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Corrigendum to the paper Internal solitary waves in a linearly stratified fluid

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1. Introduction

In resolving a discrepancy between my solutions (Weidman, 1978) and those of Miles (1979) for internal solitary waves in a fluid with linear density stratification, the author has discovered that he made an egregious error: one of the terms in the differential equation solved has an incorrect sign. The solutions to the corrected equation are found to be in full accord with the results cited by Miles (1979). The format employed here in presenting the corrected solutions is identical to that used in the original paper, including section headings and equation numbers.

2. Problem statement

Equation (2.1) in Weidman (1978) was derived by Long (1965) for particle displacements *above* the equilibrium level, i.e., $\delta' = y' - y'_0$. In going from eqs. (2.4) to (2.5) the author unwittingly used the *below* equilibrium level particle displacement $\delta' = y'_0 - y'$ as in Long's (1953) earlier work. The equation governing the particle displacement *above* equilibrium for steady waves in a linearly stratified fluid should therefore read

$$(1-\beta(y-\alpha\delta))\left(\frac{\partial^2\delta}{\partial y^2}+\gamma\frac{\partial^2\delta}{\partial x^2}\right)+\left[\frac{\alpha\beta}{2}\left\{\left(\frac{\partial\delta}{\partial y}\right)^2+\gamma\left(\frac{\partial\delta}{\partial x}\right)^2\right\}-\beta\frac{\partial\delta}{\partial y}+\sigma^2\delta\right]. \tag{2.5}$$

3. Rigid lid solutions

3.1. *Odd modes*

For $\gamma = \alpha\beta$ the sign change in (2.5) does not alter the equations for the first three eigenfunctions δ_{00}, δ_{10}, δ_{01}. At order $\alpha\beta$, however, the differential equation becomes

$$\begin{aligned}\delta''_{11}+\sigma_{00}^2\,\delta_{11} &= -\sigma_{11}^2\;\delta_{00}-\sigma_{01}^2\,\delta_{10}-\sigma_{10}^2\,\delta_{01}\\ &\quad -\delta_{00}\,\delta''_{00}+y\delta''_{10}+\delta'_{10}-\tfrac{1}{2}(\delta'_{00})^2-(\delta_{00})_{xx}\end{aligned} \tag{3.4}$$

and the solution vanishing at $y = 0$ is given by

$$\begin{aligned}\delta_{11} &= \left(f_2(x)+\frac{g_1 y}{4}\right)\sin n\pi y\\ &\quad+\frac{f^2}{4}(1+\cos 2n\pi y-2\cos n\pi y)\\ &\quad+\frac{1}{2n\pi}\left(f_{xx}+\sigma_{11}^2 f-\frac{n^2\pi^2}{2}g_1\right)y\cos n\pi y\\ &\quad+\frac{n\pi}{4}g_1 y^2\cos n\pi y.\end{aligned} \tag{3.9}$$

The upper boundary condition then requires

$$\frac{(-1)^n}{2n\pi}(f_{xx}+\sigma_{11}^2 f)+\tfrac{1}{4}f^2[1-(-1)^n] \tag{3.10}$$

which for n odd gives

$$f_{xx}+\sigma_{11}^2 f-n\pi f^2=0. \tag{3.11}$$

Written in dimensional form, the solution to this order of approximation is then

$$\left.\begin{aligned}\frac{\delta'}{h} &= -\alpha\sin n\pi\frac{y'}{h}\operatorname{sech}^2\left[\left(\frac{n\pi\alpha\beta}{3}\right)^{1/2}\frac{x'}{h}\right]\\ \sigma^2 &= n^2\pi^2-\frac{n^2\pi^2}{2}\beta-\frac{4n\pi}{3}\alpha\beta.\end{aligned}\right\} \tag{3.14}$$

3.2. *Even modes*

For $\gamma = \alpha\beta^2$ we again find that $f(x) = 0$ for n odd and $\sigma_{11}^2 = 0$ for n even. The equations for δ_{20}

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and δ_{02} remain the same, but the correct equation for δ_{12} is written

$$\begin{aligned}\delta''_{12} + \sigma^2_{00}\,\delta_{12} = {} & -\sigma^2_{12}\,\delta_{00} - \sigma^2_{10}\,\delta_{02} - \sigma^2_{01}\,\delta_{11} \\ & - \sigma^2_{11}\,\delta_{01} - \sigma^2_{02}\,\delta_{00} - \delta_{00}\,\delta''_{01} - \delta_{01}\,\delta''_{00} - \delta'_{00}\,\delta'_{01} \\ & + \delta'_{11} + y\delta'_{11} - (\delta_{00})_{xx}.\end{aligned} \tag{3.18}$$

The solution satisfying the lower boundary condition can again be written in the form of eq. (3.21) if one makes the following changes

$$\left.\begin{aligned} h_1(x) &= \frac{f_2}{4} - \frac{n\pi}{8} f^2 \\ h_2(x) &= \frac{(5 - n^2\pi^2)}{32} g_1 - \frac{n\pi}{8} f^2 \end{aligned}\right\} \tag{3.21a}$$

$$k_1(x) = \frac{(f_{xx} + \sigma^2_{12} f)}{2n\pi} - \frac{f^2}{8} - \frac{n\pi}{16} g_1 - \frac{n\pi}{4} f_2 \tag{3.21b}$$

$$l_0(x) = \frac{ff_1}{2}; \quad l_1(x) = \tfrac{3}{8} f^2 \tag{3.21c}$$

$$p_0(x) = -\frac{11}{24}\frac{f^2}{n\pi}; \quad p_1(x) = \frac{n\pi}{8} f^2;$$

$$p_2(x) = -\frac{n\pi}{8} f^2 \tag{3.21d}$$

$$q_0(x) = \frac{ff_1}{2}; \quad q_1(x) = \tfrac{3}{8} f^2. \tag{3.21e}$$

The differential equation for $f(x)$ determined by application of the upper boundary condition is written

$$f_{xx} + \sigma^2_{12} f + \tfrac{5}{4} n\pi f^2 = 0 \tag{3.23}$$

and yields the solution in dimensional form

$$\left.\begin{aligned} \frac{\delta'}{h} &= \alpha \sin n\pi \frac{y'}{h} \operatorname{sech}^2 \left[\left(\frac{5n\pi\alpha\beta^2}{24}\right)^{1/2} \frac{x'}{h}\right] \\ \sigma^2 &= n^2\pi^2 - \frac{n^2\pi^2}{2}\beta + \frac{(1 + n^2\pi^2)}{16}\beta^2 \\ &\quad - \frac{5n\pi}{6}\alpha\beta^2. \end{aligned}\right\} \tag{3.26}$$

4. Free surface solutions

The linearized free surface boundary conditions (4.1)–(4.4) are not affected by the sign change in eq. (2.5) but we take this opportunity to correct a typographical error which went unnoticed in the original publication; eq. (4.3) should read

$$\sigma^2_{01}\,\delta_{00} + \sigma^2_{00}\,\delta_{01} = \delta'_{00} \qquad @\, y = 1. \tag{4.3}$$

We note that equations (4.9) and (4.10) which agree with the solutions obtained by Benjamin (1966) for an exponential density stratification of a free surface fluid are indeed correct since they are not affected by the author's error.

For the linear density stratification we again find that solutions (4.5), (4.6), (4.7) and (4.11) are correct, but that the solution at order $\alpha\beta$ satisfying the condition $\delta_{11} = 0$ @ $y = 0$ becomes

$$\delta_{11} = \bar{\delta}_{11} + \frac{g_1}{n\pi} y \cos n\pi y \tag{4.8}$$

where $\bar{\delta}_{11}$ is the *corrected* rigid lid solution given in this paper by equation (3.9). Application of the linearized free surface condition (4.4) then requires

$$f_{xx} + \sigma^2_{11} f - 5n\pi f = 0 \qquad (n \text{ odd}) \tag{4.9}$$

$$f_{xx} + \sigma^2_{11} f + 3n\pi f = 0 \qquad (n \text{ even}) \tag{4.10}$$

and the solutions to the present order of approximation are

$$\left.\begin{aligned} \frac{\delta'}{h} &= -\alpha \sin n\pi \frac{y'}{h} \operatorname{sech}^2 \left[\left(\frac{5n\pi\alpha\beta}{6}\right)^{1/2} \frac{x'}{h}\right] \\ \sigma^2 &= n^2\pi^2 - 2\left(\frac{n^2\pi^2}{4} - 1\right)\beta - \frac{10n\pi}{3}\alpha\beta \end{aligned}\right\} (n \text{ odd}) \tag{4.14}$$

$$\left.\begin{aligned} \frac{\delta'}{h} &= \alpha \sin n\pi \frac{y'}{h} \operatorname{sech}^2 \left[\left(\frac{n\pi\alpha\beta}{2}\right)^{1/2} \frac{x'}{h}\right] \\ \sigma^2 &= n^2\pi^2 - 2\left(\frac{n^2\pi^2}{4} - 1\right)\beta - 2n\pi\alpha\beta. \end{aligned}\right\} (n \text{ even}) \tag{4.15}$$

5. Discussion and conclusion

The nondimensional streamline displacements above equilibrium may be written in the form

$$\delta(x, y) = \operatorname{sgn}(\alpha)|\alpha| \sin n\pi y \operatorname{sech}^2 \omega x \tag{5.1}$$

and the nonlinear phase speeds are (note the correction)

$$c^2 = c_n^2\left(1 + \frac{4\omega^2}{n^2\pi^2}\right) \tag{5.2}$$

Table 1

	Exponential stratification		Linear stratification		
	Top plate	Free surface	Top plate	Free surface	
$\dfrac{c_n^2}{\beta gh/n^2\pi^2}$	$1-\dfrac{\beta^2}{4n^2\pi^2}$	$1-\dfrac{2\beta}{m^2\pi^2}$	$1+\dfrac{\beta}{2}$	$1+\left(\dfrac{m^2\pi^2-4}{2m^2\pi^2}\right)\beta$	All modes
ω^2	$\dfrac{n\pi}{9}\lvert\alpha\rvert\beta$	$\dfrac{7n\pi}{18}\lvert\alpha\rvert\beta$	$\dfrac{n\pi}{3}\lvert\alpha\rvert\beta$	$\dfrac{5n\pi}{6}\lvert\alpha\rvert\beta$	Odd modes
Sgn (α) Mode 1 wave	(+) Elevation	(−) Depression	(−) Depression	(−) Depression	
ω^2	$\dfrac{n\pi}{36}\lvert\alpha\rvert\beta^2$	$\dfrac{n\pi}{2}\lvert\alpha\rvert\beta$	$\dfrac{5n\pi}{24}\lvert\alpha\rvert\beta^2$	$\dfrac{n\pi}{2}\lvert\alpha\rvert\beta$	Even modes
Sgn (α) Mode 2 wave	(−) Bulge	(+) Pinch	(+) Pinch	(+) Pinch	

where c_n is the linear wave speed. The results are summarized for both the exponential and linear stratification models in Table 1.

For fluids with weak exponential stratification there is clearly a profound effect in going from a rigid upper plate to a free surface: the vertical modal structure becomes inverted so that a mode 1 elevation wave becomes a depression wave, and a mode 2 bulge becomes a pinch. The changes wrought by a removal of the upper lid for the linearly stratified fluid are much less profound. In this case the modal structures of the waves are qualitatively the same, the only difference being a decrease in wavelength for both the even and odd modes.

The results presented here in Table 1 are in agreement with those obtained by Miles (1979).

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