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On internal solitary waves

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ABSTRACT

Earlier investigations of internal solitary waves in stable, stratified shear flows are generalized on the hypothesis that cubic and quadratic nonlinearity may be of comparable significance—or, equivalently, that $\beta = O(\alpha^2)$, where $\alpha \equiv a/h \ll 1$ and $\beta \equiv (h/l)^2 \ll 1$ for a wave of amplitude a (which may be positive or negative) and length l in a fluid of depth h. An infinite, discrete set of internal solitary waves is possible for prescribed density and horizontal velocity in the primary flow, and to each of these modes there corresponds a relation $\beta = \beta(\alpha)$, $0 < |\alpha| < |\alpha_n|$. Cubic nonlinearity is significant *vis-à-vis* quadratic nonlinearity if and only if $\alpha_n = O(\alpha)$, and $h|\alpha_n|$ then appears as a maximum achievable amplitude. The limit $\alpha_n \to 0$ (which corresponds to a critical combination of the basic flow parameters) implies $a \to 0$ and $l \to \infty$ if the mass carried by the wave is fixed.

The corresponding generalization of the Korteweg–deVries (KdV) equation also is determined and transformed to the normal KdV equation, by virtue of which it admits an infinite number of integral invariants and can be solved by inverse-scattering theory.

1. Introduction

Solitary waves in shallow, stratified fluids are considered by Keulegan (1953), Long (1956, 1965), Peters and Stoker (1960), Benjamin (1966), Benney (1966), Leonov and Miropol'skiy (1975), Kakutani and Yamasaki (1978) and Weidman (1978). Benjamin, Benney and Long (1956) also allow for shear [see also Ter-Krikorov (1962, 1963) and Freeman and Johnson (1970)]. Keulegan (1953), Long (1956) and Kakutani and Yamasaki (1978) consider only two-layer models, whilst the other papers consider continuous stratification. All but Long (1956) and Kakutani and Yamasaki (1978) include only quadratic nonlinearity and find, either explicitly or implicitly, that its effects on solitary waves may vanish for certain critical combinations of the model parameters. [Lee and Beardsley (1974) include cubic nonlinearity in an analysis of internal waves in a weakly stratified shear flow and obtain a generalized Korteweg–deVries equation that is essentially equivalent to (5.2) below. However, they do not deal explicitly with solitary waves.]

Let a and l be a characteristic amplitude and wavelength and let h be the depth of the fluid; then

$$\alpha = a/h, \quad \beta = (h/l)^2, \tag{1.1a,b}$$

both of which are assumed to be small in the present context, are measures of amplitude and dispersion. The prototype (Boussinesq) solitary wave,

$$Y = a\{\operatorname{sech}^2 \xi + O(\alpha)\}, \quad \xi = (X + cT)/l, \tag{1.2a,b}$$

propagates on the free surface of a homogeneous liquid with the speed and length determined (for prescribed amplitude) by

$$\mathcal{F} \equiv c^2/gh = 1 + \alpha + O(\alpha^2) \tag{1.3}$$

and

$$\beta = \tfrac{3}{4}\alpha + O(\alpha^2), \tag{1.4}$$

where X and Y are Cartesian coordinates in a fixed reference frame, T is the time, c is the wave speed in the negative-X direction, and $\mathcal{F}$ is the square of the Froude number. In this classical example α is an appropriate measure of nonlinearity, and (1.4) reflects the Boussinesq balance between nonlinearity and dispersion.

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The simplest model that supports an internal solitary wave is a two-layer liquid with a small discontinuity in density. Let $\hbar h$ and $(1-\hbar)h$, $0<\hbar<1$, be the depths of the lower and upper layers and ρ_0 and $\rho_0(1-\varepsilon)$, $0<\varepsilon\ll 1$, the corresponding densities. The resulting internal solitary wave (Keulegan, 1953) is approximately independent of whether the upper boundary is rigid or free, and the interfacial displacement is described by (1.2) with

$$\operatorname{sgn}\alpha=\operatorname{sgn}(\tfrac{1}{2}-\hbar), \tag{1.5}$$

$$\mathscr{F}=\varepsilon\{\hbar(1-\hbar)+\alpha(1-2\hbar)+O(\alpha^2)\}+O(\varepsilon^2) \tag{1.6}$$

and

$$\hbar^2(1-\hbar)^2\beta=\tfrac{3}{2}(\tfrac{1}{2}-\hbar)\alpha+O(\alpha^2,\alpha\varepsilon) \tag{1.7}$$

and is a surface of elevation/depression for $\hbar \lessgtr \frac{1}{2}$. The appropriate measure of (quadratic) nonlinearity in this example is $(\frac{1}{2}-\hbar)\alpha$, rather than simply α, and vanishes for $\hbar=\frac{1}{2}$. Long (1956) finds that cubic nonlinearity restricts the maximum amplitude of such an interfacial solitary wave according to $|\alpha|<|\alpha_1|$, where (for $\varepsilon\ll 1$)

$$\alpha_1=\hbar(1-\hbar)(\tfrac{1}{2}-\hbar)[\hbar^3+(1-\hbar)^3]^{-1}, \tag{1.8}$$

but he does not comment on this possibility in dealing with internal waves in a continuously stratified fluid (Long, 1965).

I consider here the effects of cubic nonlinearity, in that parametric régime in which they are comparable with those of quadratic nonlinearity, on a solitary wave in a continuously stratified, incompressible shear flow (which comprises the two-layer model as a limiting case). The flow is steady in a reference frame moving with the wave speed c and is governed by a nonlinear partial differential equation that is commonly known as Long's equation [Long (1953); it appears to have been derived originally by Dubreil-Jacotin (1937)]. This equation may be derived from a variational integral (Long, 1953), which I choose as the basis of the present analysis (see Section 2). The required solution then may be expanded in powers of α and β, *qua* independent parameters, and the governing perturbation equations may be deduced directly from the variational integral, after which the requirements of consistency determine $\beta=\beta(\alpha)$. A particular advantage of this procedure [due essentially to Benjamin (1966)] is that the dominant perturbation equations for the solitary wave follow directly from the first approximation, whereas they appear only in the third approximation if the perturbation solution is based on Long's equation. Moreover, both quadratic and cubic nonlinearity may be incorporated in this first approximation, even though cubic nonlinearity proves to be significant if and only if $\beta=O(\alpha^2)$.

I develop this first variational approximation (in Section 3) in the form $\delta=\alpha A(\xi)\phi_n(\eta)$, where: $h\delta$ is the vertical displacement of the streamline that originates at $X=-\infty$ and $Y=h\eta$; $\xi=(X+cT)/l$, as in (1.2); $A(\xi)$ satisfies a first-order, nonlinear differential equation in which the coefficients are functionals of $\phi_n(\eta)$; ϕ_n is the solution of a linear eigenvalue problem for a certain critical value, c_n, of the wave speed (only the dominant internal mode, $\phi=\phi_1$, where $c_1>c_2>\ldots$, is likely to correspond to a physically significant internal solitary wave; the subscript zero is reserved for the free-surface mode, if any, for which $c_0>c_1$). The eigenvalue problem is essentially identical with that which appears in the formulations of Benjamin (1966) and Benney (1966), whilst the differential equation for $A(\xi)$ reduces to that derived by Benjamin if cubic nonlinearity is neglected. I find that solutions for $A(\xi)$, subject to the hypothesis $A(-\infty)=0$, are possible if and only if $|\alpha|\le|\alpha_n|$, where α_n, which is explicitly determined by $\phi_n(\eta)$, is a measure of the importance of quadratic relative to cubic nonlinearity and may vanish for certain critical configurations; e.g. α_1 is given by (1.8) for Keulegan's two-layer model. $A(\xi)$ reduces to $\operatorname{sech}^2\xi$ if $|\alpha_n|\gg|\alpha|$; it is qualitatively similar to $\operatorname{sech}^2\xi$, but falls away less sharply from the peak, if $\alpha_n=O(\alpha)$; it tends to the plateau $A=1$ as $\alpha_n\to\alpha$ (where α is prescribed). An isolated solution, $A=\frac{1}{2}(1+\tanh\xi)$, is possible for $\alpha=\alpha_n$ but implies infinite perturbations of mass and energy and therefore is of limited physical significance.

These last results may be placed in somewhat clearer perspective by prescribing the mass M associated with, rather than the amplitude of, the wave. The limit $\alpha_n\to\alpha$ then corresponds to the limit $M\to\infty$, whilst the limit $\alpha_n\to 0$ for fixed M implies $a\to 0$ and $l\to\infty$.

I apply the preceding results to a weakly stratified fluid and determine the corresponding generalizations of (1.6)–(1.8) in Section 4. The end results are sensitive to details of the density profile, as anticipated by Long's (1965) and Benjamin's (1966) comments on the failure of the Boussinesq approximation in the present context. My results for a linear profile disagree with those obtained by Weidman (1978). [Dr Weidman (personal communication) informs me that his 1978 results are in error and that his corrected results agree with those presented here.]

The incorporation of cubic nonlinearity also is of interest in the more general context of Korteweg–deVries (KdV) theory. The required generalization may be obtained by carrying Benney's (1966) formulation through to higher order, and I merely state (in Section 5) the resulting evolution equation. This generalized KdV equation may be reduced to the basic KdV equation through a transformation that was used originally by Gardner (Gardner et al., 1974) to prove the existence of an infinite number of integral invariants for the KdV equation. It follows that the present generalized KdV equation also admits an infinite number of integral invariants and that it may be solved by inverse-scattering theory.

2. Variational formulation

Following Long (1953, 1965) and Benjamin (1966), I consider a wave of permanent form moving upstream with speed c in an incompressible, inviscid, stratified shear flow of quiescent depth h, density $\rho(\eta)$ and primary velocity $U(\eta)$ with a rigid lower boundary, $\eta = 0$, and an upper boundary, $\eta = 1$, that is either rigid or free; stability of this primary flow is ensured by the restrictions $-gh\rho'/\rho \geq \frac{1}{4}U'^2$ and $U' > 0$ (unless $U \equiv 0$), where, here and subsequently, $' \equiv d/d\eta$. The complete (primary plus wave-induced) flow is steady in a reference frame moving with the speed c and may be described by[1]

$$y = \eta + \delta(x, \eta), \tag{2.1}$$

where x, η and δ are dimensionless, with h as the unit of length, and $h\delta$ is the displacement of the streamline that originates at $x = -\infty$ and $y = \eta$ (so that $\delta \equiv 0$ at $x = -\infty$). The local velocity in the moving reference frame then is given by

$$\mathbf{q} = [u, v] = (U + c)(1 + \delta_\eta)^{-1}[1, \delta_x], \tag{2.2}$$

where the subscripts imply partial differentiation. Density and stagnation pressure are conserved along streamlines, so that $\rho = \rho(\eta)$ and the local gauge pressure p is determined by

$$p + \tfrac{1}{2}\rho q^2 + Ry = P + \tfrac{1}{2}Q + R\eta, \tag{2.3}$$

where

$$Q(\eta) = \rho(U + c)^2, \quad R(\eta) = \rho g h = -P'(\eta), \tag{2.4a,b}$$

and $P' = -R$ follows from hydrostatic equilibrium.

The nonlinear partial differential equation satisfied by δ,

$$Q\left(\frac{\delta_x}{1 + \delta_\eta}\right)_x + \left[Q\frac{(\delta_\eta + \frac{1}{2}\delta_\eta^2 - \frac{1}{2}\delta_x^2)}{(1 + \delta_\eta)^2}\right]_\eta - R'\delta = 0, \tag{2.5}$$ [1]

may be deduced from the variational integral (cf. Long 1953)

$$L = \int_{-\infty}^{\infty} dx \int_0^1 \{(p + \rho q^2)y_\eta - (P + Q)\}\, d\eta \tag{2.6a}$$

$$= \int_{-\infty}^{\infty} dx \int_0^1 \left\{\tfrac{1}{2}Q\left(\frac{\delta_x^2 + \delta_\eta^2}{1 + \delta_\eta}\right) - R\delta\delta_\eta\right\} d\eta \tag{2.6b}$$

[(2.6b) follows from (2.6a) through (2.1)–(2.4) and $P\delta = 0$ at $\eta = 0, 1$] in conjunction with the natural boundary conditions

$$\delta\left\{Q\frac{(\delta_\eta + \frac{1}{2}\delta_\eta^2 - \frac{1}{2}\delta_x^2)}{(1 + \delta_\eta)^2} - R\delta\right\} = 0 \quad (\eta = 0, 1), \tag{2.7}$$

which are satisfied at either rigid ($\delta = 0$) or free ($p = P = 0$) boundaries. It follows from (2.6), through Noether's theorem and the invariance of the Lagrangian density under a translation of x, that Benjamin's (1966) 'flow-force' integral

$$I = \int_0^1 \{(p + \rho u^2)y_\eta - (P + Q)\}\, d\eta \tag{2.8a}$$

$$= \int_0^1 \left\{\tfrac{1}{2}Q\left(\frac{\delta_\eta^2 - \delta_x^2}{1 + \delta_\eta}\right) - R\delta\delta_\eta\right\} d\eta = 0 \tag{2.8b}$$

is an invariant of the flow ($Ih = S - S_0$ in Benjamin's notation). Alternatively, (2.8b) may be

[1] The coordinates x and η are not orthogonal. It is possible to write $x = x(\xi, \eta)$ and $y = \eta + \delta(\xi, \eta)$ and choose x_ξ and x_η such that ξ and η are orthogonal; however, the resulting formulation is more complicated than that considered here.

[1] This is equivalent to (what is commonly known as) Long's equation, which expresses the conservation of $\nabla^2\psi + \rho(\partial\rho^{-1}/\partial\psi)$ along streamlines, where $\psi = \psi(\eta)$ is the stream function and $U \equiv -d\psi/d\eta$. It is not required for the subsequent analysis, which is based on (2.8) or, less directly, on (2.6).

derived by multiplying (2.5) through by δ_x and integrating over $0 < \eta < 1$ and indefinitely with respect to x from $-\infty$.

3. Solitary waves

Following Benjamin (1966), I seek an approximate solution for a solitary wave in the form

$$\delta = \alpha A(\xi)\phi(\eta), \quad \xi = \beta^{1/2}x, \tag{3.1a,b}$$

where: α and β are the amplitude and dispersion parameters [see (1.1)]; $\phi \equiv \phi_n$ is determined by

$$(Q_n\phi')' - R'\phi = 0 \quad (0 < \eta < 1), \tag{3.2}$$

$$\phi = 0 \quad (\eta = 0), \quad \mathscr{S} Q_n\phi' - R\phi = 0 \quad (\eta = 1) \tag{3.3a,b}$$

and

$$\phi_{\max} = 1, \tag{3.4}$$

wherein $Q_n \equiv \rho(U + c_n)^2$, c_n, the "critical speed", is the eigenvalue, and $\mathscr{S} = 0/1$ for a rigid/free upper boundary. The substitution of (3.1) into (2.8b) followed by the expansion of the integrand in powers of α and β on the hypothesis that $c = c_n\{1 + O(\beta)\}$ yields

$$(dA/d\xi)^2 = (\beta\langle Q_n\phi^2\rangle)^{-1}\{\langle(Q - Q_n)\phi'^2\rangle - \alpha\langle Q_n\phi'^3\rangle A + \alpha^2\langle Q_n\phi'^4\rangle A^2\}A^2 \equiv A^2F(A), \tag{3.5}$$

where $O(\alpha\beta, \alpha^3)$ is neglected within $\{\,\}$, and $\langle\,\rangle$ implies averaging over η—e.g.

$$\langle Q_n\phi^2\rangle \equiv \textstyle\int_0^1 Q_n\phi^2\,d\eta.$$

The A^4 term in (3.5), which is neglected in the first approximations of Benney (1966), Benjamin (1966) and Long (1965), is significant if and only if $\beta = O(\alpha^2)$ or, equivalently, $\langle Q_n\,\phi'^3\rangle = O(\alpha)$. [Benjamin (1966) recognizes the possibility that the A^4 term might be significant if $\langle Q_n\,\phi'^3\rangle = O(\alpha)$ but then decides, on the basis of an argument that is not developed in detail but which appears to be flawed by the *a priori* assumption that $\beta = \alpha$, that the difficulty is only apparent.] That this integral may be small for internal waves if both the stratification and shear are weak is evident from the identity (wherein $Q \equiv Q_n$)

$$\langle Q\phi'^3\rangle = \langle\{\tfrac{2}{3}R'' - \tfrac{1}{2}R'(Q'/Q) + \tfrac{1}{6}(Q'^3/Q^2) - \tfrac{1}{3}(Q'Q''/Q) + \tfrac{1}{6}Q'''\}\phi^3\rangle + \mathscr{S}[\{(R^2/Q) + \tfrac{1}{2}R(Q'/Q) - \tfrac{2}{3}R' + \tfrac{1}{6}(Q'^2/Q) - \tfrac{1}{6}Q''\}\phi^3]_{\eta=1}, \tag{3.6}$$

which may be derived through integration by parts with the aid of (3.2) and (3.3).

Real, bounded solutions of (3.5), subject to the null hypothesis (which rules out cnoidal waves)

$$A \to 0 \quad (\xi \to -\infty), \tag{3.7}$$

exist if and only if $F(0) > 0$ and the zeros of $F(A)$, say A_1 and A_2, $A_1 < A_2$, are real. I proceed on the hypothesis that these conditions are satisfied and choose α and β such that $A_1 \equiv 1$ and $F(0) \equiv 4$ (this scaling simplifies the subsequent results) or, equivalently,

$$\beta = \tfrac{1}{4}\langle Q_n\phi^2\rangle^{-1}(\alpha\langle Q_n\phi'^3\rangle - \alpha^2\langle Q_n\phi'^4\rangle) \tag{3.8}$$

and

$$\langle(Q - Q_n)\phi'^2\rangle = 4\beta\langle Q_n\phi^2\rangle, \tag{3.9}$$

the latter of which determines c. It then follows that A_1 and A_2 are real, as assumed, if and only if

$$\operatorname{sgn}\alpha = \operatorname{sgn}\alpha_n, \quad 0 < |\alpha| \le |\alpha_n|, \quad 0 < \beta \le \beta_n, \tag{3.10a,b,c}$$

where

$$\alpha_n = \tfrac{1}{2}\langle Q_n\phi'^4\rangle^{-1}\langle Q_n\phi'^3\rangle,$$

$$\beta_n = \tfrac{1}{16}\langle Q_n\phi^2\rangle^{-1}\langle Q_n\phi'^4\rangle^{-1}\langle Q_n\phi'^3\rangle^2. \tag{3.11a,b}$$

If $|\alpha_n| \gg |\alpha|$ ($|\alpha| \ll 1$ is implicit) the $O(\alpha^2)$terms in (3.5) and (3.8) are negligible, and the solution of (3.5) and (3.7) then is given by

$$A = \operatorname{sech}^2\xi \quad (|\alpha_n| \gg |\alpha|), \tag{3.12}$$

which is identical in form with the Boussinesq solitary wave and corresponds to the solutions obtained by Benjamin (1966), Benney (1966) and Long (1965).

If $\alpha_n = O(\alpha)$, which implies $|\alpha_n| \ll 1$, it is expedient to introduce the parameter μ such that

$$\alpha/\alpha_n = 2\mu/(1 + \mu), \quad \beta/\beta_n = 4\mu/(1 + \mu)^2 \quad (0 < \mu < 1); \tag{3.13a,b}$$

(3.5) then reduces to

$$(dA/d\xi)^2 = 4A^2(1 - A)(1 - \mu A), \tag{3.14}$$

the only bounded solution of which, subject to (3.7), is given by either

$$A = (\cosh^2 \xi - \mu \sinh^2 \xi)^{-1} \quad (0 < \mu < 1) \tag{3.15}$$

or

$$A = \tfrac{1}{2}(1 + \tanh \xi) \quad (\mu = 1). \tag{3.16}$$

The solution (3.15) is symmetrical about $\xi = 0$ and qualitatively similar in form to the Boussinesq solitary wave (3.12). The solution (3.16), which corresponds to the critical case of equal zeros of $F(A)$, is anti-symmetrical with respect to $A = \frac{1}{2}$ and implies an internal wave of infinite mass and energy, in consequence of which it cannot represent a uniformly valid solution to a real physical problem; however, it might represent an inner approximation to such a solution.[1]

It follows from (3.1), (3.4) and (3.13) that $|\alpha_n| h$ is an upper bound to the amplitude of an internal solitary wave (within the present approximations). This limitation appears in somewhat clearer perspective if μ is regarded as a measure of the mass associated with the wave, Mh^2 per unit span, where

$$M = \int_{-\infty}^{\infty} dx \int_0^1 \rho(y_n - 1)\, d\eta = \int_{-\infty}^{\infty} dx \int_0^1 \rho \delta_n d\eta \tag{3.17a}$$

$$= \alpha\beta^{-1/2} \langle \rho\phi' \rangle \int_{-\infty}^{\infty} A\, d\xi \tag{3.17b}$$

and (3.17b) follows from (3.17a) through (3.1). It then follows from (3.13) and (3.15) that

$$\mu = \tanh^2 \{\tfrac{1}{4}(\langle Q_n \phi'^4 \rangle / \langle Q_n \phi^2 \rangle)^{1/2} (M/\langle \rho\phi' \rangle)\} \tag{3.18}$$

and hence that μ increases monotonically with $|M|$ and that $\mu \uparrow 1$ only for $|M| \uparrow \infty$.

4. Weakly stratified fluid

It suffices for most applications to assume that the scale of the density variation is large compared with h, such that the density admits the expansion

$$\rho = \rho_0 \sum_{m=0}^{\infty} (r_n/n!)(\varepsilon\eta)^n, \quad r_1 \equiv -1, \quad \varepsilon \equiv -\rho_0'/\rho_0. \tag{4.1a,b,c}$$

If, in addition, shear is neglected ($U = 0$), the solution of (3.2)–(3.4) may be expanded in powers of ε to obtain

$$\phi_n = (-)^{n-1}[\sin(\sigma_n \eta) + \tfrac{1}{4}\varepsilon\{(1 + r_2)(\eta - 1 + \tfrac{1}{2}n^{-1}) \sin(\sigma_n \eta) + (1 - r_2)\sigma_n \eta^2 \cos(\sigma_n \eta)\} + O(\varepsilon^2)] \quad (n = 1, 2, \ldots) \tag{4.2}$$

and

$$\sigma_n^2 \equiv \varepsilon gh/c_n^2 = (n\pi)^2\{1 + \tfrac{1}{2}\varepsilon(r_2 - 1)\} + 2\varepsilon\mathscr{S} + O(\varepsilon^2) \tag{4.3}$$

for the internal modes ($n = 1, 2, \ldots$; the surface mode, $n = 0$, is essentially unaffected by stratification if $\varepsilon \ll 1$). Note that $r_2 = 1/0$ for the exponential/linear profile $\rho_0 \exp(-\varepsilon\eta)/\rho_0(1 - \varepsilon\eta)$.

The substitution of (4.2) and (4.3) into (3.8), (3.9) and (3.11) yields

$$\beta = \tfrac{1}{2}n\pi\alpha\varepsilon I - 3(\tfrac{1}{2}n\pi)^4 \alpha^2, \tag{4.4}$$

$$(c/c_n)^2 - 1 = (\tfrac{1}{2}n\pi)^{-2} \beta \tag{4.5}$$

and

$$\alpha_n = \tfrac{4}{3}(n\pi)^{-3} \varepsilon I, \tag{4.6}$$

where

$$I = (\varepsilon n\pi\rho_0)^{-1} \langle \rho\phi'^3 \rangle \tag{4.7a}$$

$$= -\mathscr{S} + (\varepsilon^2 \rho_0)^{-1} n\pi \langle (\tfrac{2}{3}\rho'' - \tfrac{1}{2}\rho^{-1}\rho'^2)\phi^3 \rangle, \tag{4.7b}$$

$$= -\mathscr{S} + \tfrac{1}{3}[1 + (-)^{n-1}](\tfrac{4}{3}r_2 - 1) + (\varepsilon/36)(-15 + 13r_2 + 20r_2^2 + 16r_3), \tag{4.7c}$$

error factors of $1 + O(\varepsilon)$ are implicit for each of the terms on the right-hand sides of (4.4)–(4.7), and (4.7b) follows from (4.7a) with the aid of (3.6). The $O(\varepsilon)$ term in (4.7c) is significant if and only if n is even and the upper boundary is rigid ($\mathscr{S} = 0$), in which case it implies $\alpha_n = O(\varepsilon^2)$.

It is evident from (4.6) and (4.7) that sgn α_n is rather sensitive to the upper boundary condition and the details of the density profile. This is illustrated by the following comparison between exponential and linear profiles:

$$\begin{array}{lllll} \rho/\rho_0 = & \multicolumn{2}{l}{\exp(-\varepsilon\eta)} & \multicolumn{2}{l}{1 - \varepsilon\eta} \\ \mathscr{S} = & 0 & 1 & 0 & 1 \\ \operatorname{sgn} \alpha_n = & + & - & - & - \end{array} \tag{4.8}$$

These results agree with those cited by Weidman (1978) for an exponential profile, after allowing for an opposite (in sign) normalization for n even, but not for a linear profile.

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[1] The equivalents of (3.15) and (3.16) for a two-layer model are given by Kakutani and Yamasaki (1978).

5. Evolution equation

The preceding analysis rests on the assumption of steady flow. The corresponding evolution equation for unsteady flow may be obtained from Benney's (1966) general formulation by invoking the hypothesis that cubic and quadratic non-linearity are of comparable significance. The resulting first approximation, normalized as in Section 3, has the form

$$\delta = \alpha \mathscr{A}(\xi, \tau)\phi(\eta), \quad \xi = (X + c_n T)/l,$$

$$\tau = \gamma(gh)^{1/2}\, T/l, \tag{5.1a,b,c}$$

where X is the dimensional horizontal coordinate in the fixed reference frame, T is the dimensional time, ξ and τ are dimensionless, stretched variables in a reference frame moving with the speed c_n (rather than c), $\gamma = O(\beta)$, $\mathscr{A}$ satisfies

$$2\gamma\langle (Q_n R)^{1/2}\phi'^2\rangle \mathscr{A}_\tau = 3\alpha\langle Q_n \phi'^3\rangle \mathscr{A}\mathscr{A}_\xi$$

$$- 6\alpha^2\langle Q_n \phi'^4\rangle \mathscr{A}^2\mathscr{A}_\xi + \beta\langle Q_n \phi^2\rangle \mathscr{A}_{\xi\xi\xi}, \tag{5.2}$$

and ϕ and Q_n are determined by (3.2)–(3.4). The $O(\alpha^2)$ term is significant, *vis-à-vis* the remaining terms, if and only if $\langle Q_n \phi'^3\rangle / \langle Q_n \phi'^4\rangle = O(\alpha)$. [It is worth noting that (5.2) also may be derived by differentiating (3.6) with respect to ξ, dividing by $dA/d\xi$, differentiating again, and interpreting $(c - c_n)(d/d\xi)$ as $\gamma(gh)^{1/2}(\partial/\partial\tau)$ in a reference frame moving with the speed c_n.]

Equation (5.2) reduces to

$$\mathscr{A}_\tau = 12(\mathscr{U}_1 \mathscr{A} - 2\mathscr{U}_2 \mathscr{A}^2)\mathscr{A}_\xi + \mathscr{A}_{\xi\xi\xi} \tag{5.3}$$

through the choice

$$\gamma = \tfrac{1}{2}\beta\langle Q_n \phi^2\rangle\langle (Q_n R)^{1/2}\phi'^2\rangle^{-1} \tag{5.4}$$

and the introduction of

$$\mathscr{U}_1 = \tfrac{1}{8}(\alpha/\beta)\langle Q_n \phi^2\rangle^{-1}\langle Q_n \phi'^3\rangle,$$

$$\mathscr{U}_2 = \tfrac{1}{4}(\alpha^2/\beta)\langle Q_n \phi^2\rangle^{-1}\langle Q_n \phi'^4\rangle. \tag{5.5a,b}$$

$\mathscr{U}_1$ and $\mathscr{U}_2$ are measures of quadratic and cubic nonlinearity, respectively, relative to dispersion; $\mathscr{U}_1$ reduces to Ursell's parameter, $3al^2/4h^3$, for a homogeneous, quiescent fluid, for which $Q_0 = \rho c_0^2$ = constant and $\phi = \eta$. Equation (5.3) reduces to the Korteweg–deVries (KdV) equation if $\mathscr{U}_2 \to 0$ with $\mathscr{U}_1 = O(1)$ and to the modified Korteweg–deVries (mKdV) equation if $\mathscr{U}_1 \to 0$ with $\mathscr{U}_2 = O(1)$. $\mathscr{U}_1 = 2$ and $\mathscr{U}_2 = 1$ for $\alpha = \alpha_n$ and $\beta = \beta_n$.

The scales a and l presumably are inherent in the initial data for any particular problem; however, (5.3) reduces to the normal form

$$\mathscr{A}_\tau = 6\mathscr{A}(1 - \mathscr{A})\mathscr{A}_\xi + \mathscr{A}_{\xi\xi\xi} \tag{5.6}$$

through the choices $\alpha = \alpha_n$ and $\beta = 4\beta_n$ or, equivalently, $\mathscr{U}_1 = \frac{1}{2}$ and $\mathscr{U}_2 = \frac{1}{4}$ (but note that these choices imply $a \to 0$ and $l \to \infty$ for $\langle Q_n \phi'^3\rangle \to 0$). Finally, (5.6) reduces to the normalized KdV equation

$$\mathscr{B}_\tau = 6\mathscr{B}\mathscr{B}_\xi + \mathscr{B}_{\xi\xi\xi} \tag{5.7}$$

through the transformation

$$\mathscr{B} = \mathscr{A}(1 - \mathscr{A}) + \mathscr{A}_\xi, \tag{5.8}$$

which was originally introduced by Gardner (Gardner et al., 1974) to prove that the KdV equation admits an infinite number of integral invariants. It follows from (5.8) that (5.6) also admits an infinite number of integral invariants and that it may be solved through inverse-scattering theory (see Appendix).

6. Appendix

The evolution equation (5.6) falls in the general category considered by Ablowitz et al. (1974). Choosing $a_0 = a_1 = a_2 = 0$, $a_3 = 4i$,

$$q = \mathscr{A}, \quad r = \mathscr{A} - 1, \quad x = \xi, \quad t = \tau \tag{A1}$$

in their (2.7) yields (5.6). Substituting (A1) into their (2.1) yields the eigenvalue problem

$$v_{1\xi} + i\zeta v_2 = \mathscr{A} v_2, \quad v_{2\xi} - i\zeta v_2 = (\mathscr{A} - 1)\, v_1, \tag{A2a, b}$$

which provides the basis for the inverse-scattering solution of (5.6).

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О УЕДИНЕННЫХ ВНУТРЕННИХ ВОЛНАХ

Проведенные ранее исследования внутренних уединенных волн в устойчивом стратифицированном потоке со сдвигом обобщаются на тот случай, когда роли кубической и квадратичной нелинейностей сравнимы по значению, или, что эквивалентно, когда $\beta = O(\alpha^2)$, где $\alpha \equiv a/h \ll 1$, и $\beta \equiv (h/l)^2 \ll 1$, для волны с амплитудой a (которая может быть как отрицательной, так и положительной), длиной l в жидкости с глубиной h. Для заданной плотности и горизонтальной скорости первичного потока возможно существование бесконечного дискретного набора уединенных волн, и каждой из этих мод соответствует соотношение $\beta = \beta(\alpha)$, $0 < |\alpha| < |\alpha_n|$. Кубическая нелинейность существенна наряду с квадратичной тогда и только тогда, когда $\alpha_n = O(\alpha)$, и $h|\alpha_n|$ проявляется как максимально достижимая амплитуда. Предел $\alpha_n \to 0$ (что соответствует критической комбинации основных параметров потока) означает, что $a \to 0$, и $l \to \infty$, если перенос массы волной постоянен.

Получено обобщенное уравнение Кортевега-де Фриза, (КдВ), которое приводится к обычному уравнению КдВ; оно допускает бесконечное число законов сохранения и может быть решено методом обратной задачи теории рассеяния.