Tellus (1980), *32*, 143–163

Nonlinear tidal waves in channels: A perturbation method adapted to the importance of quadratic bottom friction

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(Manuscript received December 29, 1978; in final form July 5, 1979)

ABSTRACT

Bottom friction plays an important role in propagation and damping of long waves in shallow water. A perturbation method, well adapted to the importance of this bottom stress, is presented and applied for channels with constant mean depth and mean cross-section area. The main idea results from a development of quadratic bottom friction in a Fourier series up to the third order of approximation. A quasi-linearization of the damping effect of bottom stress is deduced from this expansion, which allows one to introduce the second-order damping effects of friction into the first-order system defining the fundamental component of the spectrum, and the main part of the third-order damping terms into the computation of the second-order harmonic components. From this expansion, the generating role of the harmonic played by friction is also identified and taken into account in the second-order approximation. Analytical and analytico-numerical solutions are presented and compared to numerical solutions of the full nonlinear equations obtained by the Lax-Wendroff finite-difference integration technique. These comparisons show that the analytical solutions limited to the second-order approximation fit very well with the L.W. solutions.

1. Introduction

The existence of nonlinear distortions of long waves in shallow water is a well-known phenomenon. In the ocean, tides can be represented with a limited number of frequencies, corresponding to those contained in the tidal potential (in islands such as Oahu about 20 components are sufficient). However, on continental shelves tidal spectra appear to be more complicated. Along the European coasts, for instance, the semi-diurnal astronomical constituents of the tides are accompanied by quarter-diurnal, sixth-diurnal and eighth-diurnal components of significant amplitude (more than 60 components are required in Le Havre on the English Channel; at least 100 in London). The production of these new frequencies is due to wave distortions associated with nonlinearities of tidal motions in shallow water.

Several models have been developed in the past for the understanding of the basic mechanisms of this phenomenon. The models investigate separately the two important causes of nonlinear behaviour: wave steepening due to small depth; damping and asymmetry due to friction. All these models are based upon perturbation methods. Lamb (1932), neglecting friction, presented an analytical solution for a long wave propagating in a channel of constant depth. This solution is the familiar Airy solution, showing the steepening of the wave front and leading to bore formation. The corresponding second-order perturbation approximation consists of a constant term and the first even harmonic, which increases in amplitude as it propagates together with the incident wave. However, tidal bore formation seems unrealistic since it has never been observed in situ. One possible explanation is certainly connected with the existence of friction in the real phenomenon.

In order to study this possibility, Kreiss (1957) took into account the damping effect of friction. He

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0040-2826/80/020143-21$02.50/0

investigated the propagation of an incident monoperiodic wave in a channel closed at one end by a vertical wall. He introduced in the classical nonlinear equations used by Lamb a linear friction term proportional to the velocity, and gave analytical solutions for the first- and second-order perturbation approximations. The first-order solution gives the fundamental oscillation, strongly influenced by the damping effect of friction. The second-order solution is compounded of a constant term and a first even harmonic, just like Lamb's solution, but here strongly influenced by friction. With this model, the damping effect of friction is well represented, but odd frequencies are still missing in the second-order perturbation approximation. More recently, Gallagher and Munk (1971) looked at the same problem, but took a quadratic friction term, which is more accurate, to represent bottom stress. In this way odd harmonics appear in the second-order solution. However, because of the quadratic formulation of a friction law, the perturbation method introduced friction effects only at the second order of approximation, and thus the analytical solution limited to the second order is rather far from the corresponding numerical solution, used as a check.

In this paper, we shall present and apply to this classical problem a perturbation method well adapted to the important role played by friction in the real phenomenon. The solutions will be developed to the second order of approximation and the results will be compared with a complete computer solution in order to check these analytical solutions.

2. Basic equations and boundary conditions. Definition of the problem

2.1. *The shallow-water equations*

In theoretical study of tidal motions in channels, one usually assumes that the mean depth and the mean cross-section area are constant. Thus, neglecting the Coriolis force, the hydrodynamic equations, written in the long-wave approximation, reduce to a set of two equations in two functions: $\tilde{u}$, the velocity, and $\tilde{\zeta}$, the sea surface elevation, of two variables: time $\tilde{t}$, and abscissa $\tilde{x}$ along the channel:

$$\begin{cases} \tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + g\tilde{\zeta}_{\tilde{x}} = \dfrac{-C}{\tilde{h} + \tilde{\zeta}}|\tilde{u}|\tilde{u} \\ \tilde{\zeta}_{\tilde{t}} + [(\tilde{h} + \tilde{\zeta})\tilde{u}]_{\tilde{x}} = 0 \end{cases} \tag{2.1}$$

Here f_t, f_x, f_{xt}, etc., are defined by:

$$f_t = \frac{\partial f}{\partial t}; \quad f_x = \frac{\partial f}{\partial x}; \quad f_{xt} = \frac{\partial^2 f}{\partial x \partial t}$$

We suppose that the channel is connected with the open sea at one end and is closed at the other end by a vertical wall; we use consequently the boundary conditions derived from the theory of characteristics:

$$\tilde{u} = 0 \quad \text{at} \quad \tilde{x} = \tilde{L} \tag{2.2}$$

$$\tilde{u} + 2\sqrt{g(\tilde{h} + \tilde{\zeta})} = 2 + 2\tilde{A}\cos\tilde{\omega}\,t \quad \text{at} \quad x = 0 \tag{2.3}$$

Condition (2.3) corresponds to the data at $\tilde{x} = 0$ of an incident wave of frequency $\tilde{\omega}$ coming in from deep water where nonlinearities are assumed negligible ($\tilde{A}$ characterizes the order of magnitude of this incident wave). We shall verify in the following that such a formulation of the open boundary condition corresponds to a kind of radiation condition for waves coming out of the channel through this limit.

It is convenient to write (2.1), (2.2) and (2.3) in a nondimensional form with the dimensionless variables

$$x = \frac{\tilde{x}}{\tilde{L}}, \quad t = \frac{\tilde{t}}{\tilde{L}/\tilde{c}}, \quad \zeta = \frac{\tilde{\zeta}}{\tilde{h}}, \quad u = \frac{\tilde{u}}{\tilde{c}}, \quad \tilde{c} = \sqrt{g\tilde{h}}$$
$$\omega = \frac{\tilde{\omega}}{\tilde{c}/\tilde{L}}, \quad A = \frac{\tilde{A}}{\tilde{c}}, \quad K = C\,\frac{\tilde{L}}{\tilde{h}} \tag{2.4}$$

In these variables the equations of motion and the boundary conditions become:

$$\begin{gathered} \zeta_t + u_x + (\zeta u)_x = 0 \\ u_t + uu_x + \zeta_x = -k\frac{|u|u}{1 + \zeta} \\ u = 0 \quad \text{and} \quad x = 1 \\ u + 2\sqrt{1 + \zeta} = 2A\cos\omega t + 2 \quad \text{at} \quad x = 0 \end{gathered} \tag{2.5}$$

2.2. *The perturbation method*

Theoretically, it is not possible to represent the solution of (2.5) by a finite number of elementary functions. But, if the parameter A is small compared with unity, the solution of (2.5) can be approximated by means of a limited number of solutions of linearized equations deduced from (2.5) with the aid of a perturbation method.

Solutions u and ζ are supposed to be developed as follows:

$$\begin{cases} \zeta(x,t) = \sum A^i \zeta_i(x,t) \\ u(x,t) = \sum A^i u_i(x,t) \end{cases} \quad \text{with } i = 1, 2, \ldots, n \qquad (2.6)$$

The classical perturbation method consists of introducing these expansions in (2.5). However, given the nonanalytical formulation of the quadratic friction term

$$F = K\frac{|u|u}{1+\zeta}$$

it is necessary to expand it beforehand with an approximate polynomial development in powers of A:

$$F = \sum A^i F_i, \quad i = 1, 2, \ldots, n \qquad (2.7)$$

We shall present this expansion later.

In the same way, for condition (2.5.4) one needs:

$$\sqrt{1+\zeta} = 1 + A\frac{\zeta_1}{2} + A^2\left(\frac{\zeta_2}{2} - \frac{\zeta_1^2}{8}\right) + A^3\left(\frac{\zeta_3}{2} - \frac{\zeta_1\zeta_2}{4} + \frac{\zeta_1^3}{16}\right) + \ldots \qquad (2.8)$$

Thus, substituting (2.6) into (2.5), taking account of (2.7) and (2.8), and arranging according to powers of A, we get

$$\begin{cases} \begin{cases} \zeta_{it} + u_{ix} = -\left(\sum\limits_{k+l=i} \zeta_k u_l\right)_x \\ u_{it} + \zeta_{ix} = -\sum\limits_{k+l=i} u_k u_{lx} + F_i \end{cases} \\ \begin{cases} u_i = 0 \quad \text{at} \quad x = 1 \\ u_i + \zeta_i = L_i \quad \text{at} \quad x = 0 \end{cases} \end{cases} \qquad (2.9)$$

with

$$\begin{cases} L_1 = 2\cos\omega t \\ L_2 = \dfrac{\zeta_1^2(0,t)}{4} \\ L_3 = \dfrac{\zeta_1(0,t)\,\zeta_2(0,t)}{2} - \dfrac{\zeta_1(0,t)^3}{8} \\ \vdots \end{cases}$$

The problem is then to solve (2.9) for each order of magnitude i.

Besides this perturbation method, it is essential to introduce another hypothesis. It has been shown by Stokes (Ursell, 1953) that tidal motions are non-dispersive waves. Thus, knowing that the movement in the channel is produced by the propagation of an incident wave of frequency ω, the solutions ζ_i and u_i can be expressed in the form

$$\begin{cases} \zeta_i = \sum\limits_{k=0}^{\infty} \bar{\alpha}_{ki} \\ u_i = \sum\limits_{k=0}^{\infty} \bar{\mu}_{ki} \end{cases} \quad \text{with} \quad \begin{cases} \bar{\alpha}_{ki} = \zeta_{ki}\cos(k\omega t + \phi_{ki}) \\ \bar{\mu}_{ki} = u_{ki}\cos(k\omega t + \psi_{ki}) \end{cases} \qquad (2.10)$$

Using complex notations, one can write (2.10) as follows:

$$\begin{cases} \bar{\alpha}_{ki} = \alpha_{ki}\, e^{jk\omega t} + \alpha_{ki}^*\, e^{-jk\omega t} \\ \bar{\mu}_{ki} = \mu_{ki}\, e^{jk\omega t} + \mu_{ki}^*\, e^{-jk\omega t} \end{cases} \quad \text{with} \quad \begin{cases} \alpha_{ki} = \tfrac{1}{2}\zeta_{ki}\, e^{j\phi_{ki}} \\ \mu_{ki} = \tfrac{1}{2}u_{ki}\, e^{j\psi_{ki}} \end{cases} \qquad (2.11)$$

f^* being the conjugate of the complex function f. Thus, the systems (2.9) can be split up into systems of differential equations of the single variable x as soon as the F_i friction terms are also expended in a form similar to (2.10).

2.3. *Numerical treatment*

This whole development is of use only if the sum of the first orders of approximation already give a sufficiently good approximation of the solution. Our aim is to restrict ourselves to the first and second orders: $A\zeta_1 + A^2\zeta_2$ and $Au_1 + A^2u_2$. To examine the accuracy of this approximation, we shall compare the second-order analytical solutions, presented below, with the complete nonlinear solutions of the same problem obtained by numerical integration of problem (2.5) by means of a second-order Lax-Wendroff scheme and by development in a Fourier series of the frequency ω.

3. Development of the quadratic friction term

A complete expansion of the two components of the quadratic friction force in the general two-dimensional problem of tidal-wave propagation has been presented by Le Provost (1973). This development assumes that the velocity field is expressed as

$$\begin{cases} u = \sum_{l=1}^{N_c} A_l\, u_l(x,y) \cos [\omega_l t + \psi_l(x,y)] \\ v = \sum_{l=1}^{N_c} A_l\, u_l(x,y) \cos [\omega_l t + \chi_l(x,y)] \end{cases} \tag{3.1}$$

N_c being the number of constituents of frequency ω_l and of order of magnitude A_l present in the tidal spectrum. It is valid only in areas where a dominant wave can be found, that is of amplitude much larger than the others; if one takes index 1 for this dominant constituent, the characteristic condition for this dominant wave is

$$\begin{cases} u_1^2 + v_1^2 \neq 0 \\ A_1(u_1^2 + v_1^2) \gg \sum_{i=2}^{N_c} A_i^2(u_i^2 + v_i^2) \end{cases} \tag{3.2}$$

Thus, f_x and f_y, the components of vector $|\vec{V}|\vec{V}$, are expanded as follows:

$$\begin{cases} f_x = \sum_{i,j} A_i A_j FX_{ij}^{(n)} \cos [\Omega_{ij}^{(n)} t + \Phi X_{ij}^{(n)}], \\ \quad n = 0, 1, 2, \ldots \\ f_y = \sum_{i,j} A_i A_j FY_{ij}^{(n)} \cos [\Omega_{ij}^{(n)} t + \Phi Y_{ij}^{(n)}], \\ \quad i,j = 1, 2, \ldots N_c \end{cases} \tag{3.3}$$

with $\Omega_{ij}^{(n)} = p_k(i,j,n) \cdot \omega_k$, p_k an integer, $k = 1, 2, \ldots, N_c$. $FX_{ij}^{(n)}$, $FY_{ij}^{(n)}$, $\Phi X_{ij}^{(n)}$ and $\Phi Y_{ij}^{(n)}$ are functions of $u_i(x,y)$, $\Psi_i(x,y)$, $v_i(x,y)$ and $\chi_i(x,y)$; they have been determined analytically for the orders of approximation A_1^2, $A_1 A_j$, and A_j^2, $j = 2, 3, \ldots, N_c$. The identification of frequency $\Omega_{ij}^{(n)}$ corresponding to these different orders of approximation allows one in particular to ascertain that the following appear in (3.3):

$$\left.\begin{array}{l} \text{(i) at the order } A_1^2\text{:} \\ \quad \text{pulsations } (2n+1)\omega_1 \\ \text{(ii) at the order } A_1 A_j\text{:} \\ \quad \text{pulsation } 2n\omega_1 + \varepsilon\omega_j \end{array}\right\} \begin{array}{l} \varepsilon = \pm 1 \\ n = 0, 1, 2, \ldots + \infty \end{array}$$

As an example, the term of order A_1^2 corresponding to the frequency ω_1 is:

$$\begin{cases} FX_{11}^{(0)} \cos (\omega_1 t + \Phi X_{11}^{(0)}) \\ \quad = \dfrac{u_1}{2} \sqrt{u_1^2 + v_1^2} \left[G_{00} \cos (\omega_1 t + \psi_1) \right. \\ \quad \left. + \dfrac{G_{02}}{2} \cos (\omega_1 t + 2\theta - \psi_1) \right] \\ FY_{11}^{(0)} \cos (\omega_1 t + \Phi Y_{11}^{(0)}) \\ \quad = \dfrac{v_1}{2} \sqrt{u_1^2 + v_1^2} \left[G_{00} \cos (\omega_1 t + \chi_1) \right. \\ \quad \left. + \dfrac{G_{02}}{2} \cos (\omega_1 t + 2\theta - \chi_1) \right] \end{cases} \tag{3.4}$$

with

$$G_{00} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} [1 + J_1 \cos (2\omega_1 t + 2\theta)]^{1/2} \times d(\omega_1 t + \theta)$$

$$G_{0n} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} [1 + J_1 \cos (2\omega_1 t + 2\theta)]^{1/2} \times \cos [n(\omega_1 t + \theta)]\, d(\omega_1 t + \theta)$$

$$J_1^2 = 1 - \frac{4u_1^2 v_1^2}{(u_1^2 + v_1^2)^2} \sin^2 (\varphi_1 - \chi_1)$$

$$\operatorname{tg} 2\theta = \frac{u_1^2 \sin 2\psi_1 + v_1^2 \sin 2\chi_1}{u_1^2 \cos 2\psi_1 + v_1^2 \cos 2\chi_1}$$

Let us notice that these expressions (3.4) can be written in a slightly different form that corresponds in fact to a quasi linearized expression of friction for the dominant component:

$$\begin{cases} FX_{11}^{(0)} \cos [\omega_1 t + \Phi X_{11}^{(0)}] = R_1 u_1 \cos (\omega_1 t + \psi_1) \\ \quad + R_1' v_1 \sin (\omega_1 t + \chi_1) \\ FY_{11}^{(0)} \cos [\omega_1 t + \Phi Y_{11}^{(0)}] = R_1 v_1 \cos (\omega_1 t + \chi_1) \\ \quad - R_1' u_1 \sin (\omega_1 t + \psi_1) \end{cases} \tag{3.5}$$

with

$$R_1 = \frac{1}{\sqrt{2}} \sqrt{u_1^2 + v_1^2} \left(G_{00} + \frac{G_{02}}{2J_1} \right)$$

$$R_1' = \frac{1}{\sqrt{2}} \varepsilon \sqrt{u_1^2 + v_1^2} \frac{G_{02}}{2J_1} (1 - J_1^2)^{1/2}$$

$$\varepsilon = +1 \quad \text{if } 0 < \psi_1 - \chi_1 < \pi$$

$$\varepsilon = -1 \quad \text{if } \pi < \psi_1 - \chi_1 < 2\pi$$

The same formula had been obtained by Dronkers (1962) in his linearization of the quadratic resistance term for a pure harmonic tide.

As can be seen, the explicit formula giving the functions FX, FY, ΦX, ΦY are rather complicated We do not present them here, but they have been used, in the one-dimensional case, to deduce the expansion (2.7) of the friction term, up to the approximation A^3 (the third order of this development will be necessary in the following).

As we shall see later, by limiting the solution (2.6) to the second order of approximation, the velocity can be written:

$$u = Au_{11}\cos(\omega t + \psi_{11}) + A^2[u_{02} + u_{12} \times \cos(\omega t + \psi_{12}) + u_{22}\cos(2\omega t + \psi_{22}) + u_{32}\cos(3\omega t + \psi_{32}) + u_{52}\cos(5\omega t + \psi_{52}) + \ldots] \tag{3.6}$$

Given the way in which problem (2.6) has been formulated, the hypothesis of "dominant wave" is fully respected for realistic values of the parameter A. Thus the preceding expansion (3.3) can be used; taking

$$\begin{array}{llll}
A_1 = A, & u_1 = u_{11}, & \omega_1 = \omega, & \psi_1 = \psi_{11} \\
A_2 = A^2, & u_2 = u_{02}, & \omega_2 = 0, & \psi_2 = 0 \\
A_3 = A^2, & u_3 = u_{12}, & \omega_3 = \omega, & \psi_3 = \psi_{12} \\
A_4 = A^2, & u_4 = u_{22}, & \omega_4 = 2\omega, & \psi_4 = \psi_{22} \\
A_5 = A^2, & u_5 = u_{32}, & \omega_5 = 3\omega, & \psi_5 = \psi_{32} \\
A_6 = A^2, & u_6 = u_{52}, & \omega_6 = 5\omega, & \psi_6 = \psi_{52}
\end{array}$$

an approximate development of F is deduced:

$$F = \frac{K}{1+\zeta} f = \frac{K}{1+\zeta} \sum_{i,j} A_i A_j f_{ij}^{(n)} \times \cos[\Omega_{ij}^{(n)} t + \phi_{ij}^{(n)}],\ i,j = 1, 2, \ldots, 6 \tag{3.7}$$

Let us first look at the different frequencies $\Omega_{ij}^{(n)}$ that appear:

(i) at the order $A_1^2 = A^2$: $(2n+1)\omega$, i.e.: ω, 3ω, $5\omega, \ldots$

(ii) at the order $A_1 A_j = A^3$, with $j = 2, \ldots, 6$: $(2n\omega + \varepsilon\omega_j)$, cf. Table 1.

In Table 1 we limit ourselves to the frequency 5ω, because we know from numerical computation (cf. Saenz, 1978) that no further significant energy exists in the spectrum for numerical values of realistic tidal waves.

The expressions for the functions $f_{ij}^{(n)}$ and $\phi_{ij}^{(n)}$ are

Table 1. *Frequencies $\Omega_{ij}^{(n)}$ appearing in development (3.7) at the order $A_iA_j = A^3$ and values of the corresponding coefficients $f_{1j}^{(n)}$*

n	0	1		2		3		4		5	
j \ ε	+1	−1	+1	−1	+1	−1	+1	−1	+1	−1	+1
2	0	2ω	2ω	4ω	4ω	$\cancel{6\omega}$	$\cancel{6\omega}$	$\cancel{8\omega}$	$\cancel{8\omega}$	$\cancel{10\omega}$	$\cancel{10\omega}$
3	ω	ω	3ω	3ω	5ω	5ω	$\cancel{7\omega}$	$\cancel{7\omega}$	$\cancel{9\omega}$	$\cancel{9\omega}$	$\cancel{11\omega}$
4	2ω	0	4ω	2ω	$\cancel{6\omega}$	4ω	$\cancel{8\omega}$	$\cancel{6\omega}$	$\cancel{10\omega}$	$\cancel{8\omega}$	$\cancel{12\omega}$
5	3ω	ω	5ω	ω	$\cancel{7\omega}$	3ω	$\cancel{9\omega}$	5ω	$\cancel{11\omega}$	$\cancel{7\omega}$	$\cancel{13\omega}$
6	5ω	3ω	$\cancel{7\omega}$	ω	$\cancel{9\omega}$	ω	$\cancel{11\omega}$	3ω	$\cancel{13\omega}$	$\cancel{5\omega}$	$\cancel{15\omega}$
$f_{1j}^{(n)}$	$4u_1u_j/\pi$	$4u_1u_j/3\pi$		$-4u_1u_j/15\pi$		$4u_1u_j/35\pi$		$-\dfrac{12u_1u_j}{189\pi}$			

considerably simplified in the monoperiodic case because, v_l being zero, $J_1 = 1$, tg $2\theta =$ tg $2\psi_1$ and

$$[1 + J_1 \cos(2\omega t + 2\theta)]^{1/2} = \sqrt{2}\,|\cos(\omega t + \psi_1)|$$

One obtains

(i) at the order A^2

$$\begin{cases} f_{11}^{(n)} = \dfrac{u_{11}^2}{2\sqrt{2}}\,(G_{0,2n} + G_{0,2n+2}) \\ \text{with} \quad G_{0,2n} = \dfrac{\sqrt{2}}{\pi}\displaystyle\int_{-\pi}^{+\pi} |\cos\alpha| \cos 2n\alpha\, d\alpha \\ \phi_{11}^{(n)} = (2n+1)\psi_{11} \end{cases}$$

(ii) at the order A^3

$$\begin{cases} f_{IJ}^{(0)} = \dfrac{u_I u_J}{2\sqrt{2}}\,(G_{10} + G_{12} + G_{00}) \\ \text{with} \quad G_{1,2n} = \dfrac{1}{\pi\sqrt{2}}\displaystyle\int_{-\pi}^{+\pi} \dfrac{\cos 2n\alpha}{|\cos\alpha|}\, d\alpha \\ \phi_{IJ}^{(0)} = \psi_J \end{cases}$$

$$\begin{cases} f_{IJ}^{(n)} = \dfrac{u_I u_J}{4\sqrt{2}}\,[2G_{0,2n} + 2G_{1,2n} + G_{1,2n-2} \\ \qquad + G_{1,2n+2}] \\ \phi_{IJ}^{(n)} = 2n\psi_I + \varepsilon\psi_J \end{cases}$$

Values of $f_{IJ}^{(n)}$ and $f_{11}^{(n)}$ are presented in Tables 1 and 2.

Table 2. *Values of the coefficients* $f_{11}^{(n)}$

n	0	1	2
	ω	3ω	5ω
$f_{11}^{(n)}$	$\dfrac{8u_1^2}{3\pi}$	$\dfrac{8u_1^2}{15\pi}$	$-\dfrac{8u_1^2}{105\pi}$

In summary, f is expanded in the form:

$$\begin{cases} = A^2\left[\dfrac{8u_{11}^2}{3\pi}\cos(\omega t + \psi_{11}) + \dfrac{8}{15\pi}u_{11}^2\cos(3\omega t + 3\psi_{11}) - \dfrac{8}{105\pi}u_{11}^2\cos(5\omega t + 5\psi_{11}) + \dots\right] \\ + A^3\left[\dfrac{4}{\pi}u_{11}u_{02} + \dfrac{4}{\pi}u_{11}u_{22}\right] \\ + A^3\left\{\dfrac{4}{\pi}u_{11}u_{12}\cos(\omega t + \psi_{12}) \right. \\ \quad + \dfrac{4}{3\pi}u_{11}[u_{12}\cos(\omega t + 2\psi_{11} - \psi_{12}) \\ \quad + u_{32}\cos(\omega t + 2\psi_{11} - \psi_{32})] \\ \quad - \dfrac{4}{15\pi}u_{11}[u_{32}\cos(\omega t + 4\psi_{11} - \psi_{32}) \\ \quad + u_{52}\cos(\omega t + 6\psi_{11} - \psi_{52})] \\ \quad \left. + \dfrac{4}{35\pi}u_{11}u_{52}\cos(\omega t + 6\psi_{11} - \psi_{52})\right\} + \\ + A^3\left\{\dfrac{4}{\pi}u_{11}u_{22}\cos(2\omega t + \psi_{22}) \right. \\ \quad + \dfrac{4}{3\pi}u_{11}2u_{02}\cos(2\omega t + 2\psi_{11}) \\ \quad \left. - \dfrac{4}{15\pi}u_{11}u_{22}\cos(2\omega t + 4\psi_{11} - \psi_{22})\right\} \\ + A^3\left\{\dfrac{4}{\pi}u_{11}u_{32}\cos(3\omega t + \psi_{32}) \right. \\ \quad + \dfrac{4}{3\pi}u_{11}[u_{12}\cos(3\omega t + 2\psi_{11} + \psi_{12}) \\ \quad + u_{52}\cos(3\omega t + 2\psi_{11} - \psi_{52})] \\ \quad - \dfrac{4}{15\pi}u_{11}u_{12}\cos(3\omega t + 4\psi_{11} - \psi_{12}) \\ \quad + \dfrac{4}{35\pi}u_{11}u_{32}\cos(3\omega t + 6\psi_{11} - \psi_{32}) \\ \quad \left. - \dfrac{12}{189\pi}u_{11}u_{52}\cos(3\omega t + 8\psi_{11} - \psi_{52})\right\} + \\ + \dots \end{cases} \tag{3.8}$$

Several points must be noticed concerning (3.8):

(i) The terms of second order can be divided in two classes. The first class corresponds to the damping of the dominant component by friction:

$$\frac{8}{3\pi} Au_{11} \times Au_{11} \cos(\omega t + \psi_{11}) \tag{3.9}$$

The damping is proportional to the coefficient $(8/3\pi)\, Au_{11}$, so that this expression gives a kind of linearization of quadratic bottom stress. The same result has been given by Lorentz (1926) and Proudman (1923).

The second class of terms corresponds to the role played by friction in the generation of harmonics:

$$\frac{8}{15\pi} A^2 u_{11}^2 \cos(3\omega t + 3\psi_{11}) - \frac{8}{105\pi} A^2 u_{11}^2 \times \cos(5\omega t + 5\psi_{11}) + \ldots \tag{3.10}$$

To the second order of approximation, only odd harmonics are generated.

(ii) For the third order it is possible to derive the same conclusions. From (3.8) some terms can be selected that correspond to a linearization of the damping of the second-order harmonic components:

$$\frac{4}{\pi} Au_{11} \times [A^2 u_{02} + A^2 u_{12} \cos(\omega t + \psi_{12}) + A^2 u_{22} \cos(2\omega t + \psi_{22}) + A^2 u_{32} \cos(3\omega t + \psi_{32}) + A^2 u_{52} \cos(5\omega t + \psi_{52}) + \ldots] \tag{3.11}$$

This damping is proportional to the coefficient $(4/\pi)\, Au_{11}$, i.e. depends on the dominant wave. These conclusions are in agreement with the results given by Bowden (1953) for a residual current and by Jeffreys (1970) for a secondary oscillation propagating together with a dominant wave. The role of the other terms of third order in (3.8) is not so easy to understand as for the second order. But it must be noticed that their amplitudes are of less importance.

(iii) These conclusions give an idea of how to combine the results of Kreiss with those of Gallagher and Munk. Knowing the essential importance of damping effects, the corresponding terms of (3.8) must be taken into account in the systems describing the propagation of each wave of the spectrum, though they are of a higher order of approximation: the damping term of second order (3.9) must be introduced in the first-order system corresponding to the dominant wave, the damping terms of third order (3.11) are to be taken in the different systems (2.9) defining the components of second-order approximation. Thus, the results of Kreiss (1957) for the dominant wave must be found with a better accuracy, because of the more precise definition of the "quasi-linear" friction coefficient $8Au_{11}/3\pi$. The results for the constant solution and the first even harmonic will also be much more accurate because their damping will be taken into account. On the other hand, odd harmonics will be present in the second-order solution as in the solution of Gallagher and Munk (1971). They will be obtained with good accuracy resulting from the accurate definition of the dominant solution of the first-order approximation and the presence of damping in the second-order corresponding systems.

(iv) Thus, the expansion of the friction term will be taken as:

$$F = \frac{1}{1+\zeta} [\lambda A u_1 + \lambda' A^2 u_2 + \lambda_{32} A^2 u_{11}^2 \cos 3(\omega t + \psi_{11}) + \lambda_{52} A^2 u_{11}^2 \cos 5(\omega t + \psi_{11})] + O(A^3)$$

$$\lambda = \frac{8K}{3\pi} Au_{11}, \quad \lambda' = \frac{4K}{\pi} Au_{11}, \quad \lambda_{32} = \frac{8K}{15\pi}, \quad \lambda_{52} = -\frac{8K}{105\pi}$$

i.e. using the development

$$\frac{1}{1+\zeta} = 1 - A\zeta_1 + O(A^2)$$

one finds

$$F = \lambda A u_1 - \lambda A^2 u_1 \zeta_1 + \lambda' A^2 u_2 + \lambda_{32} A^2 u_{11}^2 \cos 3(\omega t + \psi_{11}) + \lambda_{52} A^2 u_{11}^2 \cos 5(\omega t + \psi_{11}) + \ldots + O(A^3)$$

4. Determination of the first-order solution

4.1. *Analytical solution*

According to the preceding section, the system giving the first-order solution is deduced from (2.9):

$$\begin{cases} \zeta_{1t} + u_{1x} = 0 \\ u_{1t} + \zeta_{1x} = -\dfrac{8K}{3\pi}(Au_{11})u_1 \\ u_1 = 0 \quad \text{and} \quad x = 1 \\ u_1 + \zeta_1 = 2\cos\omega t \quad \text{at} \quad x = 0 \end{cases} \tag{4.1}$$

Motion in the channel is due only to the boundary condition at $x = 0$. Thus, the solution of first-order approximation reduces to the monoperiodic oscillation:

$$\begin{cases} \zeta_1 = \zeta_{11}\cos(\omega t + \psi_{11}) \\ u_1 = u_{11}\cos(\omega t + \psi_{11}) \end{cases}$$

Consequently, the first-order solution is given by the differential system:

$$\begin{cases} j\omega\,\alpha_{11} + \dfrac{d\mu_{11}}{dx} = 0 \\ j\omega\mu_{11} + \dfrac{d\alpha_{11}}{dx} = -\dfrac{8K}{3\pi}(Au_{11})\mu_{11} \\ \mu_{11}(1) = 0 \\ \alpha_{11}(0) + \mu_{11}(0) = 1 \end{cases} \tag{4.2}$$

(4.2) is nonlinear, because the coefficient $(8K/3\pi) \times Au_{11}$ depends upon the solution u_{11}. We shall admit in this section that this coefficient is constant for all x by choosing a mean value of Au_{11}. In a following section we shall present an iterative numerical procedure in order to solve (4.2) exactly. Thus, with

$$\frac{8K}{3\pi}(Au_{11}) = \lambda = \text{const.}$$

(4.2) becomes a linear differential system that can be written:

$$\begin{cases} (1) \quad \alpha_{11} = \dfrac{j}{\omega}\dfrac{d\mu_{11}}{dx} \\ (2) \quad \dfrac{d^2\mu_{11}}{dx^2} + \omega^2\left(1 - j\dfrac{\lambda}{\omega}\right)\mu_{11} = 0 \end{cases} \tag{4.3}$$

(4.3.2) has the following general solution:

$$\mu_{11} = M\exp\left[j\omega\left(1 - j\frac{\lambda}{\omega}\right)^{1/2}(x-1)\right] + N\exp\left[-j\omega\left(1 - j\frac{\lambda}{\omega}\right)^{1/2}(x-1)\right]$$

that obeys the boundary condition $\mu_{11}(1) = 0$, so that

$$\begin{cases} \mu_{11} = B\sin\left[\omega\left(1 - j\dfrac{\lambda}{\omega}\right)^{1/2}(x-1)\right] \\ \alpha_{11} = jB\left(1 - j\dfrac{\lambda}{\omega}\right)^{1/2} \\ \quad \times \cos\left[\omega\left(1 - j\dfrac{\lambda}{\omega}\right)^{1/2}(x-1)\right] \end{cases} \tag{4.4}$$

The constant B is fixed by the open boundary condition

$$\alpha_{11}(0) + \mu_{11}(0) = 1.$$

Let us separate the real and imaginary parts of α_{11} and μ_{11} and introduce the notations

$$\begin{cases} \left(1 - j\dfrac{\lambda}{\omega}\right)^{1/2} = a - jb \\ a = \dfrac{1}{\sqrt{2}}\left[1 + \left(1 + \dfrac{\lambda^2}{\omega^2}\right)^{1/2}\right]^{1/2} \\ b = \dfrac{1}{\sqrt{2}}\dfrac{\lambda}{\omega}\left[1 + \left(1 + \dfrac{\lambda^2}{\omega^2}\right)^{1/2}\right]^{-1/2} \\ B = B_1 + jB_2 \end{cases} \tag{4.5}$$

We find

$$\begin{cases} \mathrm{Re}(\mu_{11}) = B_1\sin\omega a(x-1)\,\mathrm{ch}\,\omega b(x-1) \\ \quad + B_2\cos\omega a(x-1)\,\mathrm{sh}\,\omega b(x-1) \\ \mathrm{Im}(\mu_{11}) = B_2\sin\omega a(x-1)\,\mathrm{ch}\,\omega b(x-1) - B_1 \\ \quad \times \cos\omega a(x-1)\,\mathrm{sh}\,\omega b(x-1) \\ \mathrm{Re}(\alpha_{11}) = (B_1 b - B_2 a)\cos\omega a(x-1) \\ \quad \times \mathrm{ch}\,\omega b(x-1) - (B_2 b + B_1 a) \\ \quad \times \sin\omega a(x-1)\,\mathrm{sh}\,\omega b(x-1) \\ \mathrm{Im}(\alpha_{11}) = (B_2 b + B_1 a)\cos\omega a(x-1) \\ \quad \times \mathrm{ch}\,\omega b(x-1) + (B_1 b - B_2 a) \\ \quad \times \sin\omega a(x-1)\,\mathrm{sh}\,\omega b(x-1) \end{cases} \tag{4.6}$$

The constant B is therefore with

$$
\left\{
\begin{aligned}
B_1 &= \frac{\left\{ \begin{aligned} &b \cos \omega a \,\mathrm{ch}\,\omega b - \\ &\qquad a \sin \omega a \,\mathrm{sh}\,\omega b - \sin \omega a \,\mathrm{ch}\, \omega b \end{aligned} \right\}}{D} \\
B_2 &= -\frac{\left\{ \begin{aligned} &a \cos \omega a \,\mathrm{ch}\, \omega b + \\ &\qquad b \sin \omega a \,\mathrm{sh}\, \omega b + \cos \omega a \,\mathrm{sh}\, \omega b \end{aligned} \right\}}{D} \\
D &= (b \cos \omega a \,\mathrm{ch}\, \omega b - a \sin \omega a \,\mathrm{sh}\, \omega b \\
&\quad - \sin \omega a \,\mathrm{ch}\, \omega b)^2 + (a \cos \omega a \,\mathrm{ch}\, \omega b \\
&\quad + b \sin \omega a \,\mathrm{sh}\, \omega b + \cos \omega a \,\mathrm{sh}\, \omega b)^2
\end{aligned}
\right. \tag{4.7}
$$

Thus, the first-order approximation of the solution of the fundamental is given by:

$$
\left\{
\begin{aligned}
\zeta_1 &= \bar{\alpha}_{11} = \zeta_{11} \cos(\omega t + \phi_{11}) \\
u_1 &= \bar{\mu}_{11} = u_{11} \cos(\omega t + \psi_{11})
\end{aligned}
\right. \tag{4.8}
$$

with

$$
\left\{
\begin{aligned}
\zeta_{11} &= 2[\mathrm{Re}(\alpha_{11})^2 + \mathrm{Im}(\alpha_{11})^2]^{1/2}, \\
u_{11} &= 2[\mathrm{Re}(\mu_{11})^2 + \mathrm{Im}(\mu_{11})^2]^{1/2} \\
\phi_{11} &= \mathrm{Arc\ tg} \frac{\mathrm{Im}(\alpha_{11})}{\mathrm{Re}(\alpha_{11})}, \\
\psi_{11} &= \mathrm{Arc\ tg} \frac{\mathrm{Im}(\mu_{11})}{\mathrm{Re}(\mu_{11})}
\end{aligned}
\right. \tag{4.9}
$$

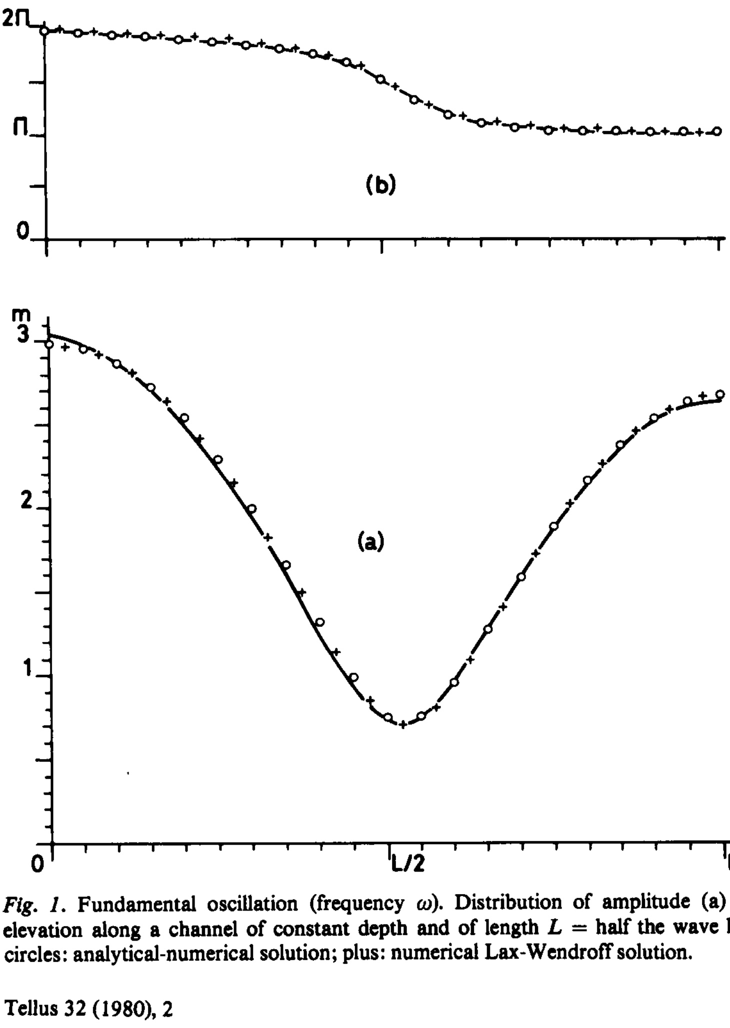

Fig. 1. Fundamental oscillation (frequency ω). Distribution of amplitude (a) and phase (b) for the sea surface elevation along a channel of constant depth and of length L = half the wave length. Full line: analytical solution; circles: analytical-numerical solution; plus: numerical Lax-Wendroff solution.

If one takes $\lambda = 0$ in this solution, we find the classical linear solution given by Gallagher and Munk:

$$\begin{cases} \mu_{11} = -je^{-j\omega} \sin \omega(x-1) \\ \alpha_{11} = e^{j\omega} \cos \omega(x-1) \end{cases}$$

4.2. *Numerical illustration*

As an example, we consider a semi-diurnal wave over a broad shallow shelf that schematically corresponds to the tidal wave in the English Channel $L = 495$ km, $H = 50$ m, $T = 12$ h 25 min, $A = 1$ m/s, $C = 3 \cdot 10^{-3}$ SI-units. With these numerical values it follows that

$$\omega = \frac{\tilde{L}}{\tilde{c}}\, \tilde{\omega} = \Pi, \quad K = C\frac{\tilde{L}}{\tilde{H}} = 29.7$$

and, taking 1 m/s as a mean value of the velocity in the Channel (in order to satisfy the simplifying hypothesis under which (4.9) is valid we find

$$\lambda = \frac{8}{3\pi} K A u_{11} = 1.14$$

The corresponding analytical solution is presented in Figs. 1a and 2a for the distribution of amplitudes and in Figs. 1b and 2b for the phases. These results are compared to those deduced from the numerical integration of problem (2.5) by the Lax-Wendroff scheme, using $\Delta\tilde{x} = \tilde{L}/40$ and $\Delta\tilde{t} = \tilde{T}/84$, i.e. $\Delta\tilde{x} = 12{,}375$ km, $\Delta\tilde{t} = 8.87$ min. Agreement is within a few per cent; even with the simplifying hypothesis $\lambda =$ const. the method used gives us a good representation of the fundamental all along the channel. Starting from this excellent approximation of the dominant oscillation, let us now look at the second-order approximation.

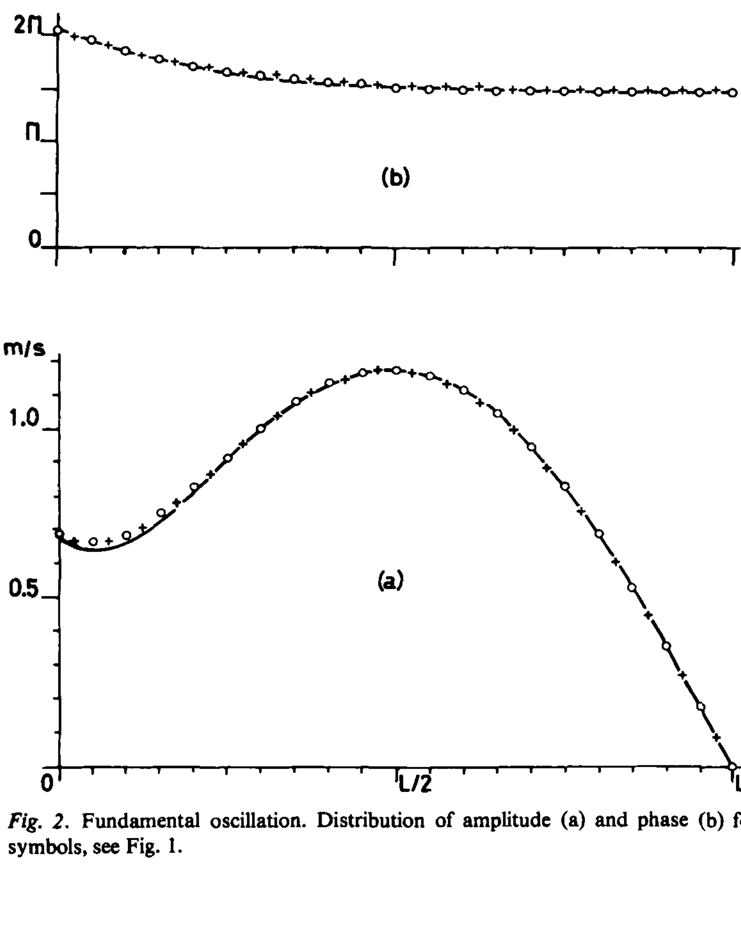

Fig. 2. Fundamental oscillation. Distribution of amplitude (a) and phase (b) for the velocity. For explanation of symbols, see Fig. 1.

5. Determination of the second-order solution

From (2.9) and (3.8) one finds the following equations for the second-order solution:

$$\begin{cases} \zeta_{2t} + u_{2x} = -(\zeta_1 u_1)_x \\ u_{2t} + \zeta_{2x} = -\frac{1}{2}(u_1^2)_x - F_2 \\ u_2 = 0 \quad \text{at } x = 1 \\ u_2 + \zeta_2 = \dfrac{\zeta_1^2}{4} \quad \text{at } x = 0 \end{cases} \tag{5.1}$$

with

$$F_2 = \lambda' u_2 - \lambda u_1 \zeta_1 + \lambda_{32} u_{11}^2 \cos 3(\omega t + \psi_{11}) + \lambda_{52} u_{11}^2 \cos 5(\omega t + \psi_{11})$$

Knowing the solution (u_1, ζ_1) from (4.8), one can write the right side of (5.1) as follows:

$$\begin{cases} (\zeta_1 u_1)_x = \dfrac{d}{dx}(\alpha_{11}\mu_{11}^* + \alpha_{11}^*) + \dfrac{d}{dx}[(\alpha_{11}\mu_{11})\, e^{2J\omega t} + \text{c.c.}] \quad \text{(complex conjugate)} \\ (u_1^2)_x = 2\dfrac{d}{dx}(\mu_{11}\mu_{11}^*) + \dfrac{d}{dx}[(\mu_{11}^2)\, e^{2J\omega t} + \text{c.c.}] \\ \zeta_1^2 = 2\alpha_{11}\alpha_{11}^* + [\alpha_{11}^2 e^{2J\omega t} + \text{c.c.}] \\ F_2 = \lambda' u_{02} - \lambda(\alpha_{11}\mu_{11}^* + \alpha_{11}^*\mu_{11}) + [\lambda' \mu_{12} e^{J\omega t} + \text{c.c.}] \\ \quad + [(\lambda' \mu_{22} - \lambda\mu_{11}\alpha_{11})\, e^{2J\omega t} + \text{c.c.}] \\ \quad + [(\lambda' \mu_{32} + 2\lambda_{32}\mu_{11}^2 e^{J\phi_{11}})\, e^{3J\omega t} + \text{c.c.}] \\ \quad + [(\lambda' \mu_{52} + 2\lambda_{52}\mu_{11}^2 e^{3J\phi_{11}})\, e^{5J\omega t} + \text{c.c.}] \end{cases} \tag{5.2}$$

since (5.1) is a linear system of equations, it can therefore be split up into differential systems of the variable x only.

5.1. *Terms of zero frequency*

From eq. (5.1) and expressions (5.2), one finds:

$$\begin{cases} (u_{02})_x = -\dfrac{d}{dx}(\alpha_{11}\mu_{11}^* + \alpha_{11}^*\mu_{11}) \\ (\zeta_{02})_x = -\dfrac{d}{dx}(\mu_{11}\mu_{11}^*) - \lambda' u_{02} + \lambda(\alpha_{11}\mu_{11}^*) \\ u_{02} = 0 \quad \text{at} \quad x = 1 \\ \zeta_{02} + u_{02} = \frac{1}{2}(\alpha_{11}\alpha_{11}^*) \quad \text{at } x = 0 \end{cases} \tag{5.3}$$

From (4.4) we have:

$$\begin{cases} \alpha_{11}\mu_{11}^* + \alpha_{11}^*\mu_{11} = (B_1^2 + B_2^2)\,[b \sin 2\omega a(x-1) - a \,\mathrm{sh}\, 2\omega b\,(x-1)] \\ \mu_{11}\mu_{11}^* = \dfrac{(B_1^2 + B_2^2)}{2}[\mathrm{ch}\, 2\omega b\,(x-1) + \cos 2\omega a(x-1)] \\ (\alpha_{11}\alpha_{11}^*)_{x=0} = (B_1^2 + B_2^2)(a^2 + b^2) \times (\cos^2 \omega a\, \mathrm{ch}^2\, \omega b + \sin^2 \omega a\, \mathrm{sh}^2\, \omega b) \end{cases}$$

Therefore

$$u_{02} = -(B_1^2 + B_2^2)\,[b \sin 2\omega a(x-1) - a\, \mathrm{sh}\, 2\omega b(x-1)] + c \tag{5.4.1}$$

from the boundary condition at the wall $u_{02}(1) = 0$ it follows that $c = 0$.

Thus, from (5.3) we have:

$$\zeta_{02} = -\mu_{11}\mu_{11}^* - (\lambda + \lambda') \int u_{02}\, dx + D$$

i.e.

$$\zeta_{02} = -(B_1^2 + B_2^2) \times \left\{ \frac{\mathrm{ch}\, 2\omega b(x-1) + \cos 2\omega a(x-1)}{2} + \frac{(\lambda + \lambda')}{\lambda}[b^2 \cos 2\omega a(x-1) + a^2\, \mathrm{ch}\, 2\omega b(x-1)] \right\} + D \tag{5.4.2}$$

with the constant D fixed by the open boundary condition:

$$D = (B_1^2 + B_2^2) \left\{ \frac{\mathrm{ch}\, 2\omega b - \cos 2\omega a}{2} - b \sin 2\omega a + a\, \mathrm{ch}\, 2\omega b + \frac{\lambda + \lambda'}{\lambda}(b^2 \cos 2\omega a + a^2\, \mathrm{ch}\, 2\omega b) + \frac{a^2 + b^2}{2}(\cos^2 \omega a\, \mathrm{ch}^2\, \omega b + \sin^2 \omega a\, \mathrm{sh}^2\, \omega b) \right\}$$

Formulas (5.4) are applied in the numerical example already used in order to check the first-order solution. Results are presented in Figs. 3a and 3b. Agreement is satisfactory for the mean Eulerian velocity. A slight discrepancy occurs for the mean level, but it must be noticed that this solution is not the exact second-order solution because

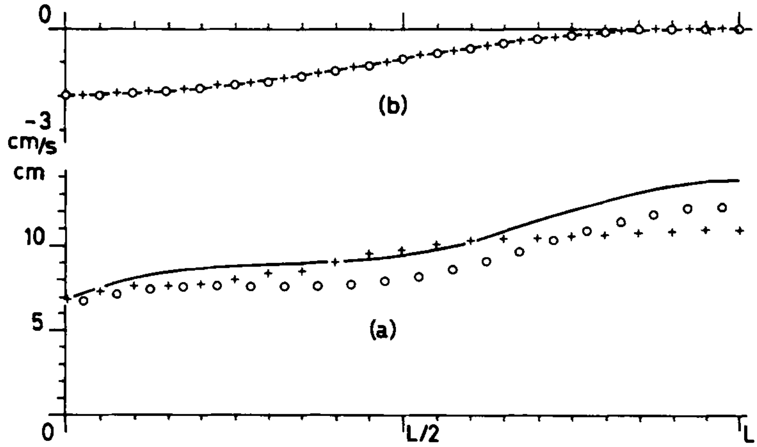

Fig. 3. Distribution of amplitude for the mean sea surface (a) and the mean velocity (b). For explanation of symbols, see Fig. 1.

of the approximation λ = const. introduced in the solution (4.3). A numerical integration of the exact equations will bring a better agreement with the Lax-Wendroff solution (see next section).

5.2. *Terms of frequency* ω

Collecting terms of frequency ω, we find:

$$\begin{cases} j\omega\alpha_{12} + \dfrac{d\mu_{12}}{dx} = 0 \\ j\omega\mu_{12} + \dfrac{d\alpha_{12}}{dx} = -\lambda'\mu_{12} \\ \mu_{12} = 0 \quad \text{at} \quad x = 1 \\ \mu_{12} + \alpha_{12} = 0 \quad \text{at}\; x = 0 \end{cases} \tag{5.5}$$

The solution is $\alpha_{12} = \mu_{12} = 0$.

5.3. *Terms of frequency* 2ω

From eqs. (5.1) and (5.2) we find:

$$\begin{cases} 2j\omega\alpha_{22} + \dfrac{d\mu_{22}}{dx} = -\dfrac{d}{dx}(\alpha_{11}\mu_{11}) \\ 2j\omega\mu_{22} + \dfrac{d\alpha_{22}}{dx} = -\dfrac{1}{2}\dfrac{d}{dx}(\mu_{11}^2) - \lambda'\mu_{22} + \lambda\alpha_{11}\mu_{11} \\ \mu_{22} = 0 \quad \text{at}\; x = 1 \\ \mu_{22} + \alpha_{22} = \dfrac{\alpha_{11}^2}{4} \quad \text{at}\; x = 0 \end{cases} \tag{5.6}$$

Knowing the first-order solution (α_{11}, μ_{11}) from (4.4), we can combine eqs. (5.6) into:

$$\begin{cases} \dfrac{d^2\mu_{11}}{dx^2} + 4\omega^2\left(1 - j\dfrac{\lambda'}{2\omega}\right)\mu_{22} = 3jB^2\omega^2 \\ \quad \times \left(1 - j\dfrac{\lambda}{\omega}\right)^{3/2} \sin\left[2\omega\left(1 - j\dfrac{\lambda}{\omega}\right)^{1/2}(x-1)\right] \\ \alpha_{22} = \dfrac{j}{2\omega}\dfrac{d}{dx}(\alpha_{11}\mu_{11} + \mu_{22}) \end{cases}$$

Given the boundary condition $\mu_{22} = 0$ at $x = 1$, the solution is

$$\begin{cases} \mu_{22} = E\sin\left[2\omega\left(1 - j\dfrac{\lambda'}{2\omega}\right)^{1/2}(x-1)\right] \\ \quad + \dfrac{3}{2}\dfrac{\omega}{2\lambda - \lambda'}B^2\left(1 - j\dfrac{\lambda}{\omega}\right)^{3/2} \\ \quad \times \sin\left[2\omega\left(1 - j\dfrac{\lambda}{\omega}\right)^{1/2}(x-1)\right] \\ \alpha_{22} = jE\left(1 - j\dfrac{\lambda'}{2\omega}\right)^{1/2} \\ \quad \times \cos\left[2\left(1 - j\dfrac{\lambda'}{2\omega}\right)^{1/2}(x-1)\right] \\ \quad + j\dfrac{\omega}{2\lambda - \lambda'}B^2\left(\dfrac{3}{2} - j\dfrac{\lambda + \lambda'}{2\omega}\right) \\ \quad \times \left(1 - j\dfrac{\lambda}{\omega}\right)\cos\left[2\omega\left(1 - j\dfrac{\lambda}{\omega}\right)^{1/2}(x-1)\right] \end{cases} \tag{5.7}$$

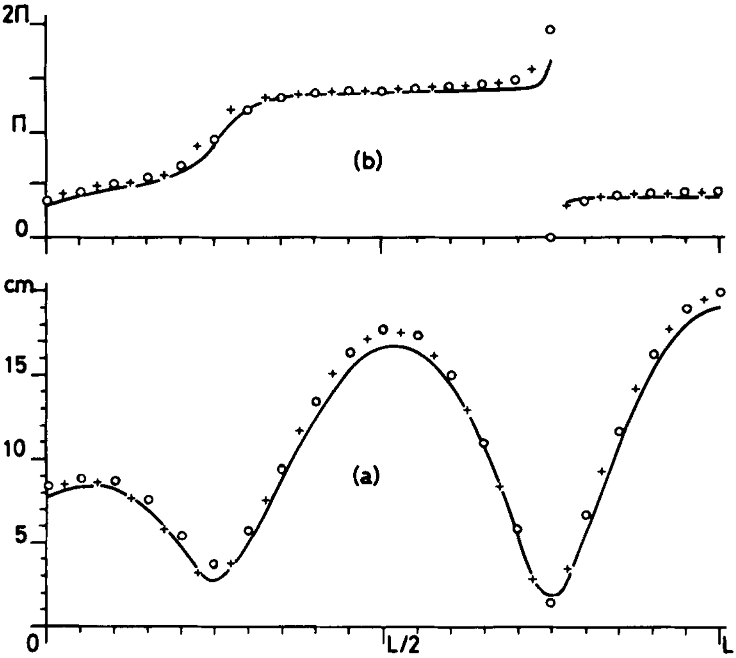

Fig. 4. Harmonic of frequency 2ω. Distribution of amplitude (a) and phase (b) for the sea-surface elevation. For explanation of symbols, see Fig. 1.

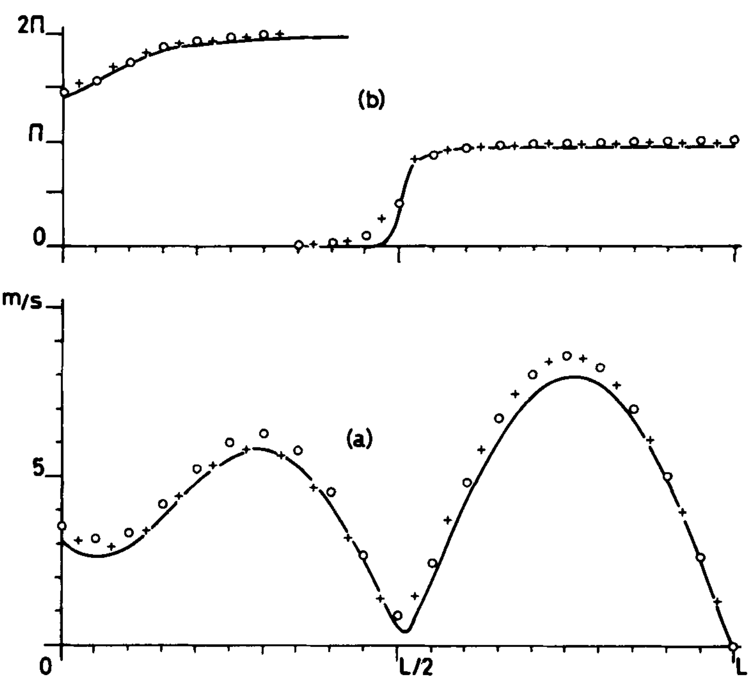

Fig. 5. Harmonic of frequency 2ω. Distribution of amplitude (a) and phase (b) for the velocity. For explanation of symbols, see Fig. 1.

The constant E is fixed by the open boundary condition.

This solution has been used in the numerical example already considered earlier. In Figs. 4a and 5a the distribution of the amplitude of ζ_{22} and u_{22} is presented. These values are in good agreement with the results obtained by harmonic analysis of the numerical solution obtained by the Lax-Wendroff finite-difference scheme; the elevations are a little too small but this discrepancy occurs because of the simplifying hypothesis introduced in the analytical calculation of the first-order solution (λ = const.). It will be shown later that with a numerical integration of the exact systems (4.2) and (5.6) the agreement is much improved. Distribution of phases is presented in Figs. 4b and 5b; they agree with the Lax-Wendroff solution.

5.4. *Terms of frequency 3ω*

Collecting terms of frequency 3ω, we find:

$$\begin{cases} 3j\omega\alpha_{32} + \dfrac{d\mu_{32}}{dx} = 0 \\ 3j\omega\mu_{32} + \dfrac{d\alpha_{32}}{dx} = -\lambda'\mu_{32} - 2\lambda_{32}\mu_{11}^2 e^{j\psi_{11}} \\ \mu_{32} = 0 \quad \text{at } x = 1 \\ \alpha_{32} + \mu_{32} = 0 \quad \text{at } x = 0 \end{cases} \tag{5.8}$$

Such a system is difficult to solve. We shall present its numerical integration in the following section. However, an analytical integration is still possible with a rough approximation of the argument ψ_{11}. From the solution (4.4) we know that, without friction,

$$\psi_{11} = \text{Arc tan}(\cot\omega) = \Omega$$

Let us take this approximation of ψ_{11}; it follows that the eqs. (5.8) can be combined into:

$$\frac{d^2\mu_{32}}{dx^2} + 9\omega^2\left(1 - j\frac{\lambda'}{3\omega}\right)\mu_{32} = -3\omega\lambda_{32}e^{-j\Omega}B^2 \times\left\{1 - \cos\left[2\omega\left(1 - j\frac{\lambda}{\omega}\right)^{1/2}(x-1)\right]\right\} \tag{5.9}$$

taking into account the solution (4.4) of μ_{11}. This has the general solution:

$$\mu_{32} = L\,e^{+\text{Arg}} + M\,e^{-\text{Arg}} + P + Q\cos\left[2\omega\left(1 - j\frac{\lambda}{\omega}\right)^{1/2}(x-1)\right] \tag{5.10.1}$$

with

$$\begin{cases} \text{Arg} = 3j\omega\left(1 - j\dfrac{\lambda'}{3\omega}\right)^{1/2}(x-1) \\ P = -\dfrac{3\omega e^{-j\Omega}}{9\omega^2 + \lambda'^2}\lambda_{32}B^2\left(1 + j\dfrac{\lambda'}{3\omega}\right) \\ Q = \dfrac{3e^{-j\Omega}}{25\omega^2 + (4\lambda - 3\lambda')^2}\lambda_{32}B^2 \\ \quad \times[5\omega - j(4\lambda - 3\lambda')] \end{cases}$$

Then

$$\alpha_{32} = \left(1 - j\frac{\lambda'}{3\omega}\right)^{1/2}[-L\,e^{\text{Arg}} + M\,e^{-\text{Arg}}] - j\tfrac{2}{3}Q\left(1 - j\frac{\lambda}{\omega}\right)^{1/2} \times\sin\left[2\omega\left(1 - j\frac{\lambda}{\omega}\right)^{1/2}(x-1)\right] \tag{5.10.2}$$

where the constants L and M are fixed by the boundary conditions at $x = 0$ and $x = 1$.

As for the solutions of frequency zero and 2ω, the solution (5.10) is applied to the numerical example considered in this paper. Results are presented in Figs. 6a and 7a for the distribution of amplitudes of ζ_{32} and u_{32}, and in Figs. 6b and 7b for their phases. A notable discrepancy appears for this analytical solution when compared with the L.W. solution, but we must remember that (5.8) has been integrated under two approximations: λ = const. and ψ_{11} = const. The question is to know if these discrepancies result from the analytical simplifications. We shall therefore consider in the following section the numerical integration of the exact systems (4.2), (5.3), (5.6) and (5.8).

6. Numerical integration of the first- and second-order approximate solutions (Analytical-numerical solutions)

From the preceding sections we have seen that the application of the method requires us to solve the differential system (4.2) for the fundamental, (5.3) for the constant terms of second order, (5.6) for the second harmonic, and (5.8) for the third harmonic. Except for (5.3), the three other systems

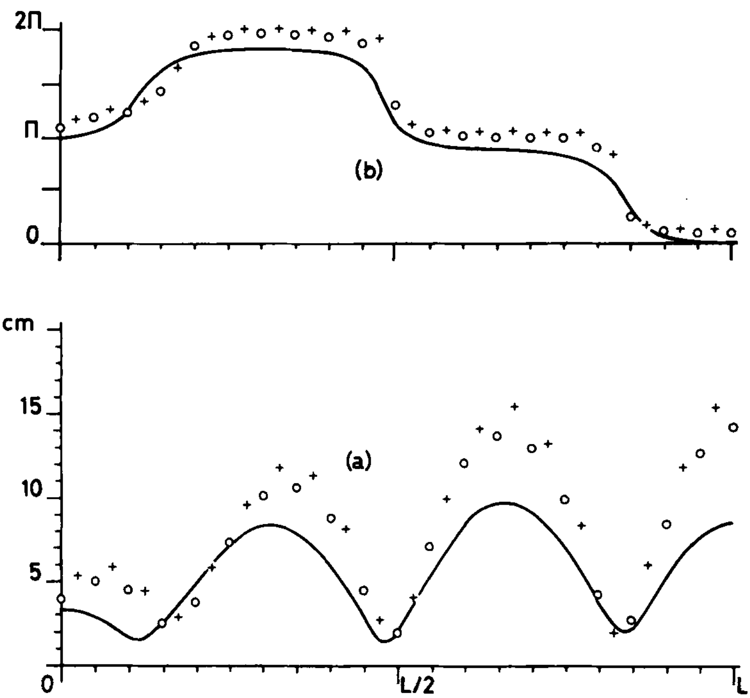

Fig. 6. Harmonic of frequency 3ω. Distribution of amplitude (a) and phase (b) for the sea-surface elevation. For explanation of symbols, see Fig. 1.

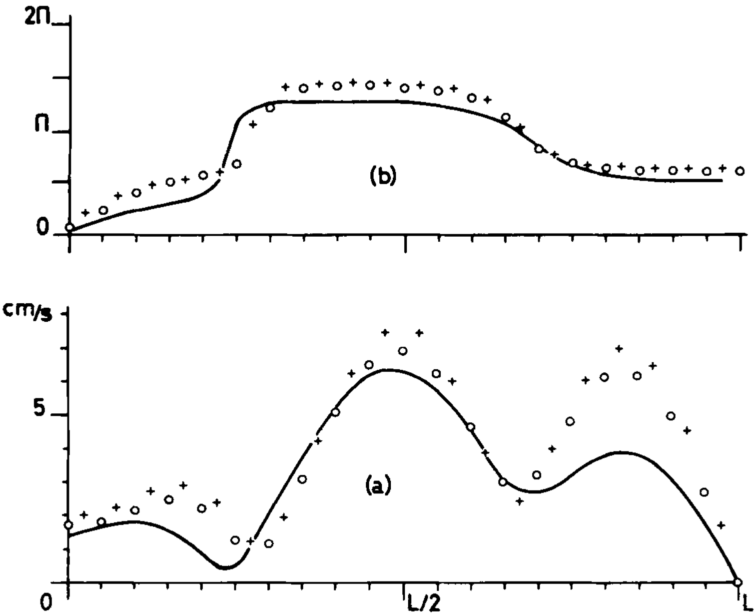

Fig. 7. Harmonic of frequency 3ω. Distribution of amplitude (a) and phase (b) for the velocity. For explanation of symbols, see Fig. 1.

are of the same form, and can be written as follows:

$$\begin{cases} jk\omega\alpha_{kl} + \dfrac{d\mu_{kl}}{dx} = -\dfrac{d}{dx}T_{kl} \\ (jk\omega + \lambda_{kl})\mu_{kl} + \dfrac{d\alpha_{kl}}{dx} = -\dfrac{d}{dx}S_{kl} + F_{kl} \\ \mu_{kl} = 0 \quad \text{at } x = 1 \\ \alpha_{kl} + \mu_{kl} = G_{kl} \quad \text{at } x = 0 \end{cases} \tag{6.1}$$

Or, for $k \neq 0$:

$$\begin{cases} (1) \quad \alpha_{kl} = \dfrac{j}{k\omega}\dfrac{d}{dx}(T_{kl} + \mu_{kl}) \\ (2) \quad \dfrac{d^2\mu_{kl}}{dx^2} + (k^2\omega^2 - jk\omega\lambda_{kl})\mu_{kl} = jk\omega\dfrac{d}{dx}S_{kl} \\ \qquad -jk\omega F_{kl} - \dfrac{d^2 T_{kl}}{dx^2} \\ (3) \quad \mu_{kl}(1) = 0 \\ (4) \quad \dfrac{d\mu_{kl}}{dx} - jk\omega\mu_{kl} = -\dfrac{dT_{kl}}{dx} - jk\omega G_{kl} \quad \text{at } x = 0 \end{cases} \tag{6.2}$$

For $k = 0$, (5.3) can be written:

$$\begin{cases} \dfrac{du_{02}}{dx} = -\dfrac{dT_{02}}{dx} \quad u_{02} = -T_{02} \\ \quad \text{(because } u_{02}(1) = 0) \\ \dfrac{d\zeta_{02}}{dx} = -\dfrac{dS_{02}}{dx} + F_{02} + \lambda' T_{02} \\ \quad \text{with} \quad \zeta_{02}(0) = G_{02} + T_{02} \end{cases} \tag{6.3}$$

Let us denote by Rf and If the real and imaginary parts of any function f. Except for the constant second-order solution, the problem is thus essentially to solve a second-order differential system (6.2.2) in two functions, $R\mu$ and $I\mu$ of the variable x, with boundary conditions (6.2.3) and (6.2.4). This can be done with a classical finite-difference scheme by developing the finite-difference analog of (6.2) at the different grid points of a net along the channel: $x_i = (i-1)\,\Delta x$ with $i = 1, 2, \ldots, n$, and $\Delta x = 1(n-1)$. In the following subscripts k and i are omitted. From (6.2.4) we obtain:

$$-\frac{3}{2\,\Delta x}R\mu(1) + \frac{2}{\Delta x}R\mu(2) - \frac{1}{2\,\Delta x}R\mu(3)$$
$$+ k\omega I\mu(1) = \frac{3}{2\,\Delta x}R_T(1) - \frac{2}{\Delta x}R_T(2)$$
$$+ \frac{1}{2\,\Delta x}R_T(3) + k\omega I_G(1) \tag{6.4.1}$$

$$\frac{3}{2\,\Delta x}I\mu(1) - \frac{2}{\Delta x}I\mu(2) + \frac{1}{2\,\Delta x}I\mu(3) + k\omega R\mu(1)$$
$$= -\frac{3}{2\,\Delta x}I_T(1) + \frac{2}{\Delta x}I_T(2) - \frac{1}{2\,\Delta x}I_T(3)$$
$$+ k\omega R_G(1)$$

and from (6.2.2), for $i = 2$ to $n-1$:

$$\begin{cases} \dfrac{1}{\Delta x^2}R\mu(i-1) + \left(k^2\omega^2 - \dfrac{2}{\Delta x^2}\right)R\mu(2) \\ \quad + \dfrac{1}{\Delta x^2}R\mu(i+1) + k\lambda(i)\omega I\mu(i) \\ \quad = -\dfrac{k\omega}{2\,\Delta x}[I_s(i+1) - I_s(i-1)] + k\omega I_F(i) \\ \quad - \dfrac{1}{\Delta x^2}[R_T(i-1) - 2R_T(i) + R_T(i+1)] \\ \dfrac{1}{\Delta x^2}I\mu(i-1) + \left(k^2\omega^2 - \dfrac{2}{\Delta x^2}\right)I\mu(i) \\ \quad + \dfrac{1}{\Delta x^2}I\mu(i+1) - k\lambda(i)\omega R\mu(i) \\ \quad = \dfrac{k\omega}{2\,\Delta x}[R_s(i+1) - R_s(i-1)] - k\omega R_F(i) \\ \quad - \dfrac{1}{\Delta x^2}[I_T(i+1) - 2I_T(i) + I_T(i+1)] \end{cases} \tag{6.4.2}$$

with condition (6.4.3): $R\mu(n) = 0$ and $I\mu(n) = 0$.

When $\lambda(i)$ is known, the problem is reduced to the solution of a linear system of $2(n-1)$ equations of the form

$$
\begin{bmatrix}
-\beta & -\gamma & -\delta & & & & k\omega & & & & & \\
A & B & A & & & & & c & & & & \\
& A & B & A & \mathrm{O} & & & & \cdot & & \mathrm{O} & \\
& & \cdot & \cdot & \cdot & & & & & \cdot & & \\
& & & \cdot & \cdot & \cdot & & & & & \cdot & \\
& & & & \cdot & \cdot & \cdot & & & & & \cdot \\
\mathrm{O} & & & & A & B & A & & \mathrm{O} & & c & \\
& & & & & A & B & 0 & & & & c \\
k\omega & & & & & & 0 & \beta & \gamma & \delta & & \\
& -c & & & & & & A & B & A & & \\
& & \cdot & \mathrm{O} & & & & & A & B & A & \mathrm{O} \\
& & & \cdot & & & & & & \cdot & \cdot & \cdot \\
& & & & \cdot & & & & & & \cdot & \cdot \\
& & & & & \cdot & & & & & & \cdot \\
\mathrm{O} & & & & & -c & & \mathrm{O} & & A & B & A \\
& & & & & & -c & & & & A & B
\end{bmatrix}
\times
\begin{bmatrix}
R\mu(1) \\ R\mu(2) \\ R\mu(3) \\ \cdot \\ \cdot \\ \cdot \\ R\mu(n-2) \\ R\mu(n-1) \\ I\mu(1) \\ I\mu(2) \\ I\mu(3) \\ \cdot \\ \cdot \\ \cdot \\ I\mu(n-2) \\ I\mu(n-1)
\end{bmatrix}
=
\begin{bmatrix}
R_g(1) \\ R_g(2) \\ R_g(3) \\ \\ \\ \\ R_g(n-2) \\ R_g(n-1) \\ I_g(1) \\ I_g(2) \\ I_g(3) \\ \\ \\ \\ I_g(n-2) \\ I_g(n-1)
\end{bmatrix}
\qquad (6.5)
$$

which can easily be solved by a "forward–backward" alogorithm coupling equations i and $i + n - 1$ at each step of the forward sweep.

6.1. *First-order solution*

For the first-order solution, the problem is complicated by the presence of the nonlinear coefficient of bottom friction,

$$\lambda(u_{11}) = \frac{8K}{3\pi} Au_{11} = \frac{16KA}{3\pi} \sqrt{R\mu_{11}^2 + I\mu_{11}^2}$$

in eqs. (4.2). The coefficient c in (6.5) is thus dependent upon the solution, and a direct resolution of this system is not possible. This difficulty has been bypassed by an iterative process:

(i) As the first step, the solution of (4.2) is calculated without the friction coefficient ($\lambda = 0$); the corresponding system (6.5) is thus a linear one without a second member (R_g, I_g) except for R_g (1) and I_g (1), the boundary condition at the mouth of the channel.

(ii) This first approximation is used to fix an approximate set of values $\lambda(i) = \lambda(u_{11})$.

A new solution of (6.5) is then calculated, giving a better value of $\lambda(i)$, and this iterative process is repeated until a stationary solution is obtained. Convergence of this iterative process has been demonstrated, and numerically it is verified that the successive solutions converge towards a stationary and unique solution. A numerical example is presented in Figs. 8, 9 and 10 for the particular case already used as an illustration of the analytical solution obtained in the preceding section.

(i) Fig. 10 shows the rapid convergence of the iterative process. After five steps, a stationary solution is obtained.

(ii) In Figs. 8 and 9, we can see how the first-step solution (frictionless) is quite different from the exact one, being twice too big.

(iii) The stationary solution (step 6) has been plotted on Fig. 1 and 2: the agreement with the L.W. solution is better than the analytical one presented in Section 4, which confirms that this solution is certainly the exact one. The good accuracy of the approximate analytical solution (4.6) must be noted once more, but one must remember that such a good fit is in fact dependent upon the choice of the approximate constant value of λ.

6.2. *Second-order solutions*

Knowing the values of the first-order solution at the n points of calculation, one can compute the forcing terms (5.2) (i.e. the quantities $R_{gk2}(i)$, $I_{gk2}(i)$, $i = 1, n$) and the damping coefficients

$$\lambda'(i) = \frac{4K}{\pi} Au_{11}[(i-1)\Delta x]$$

We have then to solve one linear system of the form (6.5) for each harmonic constituent of this approximation. A numerical application has been made for the classical example already presented. The corresponding results have been plotted on Fig. 3 for the constant component, on Figs. 4 and 5 for the second harmonic, and Figs. 6 and 7 for the third harmonic.

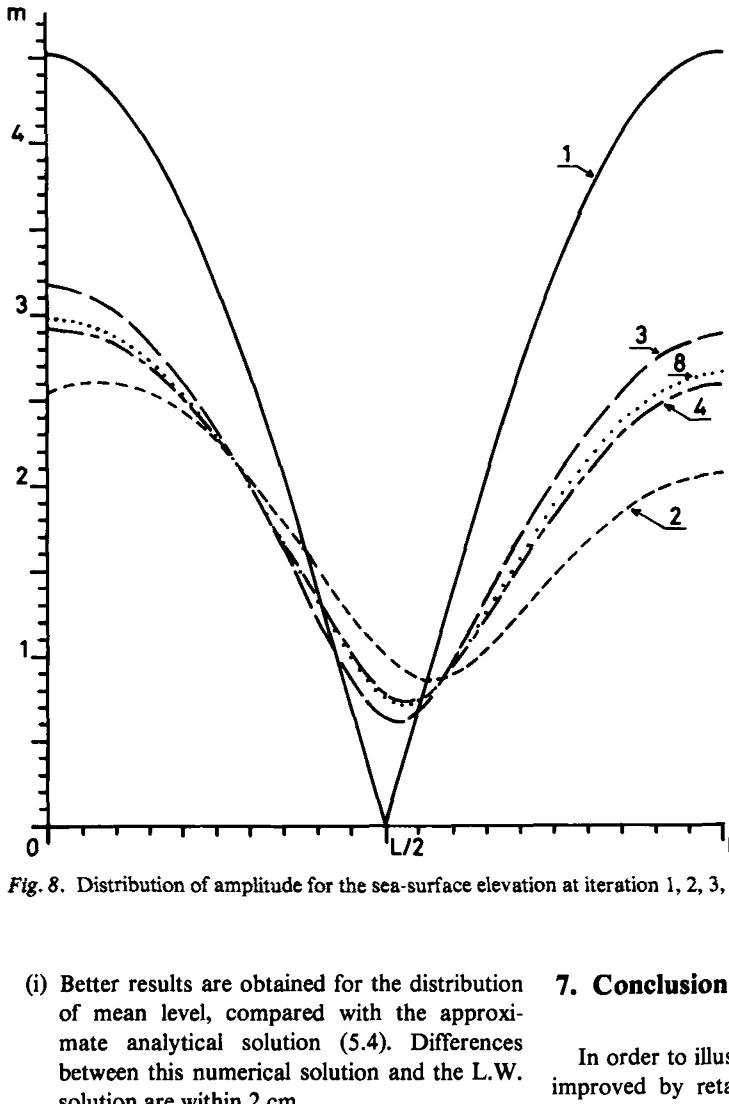

Fig. 8. Distribution of amplitude for the sea-surface elevation at iteration 1, 2, 3, 4 and 8.

(i) Better results are obtained for the distribution of mean level, compared with the approximate analytical solution (5.4). Differences between this numerical solution and the L.W. solution are within 2 cm.

(ii) For the second harmonic, this numerical solution fits almost perfectly with the L.W. solution. It should be noted that the agreement was already very good with the analytical solution (5.7).

(iii) The third harmonic solution (5.10) was quite different from the L.W. one. The numerical results here presented prove that these discrepancies were due to the simplifying approximations introduced in the analytical solution. Distributions of amplitudes are now within 2 cm for sea-surface elevations and 1 cm/s for velocities; the phases also fit perfectly.

7. Conclusions

In order to illustrate how much the solutions are improved by retaining higher-order terms of the damping friction, we have carried out several simulations in the same channel as before ($H = 50$ m, $L = 495$ km) for a fundamental wave of period $T =$ 12 h 25 min, using five different models. The first one corresponds to the numerical solution obtained with a Lax-Wendroff scheme, and is taken as a check. The four other models are perturbation solutions limited to the second-order of approximation, including (2) quadratic bottom friction introduced at the 2nd order of approximation, (3) linear bottom friction, (4) linearized bottom friction (cf. Section 5), (5) quasi-linearized bottom friction (cf. Section 6).

The amplitude of the sea-surface elevation ob-

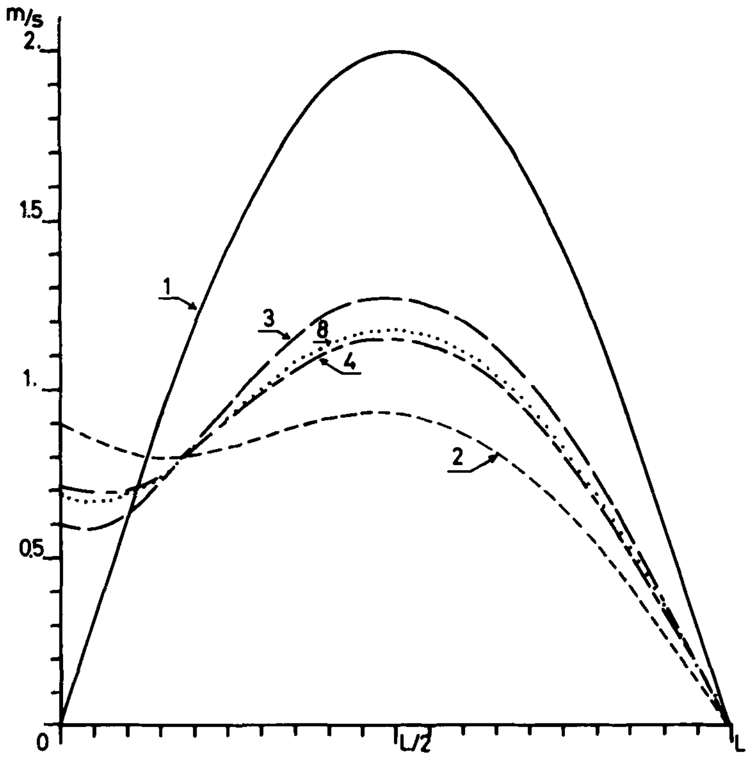

Fig. 9. Distribution of amplitude for the velocity at iteration 1, 2, 3, 4 and 8.

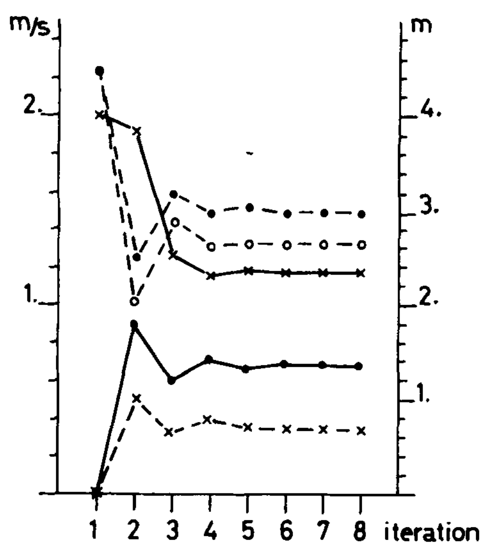

Fig. 10. Convergence at the iterative method of integration of (6.5) for the fundamental. Full line: velocity; dashed line: sea-surface elevation; points: at the mouth of the channel; crosses: at the middle of the channel; circles: at the end of the channel; for iteration 1 to 8.

tained by these methods at the entrance, the middle and end of the channel are presented in Table 3. The simulations have been done for several intensities of the fundamental oscillation in order to characterize how the degree of improvement of the different methods depends upon the size of the amplitude A of the movement.

For small wave amplitude ($A = 0.00452$, i.e. $\bar{A} = 0.1$ m/s, $\zeta \simeq 40$ cm, $\zeta/\bar{H} \simeq 0.8/100$), methods (2), (4) and (5) give correct results. The intensity of the movement is low enough, so that damping effects of friction appear to be really second-order corrections. On the other hand, method (3) gives rather bad results because of the too large value of the linear bottom friction coefficient used ($\lambda = 1.14$), which corresponds in fact only to oscillating movements of the order of magnitude of 1 m/s. We notice also that for such wave intensities nonlinear harmonic components are very small.

For larger wave amplitudes ($A = 0.01355$, i.e. $\bar{A} = 0.3$ m/s, $\zeta \simeq 140$ cm, $\zeta/\bar{H} \simeq 2/100$), the discrepancies for methods (2) and (3) are of the order of 10% for the fundamental oscillation, and 50% for

Table 3. *Amplitude in cm of the different spectral components for the sea-surface elevation in a channel of constant depth (50 m) calculated from the Lax-Wendroff numerical solution (col. 1), and from perturbation solutions up to the second order of approximation under the following assumptions: quadratic bottom friction introduced at the second-order approximation (col. 2); linear bottom friction (col. 3); linearized bottom friction (cf. Section 5 (col. 4)); quasi-linear bottom friction (cf. Section 6 (col. 5))*

		$\tilde{A} = 0.1$ m/s ($A = 0.00452$)					$\tilde{A} = 0.3$ m/s ($A = 0.01355$)					$\tilde{A} = 0.5$ m/s ($A = 0.02258$)				
		1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
α_1	$x = 0$	41.4	40.7	32.2	41.6	41.5	110.1	95.8	96.6	110.7	110.3	168.5	115.6	161.1	171.1	169.2
	$x = L/2$	1.8	2.2	6.5	1.8	1.8	12.6	19.5	19.5	12.4	12.6	28.4	54.0	32.7	27.5	28.2
	$x = L$	41.4	40.7	29.3	41.5	41.4	108.4	95.8	88.0	107.9	108.6	162.5	115.6	146.7	161.8	162.9
α_{02}	$x = 0$	0.1	0.1	0.1	0.1	0.1	0.6	0.5	0.6	0.6	0.6	1.7	1.3	1.7	1.7	1.7
	$x = L/2$	0.0	0.0	0.1	0.0	0.0	0.2	0.0	0.6	0.2	0.1	1.4	0.0	1.7	1.2	1.0
	$x = L$	0.1	0.1	0.1	0.1	0.1	0.7	0.5	1.1	0.8	0.7	2.2	1.3	3.0	2.6	2.3
α_{22}	$x = 0$	0.4	0.5	0.3	0.4	0.4	2.1	4.3	2.4	2.0	2.1	4.1	12.1	6.8	3.8	4.2
	$x = L/2$	0.4	0.5	0.5	0.4	0.4	2.8	4.3	4.1	2.7	2.8	6.3	12.1	11.5	6.1	6.5
	$x = L$	0.4	0.5	0.5	0.4	0.4	3.0	4.3	4.6	2.9	3.0	7.0	12.1	12.9	6.8	7.2
α_{32}	$x = 0$	0.2	0.2	0.0	0.1	0.1	1.0	1.6	0.0	0.7	0.9	2.2	4.4	0.0	1.4	1.8
	$x = L/2$	0.1	0.1	0.0	0.1	0.1	0.4	0.8	0.0	0.3	0.3	0.7	2.1	0.0	0.5	0.6
	$x = L$	0.2	0.2	0.0	0.2	0.2	1.7	1.6	0.0	1.2	1.6	4.6	4.4	0.0	2.9	4.2

the first harmonic. On the other hand, methods (4) and (5) fit very well with solution (1).

With increasing wave amplitudes ($A = 0.02258$, i.e. $\tilde{A} = 0.5$ m/s, $\tilde{\zeta} \simeq 170$ cm, $\tilde{\zeta}/\tilde{H} \simeq 3/100$), solution (2) appears to be less and less significant, either for the fundamental or the harmonic components; method (3) gives rather good values for the fundamental wave, but it is not able to represent correctly the order of magnitude for the harmonic components (1st harmonic too big, no 2nd harmonic). On the other hand, methods (4) and (5) continue to give very good solutions, except for method (4) for the 2nd harmonic, as was already noted in the preceding sections.

We can conclude consequently that the method proposed in this paper is well adapted for tidal-wave computations in channels. The method used to introduce quadratic bottom friction in the perturbation theory is successful from two points of view. Firstly, the separation of the main damping effects of bottom stress from its role in generating harmonics, and the introduction of the damping terms of higher order in the perturbation systems describing the propagation of each component appear to be essential to the success of the method. Secondly, the integration of the perturbation systems corresponding to the generating role of harmonics played by friction allows one to calculate odd harmonics that can reach nonnegligible amplitudes in shallow-water areas.

This method can be particularly useful for the understanding of the nonlinear process observed in the European seas where tidal waves reach amplitudes of several meters. At a rough approximation, the English Channel can be modelled by a rectangular channel of constant depth (50 m), 500 km long, closed at one end, and connected to the ocean at the other end. The amplitudes of the waves presented in Figs. 1 to 7 correspond to the order of magnitude of the real tidal components observed in situ: 3 m for the basic component M_2, with velocities of 1 m/s, 15 to 20 cm for the first harmonic component M_4, and 10 to 14 cm for the second one M_6. These results illustrate how well adapted our model is to represent real situations. For the moment, the method is limited by the hypothesis of a "dominant wave", but an extension to a situation with two dominant constituents is under investigation and will be applied to the case where diurnal and semi-diurnal tidal components are of the same order.

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НЕЛИНЕЙНЫЕ ПРИЛИВНЫЕ ВОЛНЫ В КАНАЛАХ: МЕТОД ВОЗМУЩЕНИЙ УЧИТЫВАЮЩИЙ КВАДРАТИЧНОЕ ТРЕНИЕ О ДНО

Трение о дно играет важную роль в распространении и затухании длинных волн в мелкой воде. Представлен метод возмущений, хорошо приспособленный к учету напряжений на дне, который применяется к каналам с постоянными средними глубинами и средней площадью поперечного сечения. Основная идея метода появляется при разложении квадратичного трения в ряд фурье до третьего приближения. Из этого разлжения возникает квазилинеаризация эффекта затухания из-за напряжения на дне, что позволяет ввести эффекты затухания в систему первого порядка, определяя главную компоненту спектра, и учесть основную часть членов затухания третьего порядка при вычислении гармоник второго порядка. Из этого разложения также устанавливается генерирующее действие гармоник из-за трения, что учитывается в приближении второго порядка. Аналити-ческие и аналитически-численные решения представлены и сравнены с численными решениями полных нелинейных уравнений, полученными с помощью интегрирования конечно-разностных систем методом Лакса Вендроффа. Это сравнение показывает, что аналитические решения, ограниченные вторым приближением, очень хорошо согласуются с численными.