Tellus (1980), *32*, 284–295

The influence of varying bathymetry on inertial boundary currents

By DORON NOF, *Cooperative Institute for Marine and Atmospheric Studies, University of Miami/NOAA, 4600 Rickenbacker Causeway, Miami, Florida 33149, U.S.A.*

(Manuscript received July 18; in final from December 19, 1979)

ABSTRACT

The response of steady oceanic boundary currents to topographical forcing is examined by a simplified inviscid one-layer model on a beta-plane. The currents are confined between a solid vertical boundary on one side and a free streamline, beyond which the ocean is motionless, on the other. Approximate solutions for both zonal and meridional currents are obtained using power series expansions in the ratio between the width of the current to the length scale of the topography.

It is found that upon encountering a longitudinal ridge, an eastward flowing current confined between a solid vertical boundary on the left (north) and a free streamline on the right (south) may deflect to either direction (toward or away from the coast) depending on the Rossby number, the relative depth change and the variation of the Coriolis parameter with latitude. In mid-latitude a small Rossby number flow narrows and deflects northward as it crosses a ridge in the northern hemisphere. By contrast, when the current is located close to the equator, the ridge causes broadening and a southward deflection. Consequently, there exists a latitude at which a given decrease in depth does not cause any deflection.

Similarly, a meridional southward flowing current confined between a solid wall on the left (east) and a free streamline on the right (west) intensifies and deflects eastward or broadens and deflects westward depending on the range of parameters.

The parameters of both zonal and meridional currents may combine in such a way that the currents turn sharply away from the coast. Under certain conditions, the sharp turning is associated with a separation of the current from the coast and a subsequent formation of a "blocked" region near the wall.

1. Introduction

This study is concerned with the interaction of inertial barotropic boundary currents on a beta-plane with longitudinal topographical features whose axes are perpendicular to the direction of the approaching currents. The currents are bounded by a solid wall on one side and a free streamline, beyond which the ocean is stagnant, on the other.

There have been a number of investigations of the topographic effects on inertial infinitely wide unbounded barotropic flows and the topographic effects on finite width currents. Among the former studies are those of Clark and Fofonoff (1964), Porter and Rattray (1964) and McIntyre (1968). These investigations show that a uniform, infinitely wide unbounded eastward current deflects initially to the right[1] (southward) when the current crosses a ridge. The β effect provides a restoring force and causes a stationary Rossby-wave pattern downstream.

The effect of varying bathymetry on free inertial oceanic jets have been studied by Warren (1963), Robinson and Niiler (1967), and Robinson and Taft (1972), who analyzed jets integrated across their width. Huppert and Stern (1974) examined the influence of bottom topography on flows in uniformly rotating channels. They showed that side walls play an important role in determining the steady response of a channel flow to varying

[1] Hereafter, right and left are with reference to an observer looking downstream in the northern hemisphere.

0040-2826/80/030284-12$02.50/0

bathymetry even if the walls are at large distance from each other. Røed (1979) considered the influence of varying geometry on hydraulically driven inertial boundary currents whose free surface tilt is large so that the free surface intersects the bottom. Under such conditions the current occupies only a portion of the ocean leaving the remaining floor dry. While these investigations are informative, they do not deal directly with the problem considered in this study. The same can be said of Ekman (1923), Sverdrup (1961), Neumann (1960), Boyer (1971a, b), and McCartney (1976). None of these investigations have dealt with the problem posed in our study where inertial currents bounded by a wall and a free streamline on a β plane adjust themselves to variations in depth.

The purpose of this study is to examine the general characteristics of the currents interactions with varying bathymetry. To examine the dynamics of the flow, we consider zonal and meridional geostrophically balanced currents interacting with a gently sloping bottom on a β plane. The depth change causes the currents to deviate from their exact upstream geostrophic balance and the flow reaches a new balance above the new depth. To simplify the analysis, the hydrostatic assumption is invoked and we consider bathymetric changes which are much larger than the free surface displacement. However, the Rossby number characterizing the problem is not constrained to be small. The general analysis is applicable to various types and shapes of bathymetry but only the cases of flows over sinusoidal ridges are given as examples.

Although the potential vorticity equation which governs the flow is linear, the problem is nonlinear. This is because of the boundary condition along the edge of the currents which is obtained from the Bernoulli integral and is necessary since the shape and location of the free bounding streamline are not given *a priori*. The nonlinearity of the boundary condition is removed by using power series expansion in the ratio between the width of the current to the width of the ridge. The formulation of the problem is given in Section 2 where the governing equations are derived and the approximations are discussed. Sections 3 and 4 contain the mathematical solution and the perturbation analysis for zonal and meridional flows. The paper concludes with a summary and discussion (Section 5).

2. Formulation

We shall first consider a steady, zonal, eastward flowing barotropic current. The formulation for steady, barotropic meridional currents will be presented at the end of this section.

The x axis is taken positive toward the east and the y axis is directed toward the north (Fig. 1). The

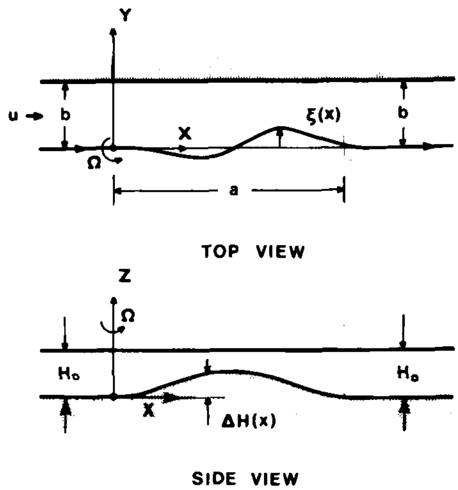


Fig. 1. Schematic diagram of the model under study. The current is confined between a solid wall on the north and a free streamline, beyond which the ocean is stagnant, on the south. ΔH has a sinusoidal distribution $[\Delta H = \widehat{\Delta H}[1 + \sin \pi(2x/a - 1/2)]/2$, where $\widehat{\Delta H}$ is the ridge height at $x = a/2]$ and is measured from the bottom of the ocean. ξ, the free streamline horizontal displacement, is measured in the y direction from the upstream undisturbed edge.

system rotates counterclockwise about a vertical axis and the topography is represented by an irregularity in the lower boundary. For hydrostatic motions the pressure is a linear function of z and the horizontal pressure gradients depend on x and y alone. Therefore, the horizontal velocity components (u and v) are taken to be independent of depth. For such conditions the potential vorticity equation and the Bernoulli integral are [see e.g., Gutman (1972)]:

$$\nabla_H \cdot (\nabla_H \psi/h) + f = hK(\psi) \tag{2.1}$$

$$(\nabla_H \psi/h)^2/2 + g\eta = G(\psi) \tag{2.2}$$

where $K(\psi) = dG(\psi)/d\psi$, ∇_H is the horizontal del-operator, f the Coriolis parameters, h the total depth of the fluid column, g the acceleration of gravity, η the free surface displacement, and ψ a transport function defined by:

$$-uh = \partial\psi/\partial y; \quad vh = \partial\psi/\partial x. \tag{2.3}$$

The free surface displacement $\eta(x,y)$ is measured upward from the free surface height directly above the origin [i.e., $\eta(0,0) = 0$], and the transport function $\psi(x,y)$ is taken to be zero at the coast [i.e., $\psi(x,b) = 0$, where b is the distance between the coast and the origin (see Fig. 1)].

To simplify the analysis we shall consider a bathymetry which varies only with x [i.e., $\Delta H = \Delta H(x)$, where ΔH is the deviation of the bottom height from its upstream level (see Fig. 1)]. The functions $K(\psi)$ and $G(\psi)$ are to be determined from the upstream conditions which must be specified. To simplify the problem we consider flows with an upstream geostrophic balance and a horizontal shear, $\partial u/\partial y = \beta y$, where β is the variation of the Coriolis parameter with latitude. Such an upstream structure is chosen because, as we shall see, it leads to an approximately uniform potential vorticity; this simplifies the mathematical analysis.

In view of the assumed horizontal shear, the velocity distribution upstream is:

$$u = u_0 + \beta y^2/2 \tag{2.4}$$

where u_0 is the upstream velocity at the edge of the current along the free bounding streamline. The velocity at the edge of the current is taken to be finite since the model is inviscid and there is no reason for the velocity to vanish there even though the rest of the ocean is stagnant. We shall return to this point in Section 5.

For such upstream conditions the potential vorticity is found [from (2.1)] to be:

$$K(\psi) = f_0/[H_0 + \eta_u(\psi)] \tag{2.5}$$

where f_0 is the Coriolis parameter at $y = 0$, H_0 the upstream free surface height at $y = 0$ (see Fig. 1), and $\eta_u(\psi)$ is the upstream free surface displacement expressed as a function of ψ. This upstream dependence of η on ψ can be found by eliminating y from the geostrophic relationship

$$\eta_u = -\frac{1}{g}\int_0^y (f_0 + \beta y)(u_0 + \beta y^2/2)\, dy$$

and the definition of the transport function

$$\psi = -\int_b^y (u_0 + \beta y^2/2)[H_0 + \eta(y)]\, dy$$

However, as we shall see, the detailed structure of $\eta_u(\psi)$ is not important for the present analysis because we shall eventually neglect the contribution of η to the potential vorticity. In view of (2.5) the potential vorticity equation (2.1) is:

$$\nabla_H \cdot \left[\frac{\nabla_H \psi}{H_0 - \Delta H(x) + \eta(x,y)}\right] + f_0 + \beta y$$

$$= f_0 [H_0 - \Delta H(x) + \eta(x,y)]/[H_0 + \eta_u(\psi)] \tag{2.6}$$

where $\Delta H(x)$ is taken positive for a ridge so that $(H_0 - \Delta H + \eta)$ represents the total depth of a fluid column (h).

The Bernoulli function $G(\psi)$ can be determined directly from the upstream conditions but, it is more convenient to calculate it from the known relationship between $K(\psi)$ and $G(\psi)$ mentioned earlier [$K(\psi) = dG(\psi)/d\psi$]. This relationship gives:

$$G(\psi) = \int_{\psi_0}^{\psi} K(\psi)\, d\psi + C_1$$

where ψ_0 is the upstream value of ψ at the free edge of the current [i.e., $\psi_0 = \psi(x,0)$ for $x \leqslant 0$] and C_1 is an integration constant to be determined. Since η was defined to be zero upstream at $y = 0$ [i.e., $\eta(x,0) = 0$ for $x \leqslant 0$] it follows from (2.2) and (2.4) that $C_1 = u_0^2/2$. Therefore, we may write the Bernoulli equation (2.2) in the form:

$$\frac{1}{2}\left[\frac{\nabla_H \psi}{H_0 - \Delta H(x) + \eta(x,y)}\right]^2 + g\eta(x,y)$$

$$= \int_{\psi_0}^{\psi} \frac{f_0\, d\psi}{H_0 + \eta_u(\psi)} + \frac{u_0^2}{2} \tag{2.7}$$

This equation shows that the sum, $\frac{1}{2}(u^2 + v^2) + g\eta$, is conserved along a line of constant ψ, and that the maximum variation of the free surface vertical displacement $\eta(x,y)$ is of the order of (u_0^2/g) or $(f_0 u_0 b/g)$ which ever is larger. Hence, for $f_0 \sim 10^{-4}$ s^{-1}, $u_0 \sim 0{\cdot}1$ m s^{-1} and $b \sim 100$ km, the free surface displacement η is about 10 cm. That is, η is about three orders of magnitude smaller than both ΔH and H_0 which are of 0(100–1000 m) for most currents of practical interest. Consequently, terms involving $(\eta/\Delta H)$ and (η/H) can be neglected without introducing errors larger than $0(10^{-3})$.

For such conditions, eq. (2.5) shows that the potential vorticity is uniform $[K(\psi) = f_0/H_0]$, and eq. (2.6) reduces to:

$$\nabla_H^2 \psi + \frac{\partial \Delta H}{\partial x}\frac{\partial \psi}{\partial x}\Big/(H_0 - \Delta H) + (\beta y + f_0 \Delta H/H_0) \times (H_0 - \Delta H) = 0 \qquad (2.8)$$

We see that the neglect of terms of $0(\eta/\Delta H)$ eliminated η from the potential vorticity equation. By using the same simplification [i.e., neglecting terms of $0(\eta/\Delta H)$], the Bernoulli equation (2.7) reduces to:

$$\left(\frac{\nabla_H \psi}{H_0 - \Delta H}\right)^2\Big/ 2 + g\eta = f_0(\psi - \psi_0)/H_0 + u_0^2/2 \qquad (2.9).$$

and the definition of the transport function (2.3) takes the simple form:

$$-u[H_0 - \Delta H(x)] = \frac{\partial \psi}{\partial y};$$
$$v[H_0 - \Delta H(x)] = \frac{\partial \psi}{\partial x} \qquad (2.10)$$

which, in a similar fashion to (2.8), does not involve the free surface vertical displacement η.

The governing equations are subject to the following boundary conditions:

$$\psi = 0; \qquad y = b \qquad (2.11a)$$
$$\psi = \psi_0 = u_0 H_0 b + \beta b^3 H_0/6; \qquad y = \xi(x) \qquad (2.11b)$$
$$[\nabla_H \psi/(H_0 - \Delta H)]^2 = u_0^2; \qquad y = \xi(x) \qquad (2.11c)$$

where $\xi(x)$ is the horizontal deviation of the free bounding streamline from its undisturbed upstream position (see Fig. 1).

Condition (2.11a) states that the coast is a streamline, and condition (2.11b) indicates that the free streamline, which bounds the current from the right is allowed to shift as the current crosses the ridge. Condition (2.11c) is nonlinear and states that the square of the velocity is constant along the free bounding streamline. This condition results from an application of the Bernoulli principle (2.9) along the free bounding streamline ($\psi = \psi_0$). It indicates that the free surface height (η) does not vary along the edge of the current which ensures that the region beyond the bounding streamline is motionless everywhere. This free streamline condition compensates for the fact that the shape and location of the free bounding streamline are not given in advance but must be determined as part of the flow. Similar free streamline conditions are discussed by Stern (1975, chap. 2.6), Batchelor (1967), Ingersoll (1969) and Garabedian (1964).

We shall now discuss the governing equations for meridional boundary currents. For such currents we rotate the coordinate system, shown in Fig. 1, by 90° counterclockwise so that the x and y axes are directed toward the east and north respectively. The current is flowing southward and has a solid boundary on the left (east) and a free streamline on the right (west). The ridge axis of symmetry is located at $y = -d/2$ and is in the x direction.

In contrast to the zonal currents case, we consider flows with a uniform velocity distribution upstream ($v = v_0$ at $y \geqslant 0$ and $0 \leq x \leq c$, where c is the current width). With the assumption $|\eta| \ll H_0$ and $|\eta| \ll |\Delta H|$, the upstream condition gives a uniform potential vorticity, $K(\psi) = f_0/H_0$. For topography which is independent of x [i.e., $\Delta H = \Delta H(y)$] the potential vorticity equation (2.1) becomes:

$$\nabla_H^2 \psi + \frac{\partial \Delta H}{\partial y}\frac{\partial \psi}{\partial y}\Big/[H_0 - \Delta H(y)] + (\beta y + f_0 \Delta H/H_0) \times (H_0 - \Delta H) = 0 \qquad (2.12)$$

The free surface vertical displacement η is defined in a similar fashion to previously [i.e., $\eta(0,0) = 0$] and the transport function equals ψ_0 upstream at $x = 0$ [i.e., $\psi_0 = \psi(0,0)$]. Under such conditions the Bernoulli equation has the same form as (2.9), where $u_0^2/2$ is simply replaced by $v_0^2/2$. The boundary conditions are similar to those of zonal currents and their details will be given in Section 4.

3. Zonal currents

3.1. *Perturbation analysis*

Equations (2.8) and (2.9) hold for all streamlines which originate upstream. In the subsequent analysis, the following non-dimensional scaled variables are used:

$$\left.\begin{aligned} &x^* = x/a; \quad y^* = y/b; \quad \xi^* = \xi/b; \\ &u^* = u/u_0 \\ &v^* = v/u_o(b/a); \quad \Delta H^* = \Delta H/H_o; \\ &\psi^* = \psi/u_0 H_0 b \\ &R_0 = u_0/f_0 b; \quad \varepsilon = (b/a)^2; \\ &\beta^* = \beta b^2/u_0 \end{aligned}\right\} \quad (3.1)$$

where a is the width of the ridge which is assumed to be much larger than the upstream current width (b). In terms of these non-dimensional variables (2.8) becomes:

$$\varepsilon\left[\partial^2\psi^*/\partial x^{*2} + \frac{\partial \Delta H^*}{\partial x^*}\frac{\partial \psi^*}{\partial x^*}\Big/(1-\Delta H^*)\right] + \frac{\partial^2\psi^*}{\partial y^{*2}} + (1-\Delta H^*)(\beta^* y^* + \Delta H^*/R_0) = 0 \quad (3.2)$$

The non-dimensional boundary conditions are found from (2.11) to be:

$$\psi^* = 0; \qquad y^* = 1 \quad (3.3a)$$

$$\psi^* = 1 + \beta^*/6; \qquad y^* = \xi^*(x^*) \quad (3.3b)$$

$$\varepsilon(\partial\psi^*/\partial x^*)^2 + (\partial\psi^*/\partial y^*)^2 = (1-\Delta H^*)^2; \qquad y^* = \xi^*(x^*) \quad (3.3c)$$

It is further assumed that the transport function ψ^* and the free streamline displacement ξ^* possess the expansions:

$$\psi^* = \psi^{(0)} + \varepsilon\psi^{(1)} + \dots \quad (3.4a)$$

$$\xi^* = \xi^{(0)} + \varepsilon\xi^{(1)} + \dots \quad (3.4b)$$

where the parameter $\varepsilon = (b/a)^2$ is small compared to unity. By substitution of (3.4) into (3.2) and (3.3) and collecting terms of order unity one obtains the zeroth-order potential vorticity equation:

$$\frac{\partial^2\psi^{(0)}}{\partial y^{*2}} + (1-\Delta H^*)(\beta^* y^* + \Delta H^*/R_0) = 0 \quad (3.5)$$

and the zeroth-order boundary conditions:

$$\psi^{(0)} = 0; \qquad y^* = 1 \quad (3.6a)$$

$$\psi^{(0)} = 1 + \beta^* 6; \qquad y^* = \xi^{(0)}(x^*) \quad (3.6b)$$

$$\partial\psi^{(0)}/\partial y^* = -(1-\Delta H^*); \qquad y^* = \xi^{(0)}(x^*) \quad (3.6c)$$

which are valid for all x^*. We see that the perturbation scheme has removed the changes in curvature vorticity ($\partial v^*/\partial x^*$) from the potential vorticity equation and the nonlinearity from the boundary condition at the edge of the current. Under such conditions the changes in the bathymetry are compensated only by generation of shear ($\partial u^{(0)}/\partial y^*$) and changes in latitude. Note that both R_0 and ΔH^* can be of order unity and are not necessarily small.

3.2. *Zeroth-order solution*

To find the zeroth-order solution we proceed as follows. The solution of (3.5) which satisfies (3.6a) is:

$$\psi^{(0)} = \frac{-\Delta H^*}{2R_0}(1-\Delta H^*)[(y^*)^2 - 1] - \frac{\beta^*}{6} \times (1-\Delta H^*)[(y^*)^3 - 1] + A(x^*)(y^* - 1) \quad (3.9)$$

where $A(x^*)$ is to be determined. Substitution of (3.9) into (3.6b) and (3.6c) gives:

$$1 + \beta^*/6 = \frac{-\Delta H^*}{2R_0}[(\xi^{(0)})^2 - 1](1-\Delta H^*) - \frac{\beta^*}{6}(1-\Delta H^*)[(\xi^{(0)})^3 - 1] + A(x^*)(\xi^{(0)} - 1) \quad (3.10)$$

and,

$$A(x^*) = (1-\Delta H^*)\left[\frac{\Delta H^*}{R_0}\xi^{(0)} + \frac{\beta^*}{2}((\xi^{(0)})^2 - 1\right] \quad (3.11)$$

The last two equations enable one to find both $A(x^*)$ and $\xi^{(0)}$ and thus to obtain an expression for $\psi^{(0)}$. By eliminating $A(x^*)$ from (3.10) and (3.11) one obtains:

$$(1 + \beta^*/6)/(1-\xi^{(0)})(1-\Delta H^*) + \beta^*(\xi^{(0)})^2/3 + \xi^{(0)}(\Delta H^*/2R_0 - \beta^*/6) - (\Delta H^*/2R_0 + \beta^*/6 + 1) = 0 \quad (3.12)$$

which is a cubic equation for $\xi^{(0)}$ and hence has an exact analytical solution [see e.g. Pearson (1974)]. We shall now discuss the properties of (3.12) for a various range of parameters. Before discussing the most general cases, where the uniform rotation ($\Delta H^*/R_0$) and the β effect are of comparable importance, we shall consider the solutions on a non-rotating plane ($R_0 \to \infty$; $\beta^* = 0$) and the solution on a uniformly rotating plane ($\beta^* = 0$). These will shed some light on the processes involved.

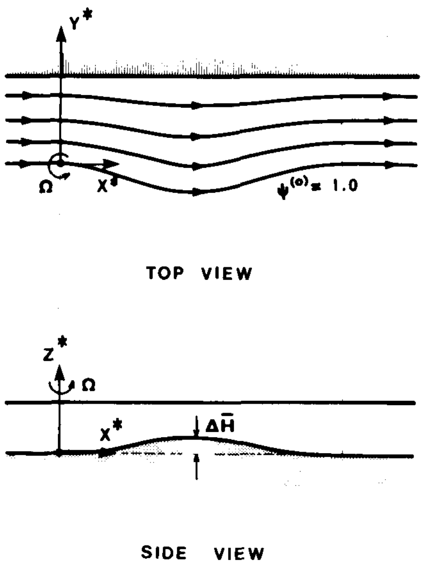


Fig. 2. Equal transport lines for a non-rotating current crossing a ridge $[\overline{\Delta H} = \widehat{\Delta H}/H_0 = 0.25; R_0 \to \infty; \beta^* = 0; (b/a)^2 = 0.10]$.

In the absence of rotation ($R_0 \to \infty$; $\beta^* = 0$), the solution of (3.12) is:

$$\xi^{(0)} = 1 - 1/(1 - \Delta H^*) \tag{3.13}$$

which by (3.11) and (3.9) gives:

$$\psi^{(0)} = -(1 - \Delta H^*)(y^* - 1) \tag{3.14}$$

A typical solution is shown in Fig. 2. The velocity x component is uniform everywhere; the current broadens as it crosses the ridge and returns to its original position and width downstream. Equation (3.13) shows that the broadening increases as the height of the ridge increases.

For a uniformly rotating plane ($\beta^* = 0$), (3.12) reduces to a quadratic equation which gives two solutions for $\xi^{(0)}$. The physically relevant solution, corresponding to $\xi^{(0)} = 0$ for $\Delta H^* = 0$, is:

$$\xi^{(0)} = \{1 + \Delta H^*/R_0 - [1 + 2\Delta H^*/R_0 \times (1 - \Delta H^*)]^{1/2}\}R/\Delta H^* \tag{3.15}$$

The transport function $\psi^{(0)}$ is found by (3.9) and (3.11) to be:

$$\psi^{(0)} = (1 - \Delta H^*)(y^* - 1) \times [\Delta H^*(2\xi^{(0)} - y^* - 1)/2R_0 - 1] \tag{3.16}$$

and the velocity x component is:

$$u^{(0)} = 1 + \Delta H^*(y^* - \xi^{(0)})/R_0 \tag{3.17}$$

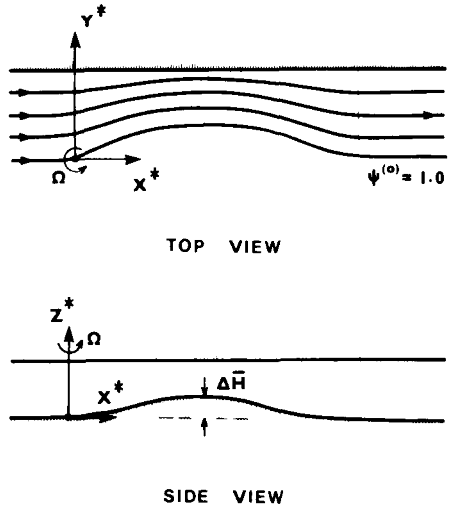


Fig. 3. Equal transport lines for a current crossing a ridge on an f plane $[\overline{\Delta H} = 0.04; R_0 = 0.02; \beta^* = 0; (b/a)^2 = 0.10]$.

We see that across the ridge the velocity x component increases with y^* and is linearly distributed. A typical solution is shown in Fig. 3 which indicates that the current narrows and deflects to the left as it crosses the ridge. Downstream the current returns to its original position and width and is in an exact geostrophic balance. The deflection increases as the Rossby number decreases as indicated by Fig. 4 which shows the dependence of the free streamline displacement on the Rossby number.

It is important to note that while the solution (3.15) is valid for all flows over ridges ($\Delta H^* > 0$), it breaks down for many flows over canyons ($\Delta H^* < 0$). As we shall see shortly, this results from the term in the square brackets in (3.15) which may approach zero or become negative for a sufficiently deep canyon or a sufficiently small Rossby number.

The first-order equation is found by inserting (3.4) into (3.2) and collecting terms of $o(\varepsilon)$, to be:

$$\partial^2 \psi^{(1)}/\partial y^{*2} = -\partial^2 \psi^{(0)}/\partial x^{*2} - \frac{\partial \Delta H^*}{\partial x^*} \times \frac{\partial \psi^{(0)}}{\partial x^*}\Big/(1 - \Delta H^*)$$

which, by (3.16) gives:

$$\frac{\partial^2 \psi^{(1)}}{\partial y^{*2}} = \frac{\partial^2 \xi^{(0)}}{\partial x^{*2}} F_1 + \frac{\partial \xi^{(0)}}{\partial x^*} F_2 + F_3$$

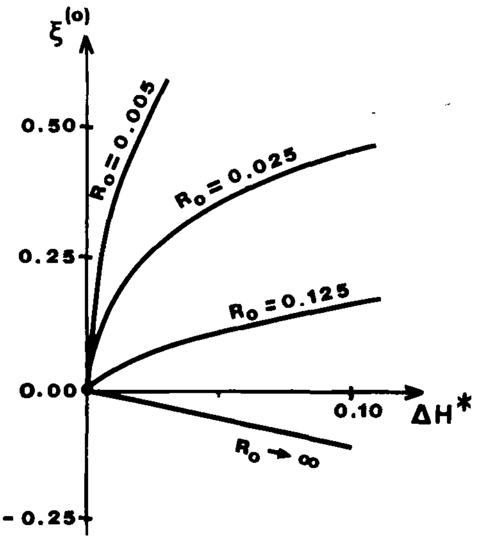


Fig. 4. Free streamline horizontal displacement ($\xi^{(0)}$) as a function of Rossby number and the ridge height for a current on an f plane ($\beta^* = 0$).

where F_1, F_2 and F_3 are known well-behaved functions of ΔH^*, R_0 and y^* whose detailed structure is not important for the present analysis. It is easy to show from (3.15) that the terms containing $\partial\xi^{(0)}/\partial x^*$ and $\partial^2\xi^{(0)}/\partial x^{*2}$ approach infinity as the term in the square brackets in (3.15) approaches zero. That is, as the term in the square brackets in (3.15) approaches zero the current turns sharply away from the coast ($\partial\xi^{(0)}/\partial x^* \to -\infty$), the perturbations become infinitely large and the solution becomes invalid. Under such conditions the curvature vorticity ($\partial v^*/\partial x^*$) and the square of the velocity y component at the edge of the current, which have been neglected, become important. This means that the current cannot cross the topographical feature without losing its structure as an intense narrow boundary current.

More insight into this point is obtained by examining the velocity field. Examination of the velocity x component (3.17) shows that when the current reaches the depth which causes the term in the square brackets in (3.15) to be identically zero, the velocity vanishes at the wall. When the current advances into deeper water the horizontal shear ($\Delta H^*/R_0$) must also increase according to (3.17) and this requires reverse velocities near the coast. However, reverse velocities near the wall do not correspond to parcels which originated upstream but to parcels which have originated downstream. It follows that the current separates from the wall near the point where the depth causes the velocity to vanish at the wall (i.e., when the term in the square brackets in (3.15) is identically zero). Beyond the separation point the current is bounded by two free streamlines, one at each edge; the region near the wall beyond the separation point is "blocked" in the sense that no parcels which have originated upstream can enter it. The detailed solution of such separated currents is beyond the scope of this study. Note that the location of the actual separation point will differ from that given by the condition that the term in the square brackets of (3.15) vanishes because the zeroth-order solution breaks down at this point.

So far, we have discussed the solution of (3.12) as it applies to eastward boundary currents on an f plane. We shall now consider the more general cases of eastward boundary currents interacting with topography on a β plane. Although the solution of the cubic equation (3.12) can be found analytically for any $\xi^{(0)} < 1$, we shall focus our attention on the cases where $(\xi^{(0)})^3 \ll 1$ because for such cases the solution can be simplified by reducing (3.12) to a quadratic equation. To do so one replaces the term $(1 - \xi^{(0)})^{-1}$ in (3.12) by $[1 + \xi^{(0)} + (\xi^{(0)})^2]$ to find, as previously, two solutions for $\xi^{(0)}$. The physically relevant solution, corresponding to $\xi^{(0)} = 0$ for $\Delta H^* = 0$, is:

$$\begin{aligned}\xi^{(0)} \approx \Big\{ &\beta^*/6 - \Delta H^*/2R_0 - (1 + \beta^*/6)/(1 - \Delta H^*) \\ &+ [(1 + \beta^*/6)(3\Delta H^*/R_0 + 4 - \beta^*)/(1 - \Delta H^*) \\ &- 3(1 + \beta^*/6)^2/(1 - \Delta H^*)^2 + (\Delta H^*/R_0 \\ &+ \beta^*)^2/4 \\ &+ 4\beta^*/3]^{1/2} \Big\} \{2\beta^*/3 + 2(1 + \beta^*/6)/ \\ &(1 - \Delta H^*)\}^{-1} \end{aligned} \tag{3.18}$$

The transport function $\psi^{(0)}$ is again given by (3.16) and $A(x^*)$ by (3.11). The velocity x component is found to be:

$$\begin{aligned} u^{(0)} = 1 &+ \Delta H^*(y^* - \xi^{(0)})/R_0 \\ &+ \beta^*[(y^*)^2 - (\xi^{(0)})^2]/2 \end{aligned} \tag{3.19}$$

We see that across the ridge the velocity x component increases northward and has a parabolic distribution. As compared to the upstream velocity distribution $[1 + \beta^*(y^*)^2/2]$, the uniform rotation term ($\Delta H^*/R_0$) increases the velocity in the field by $\Delta H^*(y^* - \xi^{(0)})/R_0$ while the β effect decreases the velocity by $\beta^*(\xi^{(0)})^2/2$. A typical

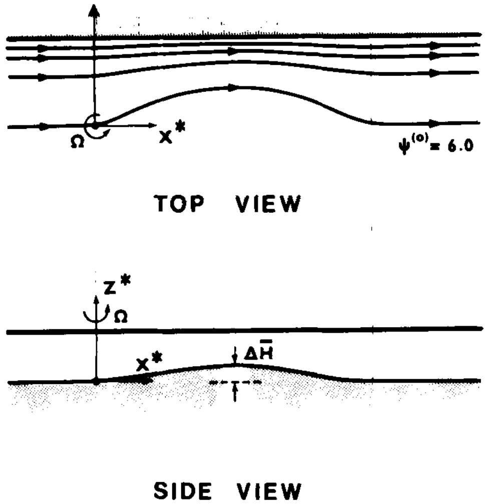


Fig. 5. Equal transport lines for a zonal mid-latitude eastward current on a β plane [$\overline{\Delta H} = 0.02$; $R_0 = 1.25 \times 10^{-3}$; $\beta^* = 30$; $(b/a)^2 = 0.10$; $\beta^* < 3(1 - \overline{\Delta H})/R_0 - 6$].

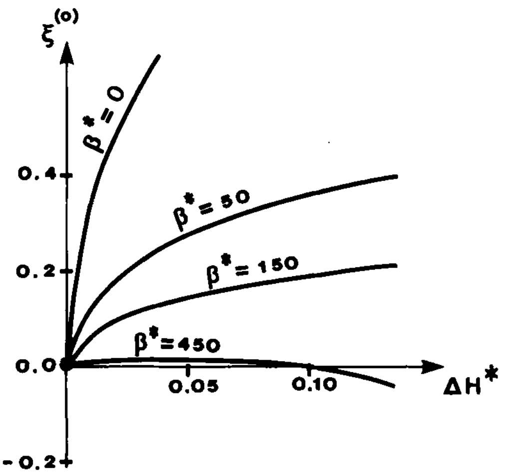


Fig. 6. Free streamline displacement of a zonal eastward current as a function of β^* and the ridge height ($R_0 = 0.006$).

solution on a β plane is shown in Fig. 5. This figure indicates that, as on an f plane, the current intensifies and deflects to the left (northward) as it crosses the ridge. The β effect reduces the deflection toward the left and tends to deflect the current toward the equator as can be seen from Fig. 6 which shows the sensitivity of the free streamline displacement $\xi^{(0)}$ to β^*. This tendency of the β effect is also apparent from the distribution of the velocity x component. As mentioned above, the

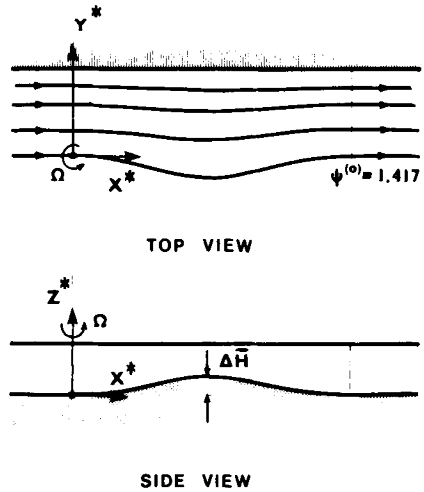


Fig. 7. Equal transport lines for a zonal low-latitude eastward current on a β plane [$\overline{\Delta H} = 0.10$; $R_0 = 4.0$; $\beta^* = 2.5$; $(c/d)^2 = 0.10$; $\beta^* > 3(1 - \overline{\Delta H})/R_0 - 6$].

β effect reduces the velocities above the ridge by $\beta^*(\xi^{(0)})^2/2$; by continuity, this causes relative broadening of the current which results in a reduced deflection to the left.

If β^* is relatively large the β effect may overcome the effect of the uniform rotation and cancel the deflection or even cause a net deflection toward the equator. The condition for a cancellation of the deflection by the β effect is found by substituting $\xi^{(0)} = 0$ into (3.12) to be:

$$\beta^* = 3(1 - \Delta H^*)/R_0 - 6$$

which for $R_0 \ll 1$ can be approximated by the dimensional form:

$$\beta b \approx 3 f_0 (1 - \Delta H^*)$$

We see that for small Rossby number flows a cancellation will not occur in mid-latitude because βb is usually smaller than f_0 and ΔH^* does not approach unity. However, close to the equator the condition $\beta^* > [3(1 - \Delta H^*)/R_0 - 6]$ can be easily satisfied. Under such circumstances, the boundary current will be deflected southward as shown in Fig. 7.

We have seen that mid-latitude small Rossby number flows are dominated by the uniform rotation effect and deflect northward while low latitude flows can be dominated by the β effect and be deflected southward. As in the uniform rotation case ($\beta^* = 0$) the solution to the general problem ($\beta^* \neq 0$) (3.18) breaks down when the term in the square brackets in (3.18) approaches zero or becomes negative. For mid-latitude flows over ridges the term is positive but the term may vanish or be negative for flows over canyons or when β^* is relatively large. Under such conditions, the current turns sharply away from the coast $[\partial\xi^{(0)}/\partial x^* \to -\infty]$ and the perturbations become infinitely large. That is, the current cannot retain its structure of a narrow boundary current as it crosses the topographical feature. It is easy to show that, in contrast to the uniform rotation case, the sharp turning of the current on a β plane is not necessarily associated with a vanishing velocity at the coast and a subsequent separation from the wall.

4. Meridional currents

To analyze meridional boundary currents, we shall follow the procedure given in the previous section. The non-dimensional parameters characterizing the problem are:

$$\left.\begin{array}{lll} x^* = x/c; & y^* = y/d; & \Delta H^* = \Delta H/H_0 \\ v^* = v/v_0; & \xi^* = \xi/c; & \varepsilon = (c/d)^2 \\ \beta^* = \beta c d/v_0; & R_0 = v_0/f_0 c; & \\ \psi^* = \psi/v_0 H_0 c & & \end{array}\right\} \quad (4.1)$$

In terms of these non-dimensional variables the potential vorticity equation (2.12) becomes:

$$\frac{\partial^2 \psi^*}{\partial x^{*2}} + \varepsilon\left[\frac{\partial^2 \psi^*}{\partial y^{*2}} + \frac{\partial \Delta H^*}{\partial y^*}\frac{\partial \psi^*}{\partial y^*}\bigg/(1-\Delta H^*)\right] + (1-\Delta H^*)(\beta^* y^* + \Delta H^*/R_0) = 0 \quad (4.2)$$

The boundary conditions [equivalent to (3.3)] are:

$$\psi^* = 0;$$

$$\psi^* = 1; \qquad x^* = 1 \quad (4.3a)$$

$$\varepsilon(\partial\psi^*/\partial y^*)^2 + (\partial\psi^*/\partial x^*)^2 \qquad x^* = \xi^*(y^*) \quad (4.3b)$$

$$= (1-\Delta H^*)^2; \qquad x^* = \xi^*(y^*) \quad (4.3c)$$

Using the expansions defined by (3.4) [with $\varepsilon = (c/d)^2$] we find the zeroth-order potential vorticity equation:

$$\partial^2\psi^{(0)}/\partial x^{*2} + (1-\Delta H^*)[\beta^* y^* + \Delta H^*/R_0] = 0 \quad (4.4)$$

and the zeroth-order boundary conditions:

$$\psi^{(0)} = 0; \qquad x^* = 1 \quad (4.5a)$$

$$\psi^{(0)} = 1; \qquad x^* = \xi^{(0)}(y^*) \quad (4.5b)$$

$$\frac{\partial\psi^{(0)}}{\partial x^*} = -(1-\Delta H^*); \qquad x^* = \xi^{(0)}(y^*) \quad (4.5c)$$

As in the zonal currents case, the perturbation analysis removed the changes in curvature vorticity $(\partial u^*/\partial y^*)$ from the potential vorticity equation and the non-linearity from the boundary condition at the edge of the current. The solution of (4.4) which satisfies (4.5a) is:

$$\psi^{(0)} = -(1-\Delta H^*)(\beta^* y^* + \Delta H^*/R_0) \times [(x^*)^2 - 1]/2 + A(y^*)(x^* - 1) \quad (4.6)$$

where $A(y^*)$ is to be determined. Substitution of (4.6) into (4.5b) and (4.5c) enables one to find both $\xi^{(0)}$ and $A(y^*)$. Such a substitution gives a quadratic equation for $\xi^{(0)}$ whose physically relevant solution (corresponding to $\xi^{(0)} = 0$ for $\Delta H^* = 0$ and $y^* = 0$) is:

$$\xi^{(0)} = \{1 + \gamma - [1 + 2\gamma/(1-\Delta H^*)]^{1/2}\}/\gamma \quad (4.7)$$

where:

$$\gamma = \beta^* y^* + \Delta H^*/R_0 \quad (4.8)$$

The function $A(y^*)$ and the velocity y component are found to be:

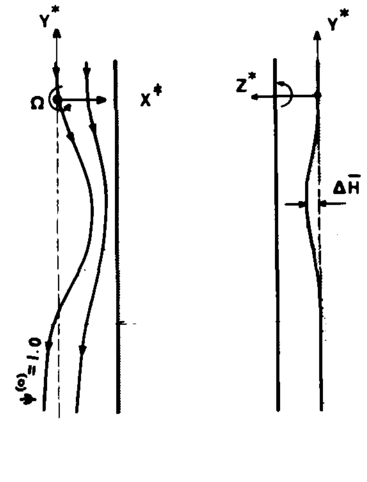
$$A(y^*) = (1-\Delta H^*)(\gamma\xi^{(0)} - 1) \quad (4.9)$$

and

$$v^{(0)} = \gamma(\xi^{(0)} - x^*) - 1 \quad (4.10)$$

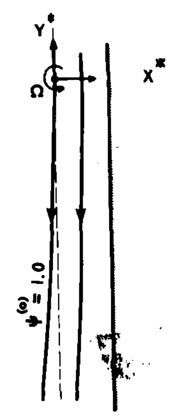
A typical solution is shown in Fig. 8A; Fig. 8B shows the behavior of the same meridional current over a flat bottom. Above the ridge, the current intensifies and migrates toward the coast. As in the zonal currents case, the β effect causes a deflection away from the coast, in the opposite direction to that caused by the uniform rotation. This can be seen from Fig. 9 and eq. (4.8); Fig. 9 indicates that $\xi^{(0)}$ increases as γ increases and (4.8) shows that γ decreases due to the β effect because y^* is negative in the field. Fig. 8A shows that due to the β effect there is a net westward displacement downstream. This displacement is identical to the one that would be caused in the absence of varying bathymetry as shown by Fig. 8B.

As in the zonal currents case, the solution breaks down when the term in the square brackets in (4.7) approaches zero. We shall see shortly that this



TOP VIEW SIDE VIEW

Fig. 8A. Equal transport lines for a meridional southward flowing current crossing a ridge on a β plane $[\Delta\bar{H} = 0.01; R_0 = 2 \times 10^{-3}; \beta^* = 0.25; (c/d)^2 = 0.10]$.



TOP VIEW

Fig. 8B. The same as (8A) but over a flat bottom.

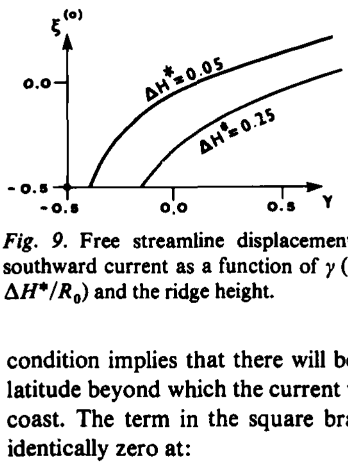


Fig. 9. Free streamline displacement of a meridional southward current as a function of γ (where $\gamma = \beta^* y^* + \Delta H^*/R_0$) and the ridge height.

condition implies that there will be a southernmost latitude beyond which the current will not touch the coast. The term in the square brackets in (4.7) is identically zero at:

$$y^* = -\Delta H^*(1/R_0 - 1/2)/\beta^* - 1/2\beta^*. \tag{4.11}$$

By examining the velocity field (4.10), it is easy to show that if the parameters of the problem are combined in such a way that (4.11) is satisfied then the velocity vanishes at the coast. As the current advances farther south, the velocity becomes negative near the wall according to (4.10). Because reverse velocities correspond to parcels which have originated downstream and not upstream, it follows that at this latitude the current must separate from the wall. It is important to note that the actual separation latitude will differ from (4.11) since our model breaks down at this latitude and higher order approximations are necessary to determine the correct separation latitude.

5. Summary and conclusions

Before listing our conclusions it is perhaps appropriate to stress once more the main limitations of this study. The applicability of our model is limited to those cases where the potential vorticity is uniform and does not vary across the current. In reality, the potential vorticity may not be uniform and this may alter the results. In addition, the model is limited to those cases where the topography induced velocities in the direction perpendicular to the coast are small compared to velocities along the coast. Otherwise, the perturbation scheme breaks down and the changes in the bathymetry cannot be compensated by generation of shear and changes of latitude alone.

We have considered currents with finite velocities at the bounding streamline since it is the most general boundary condition for an inviscid model. Solutions for the special cases of vanishing velocity at the edge of the currents have been obtained and were found to be similar to the solutions given earlier. They do not provide any new physical insights and therefore are not presented.

The results of the theory can be summarized as follows:

(i) A zonal eastward flowing current confined between a solid vertical wall on the left and a free streamline on the right will be deflected northward or southward as it crosses a ridge depending on the relative importance of the β effect.

(ii) An eastward mid-latitude small Rossby number flow is dominated by the uniform rotation and deflects northward but in low latitudes the β effect dominates and the current deflects southward if $\beta^* > [3(1 - \Delta H^*)/R_0 - 6]$. Consequently, there exists a latitude at which a given decrease in depth will not cause any deflection.

(iii) Downstream the eastward current returns to its original upstream position and there is no net deflection.

(iv) As in the zonal currents case, a meridional southward flowing current confined between a solid boundary on the left (east) and a free streamline on the right (west) can be deflected in either direction (left or right) as it crosses a ridge.

(v) The uniform rotation causes a deflection toward the left (east) while the β effect causes a deflection toward the right (west). The current is dominated by the uniform rotation and deflects eastward if $\Delta H^* f_0/\beta d > 1$. When $\Delta H^* f_0/\beta d < 1$, the β effect overwhelms the uniform rotation effect and the current migrates westward.

(vi) Downstream, the meridional southward flowing current does not return to its original position but rather has a net westward deflection due to the β effect.

(viii) The parameters of the problem may combine in such a way that the currents turn sharply away from the coast and cannot retain their original structure of boundary currents. Under certain circumstances the sharp turning is accompanied by a separation of the currents from the wall followed by a formation of a "blocked" region near the wall.

These results demonstrate that the response of finite width boundary currents to topographical forcing crucially depends on the range of the parameters characterizing the problem. We have considered only two cases, an eastward flowing current and a southward current, but our methods and techniques could be applied to currents flowing in any direction.

The model could be tested in the field by various methods such as comparing surface drifters tracks to the shape and location of the predicted streamlines. Although the model is barotropic, it can also be applied to currents flowing along a continent underneath a slightly less dense stagnant fluid provided that the interface displacements are small compared to the variation of the bottom height.

6. Acknowledgements

This study was supported by NSF under Grant ATM 77-28126 and Grant OCE 75-22940.

REFERENCES

Batchelor, G. K. 1967. *An introduction to fluid dynamics.* Cambridge University Press.

Boyer, D. L. 1971a. Rotating flow over long shallow ridges. *Geophys. Fluid Dyn. 2*, 165–184.

Boyer, D. L. 1971b. Rotating flow over a step. *J. Fluid Mech. 50*, 675–687.

Clark, R. A. and Fofonoff, N. P. 1969. Oceanic flow over varying bottom topography. *J. Mar. Res. 27*, 226–240.

Ekman, V. M. 1923. Über horizontalzirkulation beiwinderzeugten Meeresströmunge. *Ark. F. Mat., Astron. och Fysik 17*, 1–74.

Garabedian, P. R. 1964. *Partial differential equations.* John Wiley & Sons, N.Y.

Gutman, L. N. 1972. *Introduction to the nonlinear theory of mesoscale meteorological processes.* NOAA, U.S. Dept. of Commerce.

Huppert, H. E. and Stern, M. E. 1974. The effect of side walls on homogeneous rotating flow over two-dimensional obstacles. *J. Fluid Mech. 62*, 417–436.

Ingersoll, A. R. 1969. Intertial Taylor columns and Jupiter's Great Red Spot. *J. Atmos. Sci. 26*, 744–752.

McCartney, M. S. 1976. The interaction of zonal currents with topography with applications to the Southern Ocean. *Deep-Sea Res. 23*, 413–427.

McIntyre, M. E. 1968. On stationary topography induced Rossby-wave patterns in a barotropic zonal current. *Deut. Hydrogr. Zeit.*, 203–214.

Neumann, G. 1960. On the effect of bottom topography on ocean currents. *Deut. Hydrogr. Zeit. 13*, 132–141.

Niiler, P. P. and Robinson, A. R. 1967. The theory of free inertial jets. II. A numerical experiment for the path of the Gulf Stream. *Tellus 19*, 601–619.

Pearson, C. E. 1974. *Handbook of applied mathematics.* Von Nostrand Reinhold.

Porter, G. H. and Rattray, M. 1964. The influence of variable depth on steady zonal barotropic flow. *Deut. Hydrog. Zeit. 17*, 164–174.

Robinson, A. R. and Niiler, P. P. 1967. Theory of free inertial currents. I. Path and structure. *Tellus 19*, 269–291.

Robinson, A. R. and Taft, B. A. 1972. A numerical experiment for the path of the Kuroshio. *J. Mar. Res. 30*, 65–101.

Røed, L. P. 1979. Curvature effects on hydraulically driven inertial boundary currents. *J. Fluid Mech.* In press.

Stern, Melvin E. 1974. *Ocean circulation physics.* Academic Press.

Sverdrup, H. U. 1941. The influence of bottom topography on ocean currents. *Applied Mechanics. Th. von Karman Anniversary Volume*, 66–75.

Warren, B. A. 1963. Topographic influence on the path of the Gulf Stream. *Tellus 15*, 167–183.

ВЛИЯНИЕ МЕНЯЮЩЕЙСЯ БАТИМЕТРИИ НА ИНЕРЦИОННЫЕ ПОГРАНИЧНЫЕ ТЕЧЕНИЯ

С помощью упрощенной нелинейной невязкой однослойной модели на бэта-плоскости исследуется реакция стационарных инерционных пограничных течений на изменения топографии дна. Течения заключены между твердой вертикальной границей с одной стороны и свободной линией тока с другой, за которой океан неподвижен. При разложении в ряд по отношению ширины течения к масштабу длины топографии получены приближенные решения как для зональных, так и меридиональных течений.

Найдено, что при натекании на хребет в долготном направлении течение, идущее на восток и ограниченное твердой вертикальной стенкой слева (с севера) и свободной линией тока справа (с юга), может отклоняться в любом направлении (к берегу или от него) в зависимости от числа Россби, относительного изменения глубины и изменения параметра Кориолиса с широтой. В средних широтах течение с малым числом Россби сужается и отклоняется к северу по мере того, как оно пересекает хребет в северном полушарии. Когда течение находится вблизи экватора, наоборот, оно расширяется и отклоняется к югу. Следовательно, существует широта, гда уменьшение глубины не приводит к какому-либо отклонению.

Подобным же образом, меридиональное направленное на юг течение, ограниченное твердой стенкой слева (с востока) и свободной линией справа (с запада) усиливается и отклоняется к востоку или расширяется и отклоняется к западу в зависимости от интервала изменений параметров задачи.