suggests a possible iterative method of solution, starting with the geostrophic value of $\mu(r, \alpha)$. As another alternative, an equation in the single variable ω is obtained by differentiating (7.6) with respect to r and making use of (7.8) and (7.17):

$$\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(r^2\omega) + fQ\frac{\partial^2}{\partial\alpha^2}(\omega + \omega^2) = 0 \tag{7.20}$$

In formulating the lower boundary condition, distinction must be made between the isentropic obstacle case ($\alpha = \alpha_B$ over the entire lower boundary) and the "warm" obstacle case ($\alpha > \alpha_B$ over part of the obstacle). In the latter case, the absolute circulation must vanish ($\omega = -1/2$, $\omega_g = -1/4$) over that part of the boundary where $\alpha > \alpha_B$.

8. Concluding remarks

The use of quasi-Lagrangian isopycnic coordinates appears advantageous in the problem considered. The theory makes use of the exact potential vorticity theorem, rather than the quasi-geostrophic pseudo-potential vorticity. Generalization to isentropic coordinates in the case of adiabatic motion of a gas is straightforward.

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СБАЛАНСИРОВАННОЕ ДВИЖЕНИЕ СТРАТИФИЦИРОВАННОЙ ВРАЩАЮЩЕЙСЯ ЖИДКОСТИ, ВОЗБУЖДАЕМОЕ ТОПОГРАФИЕЙ ДНА

Изучается ограниченное топографией сбалансированное движение несжимаемой, невязкой и стратифицированной жидкости на f-плоскости. Потенциальный вихрь жидкости определяется как положительная функция ее плотности.

Показано, что для бесконечного прямого гребня на дне решение существует только когда высота гребня меньше критического значения, зависящего от формы гребня и отношения его ширины к радиусу деформации. Если высота гребня превосходит критическое значение, его верхняя часть должна вклиниться в менее плотные верхние слои, разрывая нижние изэнтропические поверхности. Для некоторых простых профилей гребня вычислена его критическая высота.

Представляется, что ситуация качественно такая же для круговой топографии дна. В этом случае критическая высота должна определяться из нелинейного уравнения градиентного ветра, так как геострофическая аппроксимация не действует вблизи критической высоты. Хотя явное выражение для критической высоты не предлагается, представляется правдоподобным, что ее значение будет меньше, чем в случае гребня подобного профиля.