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On the preferred mode of Ekman-CISK

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ABSTRACT

The relationship between the vertical partitioning of the diabatic heating and the preferred mode of motion is explored for an Ekman-CISK model. First an unrealistic, but heuristically useful, sinusoidal heating distribution in the vertical is used to establish the link between the growth rate of unstable modes, the Rossby radius of deformation and the diabatic heating profile. The results of the analysis, augmented with those obtained with a more realistic heating distribution, indicate that a rapidly growing, tropical cyclone scale, preferred mode will result from a diabatic heating profile that has a maximum at a low-level. The vertical structure of this mode possesses some realistic features. In particular it has a large vertical gradient of relative vorticity compared with modes associated with mid-tropospheric diabatic heating maximae.

1. Introduction

The Ekman-CISK theory was elegantly announced by Charney and Eliassen (1964) to account for hurricane development. In this theory a cumulus cloud ensemble is regarded as being forced by the moisture convergence in the surface Ekman layer, and the moisture convergence itself is viewed as a dynamical response to the diabatic heating. There have been numerous subsequent studies and generalizations of this scale-interaction instability mechanism and a concise summary of the results is to be found in the theoretical studies section of GARP (1976).

In this study we examine the stability to essentially quasi-geostrophic perturbations of a basic state that represents a conditionally unstable atmosphere at rest above a constant f-plane. The perturbations are taken to be subject to an unconditional heating (i.e. ascent is associated with a diabatic heat source and descent with a diabatic heat sink) that is related to the vertical velocity at the top of the Ekman boundary layer. The adoption of such a simple model requires some comment. Firstly it is to be noted that inertia-gravity waves are excluded from our system. In effect we assume explicitly that there is some additional mechanism that inhibits the instability of these waves without significantly modifying the quasi-geostrophic perturbations. Kuo (1975) postulated that moisture budget constraints would produce such an effect. Another mechanism with this kind of response has recently been studied by Davies (1979) and is based upon the observed lag between the diabatic heating and vertical velocity field at the top of the boundary layer (Cho and Ogura, 1974; Johnson, 1978; Nitta, 1978). Again the assumption of an f-plane geometry is only justifiable if the horizontal length scale of the preferred mode is comparatively small. We shall examine this restriction *a posteriori*. In the same manner we shall later comment briefly upon the physical relevance of our unconditional heating assumption which is invoked principally on grounds of expediency.

For the present we confine our attention to the simplified problem. It has been shown (Chang, 1971; Chang and Williams, 1974; GARP, 1976) that the amplitude of the perturbation changes with time such that, the growth due to diabatic heating

$\propto k^2/[(\mu/R)^2 + k^2]$

and

the decay due to Ekman suction $\propto k$,

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where k is the total horizontal wavenumber and R is the Rossby radius of deformation. It follows that there is a possibility of a short-wave cut-off as $k \to \infty$. Also, if $(k^2 \gg \mu/R^2)$, then the growth rate will asymptote to a constant as $k \to 0$. It has been stated that this inequality is *usually* satisfied in the tropics and moreover that there is no preferred mode at an intermediate scale. However, we shall demonstrate that the pertinent inequality is not satisfied for certain, *perhaps rare but not unreasonable*, diabatic heating profiles. In this case the preferred mode that ensues has a large growth rate and an horizontal length scale of the order of 1000 km.

2. Model formulation

We consider linear axi-symmetric perturbations about a basic isothermal state of no motion, and furthermore make the hydrostatic and balance assumptions. Then in cylindrical polar coordinates our momentum equations take the form,

$$-fv' = -\frac{\partial}{\partial r} p^*, \tag{1}$$

$$\frac{\partial v'}{\partial t} + fu' = 0, \tag{2}$$

$$\left(\frac{\partial}{\partial z} - \frac{1}{g} N^2\right) p^* = g \frac{1}{\bar{\theta}} \theta', \tag{3}$$

and the continuity, thermodynamic equation and the equation of state can be written as,

$$\frac{1}{\bar{\rho}} \frac{\partial \rho'}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (ru') + \left(\frac{\partial}{\partial z} - \frac{1}{H}\right) w' = 0, \tag{4}$$

$$\frac{g}{\bar{\theta}} \frac{\partial \theta'}{\partial t} + N^2 w' = \frac{g}{(c_p \bar{T})} Q, \tag{5}$$

$$\frac{1}{\bar{\rho}} \rho' - (\gamma R \bar{T})^{-1} p^* = -\frac{1}{\bar{\theta}} \theta'. \tag{6}$$

Here the primed variables denote perturbations, the barred variables refer to the basic state, $p^* = (p'/\bar{\rho})$. The parameters N^2 and H represent respectively the square of the buoyancy frequency and the atmospheric scale height, and are assumed to take the constant values $1.25 \times 10^{-5}\ \mathrm{s}^{-2}$ and 7 km. The remaining variables assume their conventional meaning. The diabatic heating will be related to the vertical velocity (w_s') at the top of the boundary layer as follows,

$$\frac{g}{c_p \bar{T}} Q = m\eta(z) N^2 w_s', \tag{7}$$

where $\eta = \eta(z)$ is the form factor in the vertical for the diabatic heating due to cloud convection and m denotes the dimensionless amplitude of that heating.

Further we shall apply the Ekman relationship at the top of the surface boundary layer, and in our notation it takes the form,

$$w_s' = \frac{1}{f} \lambda \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} p_s^*\right), \tag{8}$$

Here $\lambda = \frac{1}{2} D_E \sin 2\alpha$, with D_E denoting the scale depth of the boundary layer and α the angle between the geostrophic wind and isobars at the surface.

Our system of equations reduces to the single equation

$$\frac{\partial}{\partial t} \left\{\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial p^*}{\partial r}\right) + f^2 \left(\frac{\partial}{\partial z} - \frac{1}{H}\right) \frac{1}{N^2} \frac{\partial p^*}{\partial z}\right\}$$
$$= mf^2 \left(\frac{\partial}{\partial z} - \frac{1}{H}\right) \eta w_s'. \tag{9}$$

This equation is separable and we seek solutions of the form,

$$p^* = e^{-i\sigma t} F(r)\, \chi(z)\, e^{z/2H}, \tag{10}$$

with the function $F(r)$ taking the form of the Bessel function $J_0(K^* r)$ since it satisfies the differential equation,

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dF}{dr}\right) = -K^{*2} F.$$

The vertical structure equation for $\chi(z)$ is

$$\frac{d^2 \chi}{d\tilde{z}^2} - K^2 \chi = -i(K^2/\sigma)\,(m\lambda f/H)\, \chi_s \left(\frac{\partial}{\partial \tilde{z}} - \frac{1}{2}\right) \tilde{\eta}. \tag{11}$$

Here the overbarred variables are given by $\tilde{z} = (z/H)$, $\tilde{\eta} = \eta e^{-\tilde{z}/2}$, and

$$K^2 = \tfrac{1}{4} + (HK^*/f)^2 N^2 \simeq (HK^*/f)^2 N^2. \tag{12}$$

Thus $K \simeq RK^*$, where $R (= NH/f)$ is the Rossby radius of deformation.

In the next section we shall solve Eq. (11) with $\eta = \eta(z)$ taking a specified non-zero form from the top of the boundary layer to the tropopause ($0 < \tilde{z} < 2$), and $\eta \equiv 0$ above this region. The procedure is to derive solutions for these regions subject to Eq. (8) as a lower boundary condition, an upper boundary conditions that $\chi(\tilde{z})$ decays with height as $z \to \infty$, and inter-region conditions that imply that the pressure and the vertical velocity are continuous at such a level.

3. Model solutions

The sensitivity of CISK solutions to the vertical partitioning of the diabatic heating was a central feature of the early study of Syono and Yamasaki (1966), and various related aspects have been further considered by subsequent workers, e.g. Koss (1976). The latter study delineated the desirability of an adequate vertical resolution to represent the heating profile, and gave a detailed account of the nature of the solutions with a relatively high resolution model for cases when the heating distribution was centred predominantly in the middle and upper troposphere. We shall examine this aforementioned sensitivity by considering two basic and complementary profiles for the form factor $\eta(z)$.

3.1. *Sinusoidal diabatic profile*

We assume $\tilde{\eta}$ takes the form,

$$\tilde{\eta} = \sin\left(n\frac{\pi}{2}\tilde{z}\right),$$

with $n = 1,2,3\ldots$ for $0 \leqslant \tilde{z} \leqslant 2$,

$$\tilde{\eta} = 0 \qquad \text{for } \tilde{z} \geqslant 2. \tag{13}$$

In general this is a highly unrealistic profile since it represents a succession of heat sources and sinks surmounting one another in the vertical (the $n = 1$ profile is the only exception). However it provides a useful insight to the behaviour of more realistic profiles inasmuch as realistic profiles can be constructed from the sum of functions of this form.

Proceeding as outlined in the previous section we obtain after some algebraic manipulation the following expression for the frequency,

$$\sigma = i(\lambda f/H)\,A\left\{-K + m(n\pi/2)\frac{K^2}{[(n\pi/2)^2 + K^2]}B_1\right\} \tag{14}$$

where

$$A = \left[1 - \frac{\frac{1}{2} - HN^2/g}{K}\right]^{-1}$$

$$B_1 = \left(1 - \frac{1}{2K}\right)[1 - (-1)^n e^{-2K}].$$

The frequency is seen to be purely complex and the growth/decay rate of the perturbations is determined primarily by the relative values of the two terms in the curly brackets in Eq. (14). A comparison of this expression with our remarks in the introduction reveals that the role of the Rossby radius of deformation (R) is modified by a factor (μ) proportional to the characteristic wavenumber (n) of the diabatic heating profile. Moreover, the growth rate due to diabatic effects is only positive if m is positive, and this requirement corresponds to heating in the lowest layer of the ($0 \leqslant \tilde{z} \leqslant 2$) region. Thus we assert, and indeed shall substantiate in the next sub-section, that for more general vertical heating profiles a small value of (R/μ) will be associated with a profile possessing a low-level maximum. The occurrence of a preferred mode and its dependence upon the

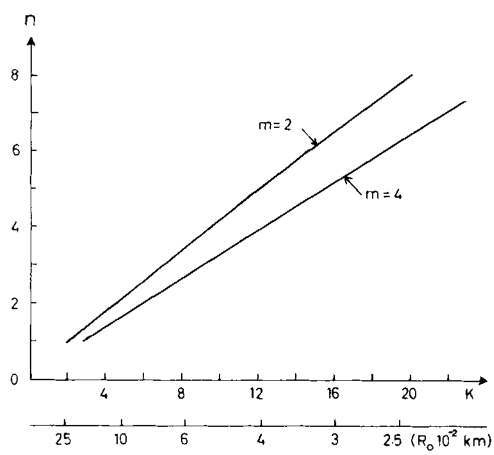

Fig. 1 The location of the most unstable mode displayed as a function of the pseudo-horizontal wavenumber (K) and pseudo-vertical wavenumber (n) of the diabatic heating profile for two values of the diabatic heating amplitude (m).

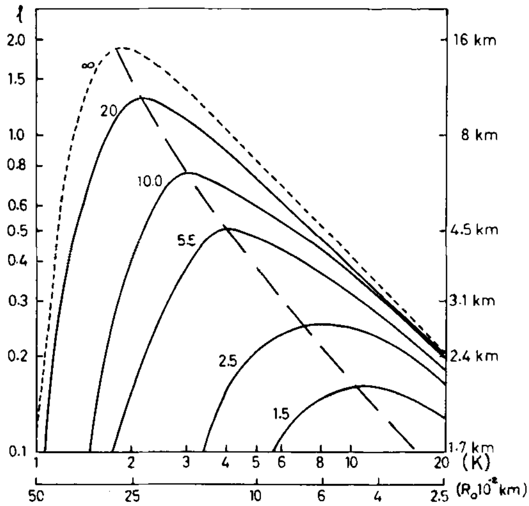

Fig. 2 The *e*-folding time (in days) as a function of the pseudo-horizontal wavenumber (K) and the non-dimensional height (l) of the maximum of the $\tilde{\eta}$ heating profile. The alternative ordinate scale on the right denotes the height above the ground.

vertical structure of the diabatic heating is displayed in Fig. 1 which shows the location of this mode in the (K, n) plane for two values of the diabatic heating amplitude (m). Here we have assumed $f = 0{\cdot}377 \times 10^{-4}\ \mathrm{s}^{-1}$ (corresponding to 15° latitude) and $\lambda = 187$ m. The alternative scale for the abscissae in the figure relates to the corresponding location of the first zero ($r = R_0$) of the $J_0(K^* r)$ function. We note that there is an order of magnitude decrease in this horizontal length scale as n increases from 1 to 8.

3.2. *A two-segment profile*

In this case the $\tilde{\eta}$ diabatic profile is composed of two segments, each representing a quarter of a sine wave such that,

$$\tilde{\eta} = \sin\left\{\frac{\pi}{2l}\,\tilde{z}\right\} \qquad \text{for } 0 \leqslant \tilde{z} \leqslant l$$

$$\tilde{\eta} = \sin\left\{\frac{\pi}{2(2-l)}\,(2-\tilde{z})\right\} \qquad \text{for } l \leqslant \tilde{z} \leqslant 2 \qquad (15)$$

and

$$\tilde{\eta} = 0 \qquad \text{for } \tilde{z} \geqslant 2.$$

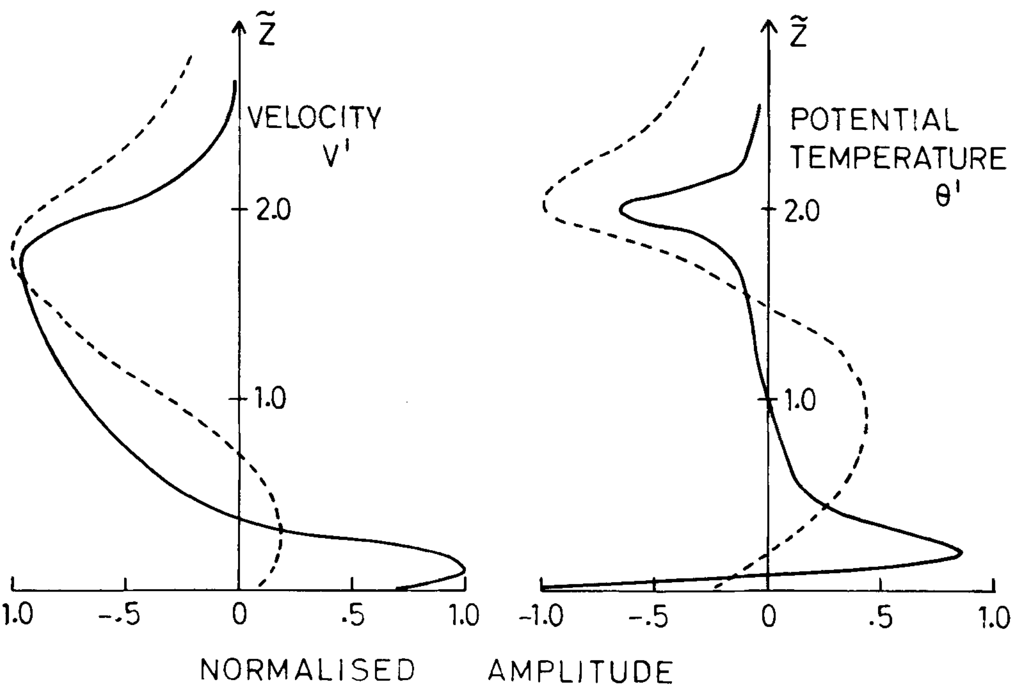

Fig. 3 The vertical structure of the amplitude of the perturbation azimuthal velocity field (v') and the perturbation potential temperature field (θ') shown for the cases $l = 0.25$ (solid lines) and $l = 1.0$ (dashed lines).

For this profile $\tilde{\eta}$ is a maximum at $\tilde{z} = l$, and there is a positive (negative) heating at all levels in the vertical below $\tilde{z} = 2$ when the Ekman suction velocity w_s' is positive (negative).

It can be deduced that the frequency relationship for this diabatic heating profile takes the form,

$$\sigma = i(\lambda f/H)\, A \left\{ -K + m(\pi/2l) \frac{K^2}{[(\pi/2l)^2 + K^2]} [B_2 + B_3] \right\} \tag{16}$$

where

$$B_2 = \left[(\pi/2l) \frac{1}{K} \left\langle 1 - \left(\frac{l}{2-l}\right)^2 \gamma \right\rangle + \tfrac{1}{2}(\pi/2l)^{-1} \langle 1 - \gamma \rangle \right] e^{-Kl},$$

$$B_3 = (1 - 1/2K) \left[1 + \left(\frac{l}{2-l}\right) \gamma e^{-2K} \right]$$

with

$$\gamma = [(\pi/2l)^2 + K^2]/[(\pi/2(2-l))^2 + K^2].$$

If we stipulate that, for a given Ekman suction, the net amount of diabatic heating must remain the same for different values of l, then m will become a function of l, such that,

$$m = (\delta q_s)\, gL/(N^2 C_p \bar{T} \mathscr{F}), \tag{17}$$

where

$$\mathscr{F} = 2H \left\{ \frac{\pi/l - e^{-l/2}}{[1 + (\pi/l)^2]} + \frac{\pi e^{-1}/(2-l) + e^{-l/2}}{[1 + \pi^2/(2-l)^2]} \right\}$$

and q_s denotes the specific humidity (assumed constant) in the boundary layer, and δ is the ratio of the total diabatic heat released in a column to that derived from the moisture pumped out of the Ekman layer. Observational studies (e.g. Cho and Ogura, 1974) suggest that in the environment of tropical wave disturbances $q_s \sim$ 12–20 gm/K gm and $\delta \lesssim$ (1–2). The other symbols have their conventional meaning.

In Fig. 2 the e-folding time in days derived from the growth rate σ as obtained from Eqs. (16) and (17), with $(\delta q_s) = 2.8 \times 10^{-2}$, and λ and f taking the values assigned previously, is shown as a function of l and K. The growth rate curves in the figure are in harmony with the inferences drawn in the previous sub-section. The e-folding time decreases as the location of the maximum diabatic heating is lowered (i.e. as l is decreased), and the changes are most pronounced as l tends to zero. In particular we note the small e-folding time ($\sim$2 days) of the most preferred modes for $l \leqslant 0.3$—corresponding to a maximum diabatic heating at, or below, the 3 km level. For these latter modes the adoption of an f-plane geometry is not unreasonable since the perturbation variables are constrained to exhibit the $(2/\pi K^* r)^{\frac{1}{2}}$ radial decay structure of Bessel functions.

If the value of (δq_s) is doubled, then the growth rates increase substantially but there is only a relatively slight change in the horizontal length scale of the preferred modes. On the other hand if (δq_s) is decreased substantially then the growth rates become very weak and the preferred modes for $l \ll 1$ shift to much smaller values of K.

The vertical structure of the most unstable modes corresponding to $l = 1$ and $l = 0.25$ for the case $(\delta q_s) = 2.8 \times 10^{-2}$ is compared in Fig. 3. Both modes possess a low level warm core and an upper level cold core structure, but the $l = 0.25$ mode has a much richer thermal structure. Also the low level heating maximum generates in relative terms a much larger low level azimuthal velocity maximum. This implies a much larger vertical gradient of relative vorticity in the troposphere. This latter feature is regarded as the distinctive feature of an incipient tropical cyclone or hurricane (Gray, 1979).

4. Further remarks

In the context of an Ekman-CISK model a low-level diabatic heating maximum has been shown to be conducive to the generation of a rapidly growing, tropical cyclone-scale mode of motion. Growth of this mode requires an almost saturated boundary layer and considerable enhancement of the diabatic heating due to moisture convergence in the free atmosphere as evidenced by the comparatively large value required for (δq_s). The desired heating profile with a maximum at, or below the 3 km ($\sim$700 mb) level may not be unusual (c.f. Reed et al., 1977), and the present study again underlines the requirement for a satisfactory theory for cumulus parameterization.

Our adoption of an unconditional heating scheme is of questionable validity. However, some arguments can be advanced to indicate its usefulness. Numerical experiments with crude vertical resolution undertaken by Syono and Yamasaki (1966) and Koss (1975) suggest that the results from an unconditional scheme are a useful guide to the behaviour of the conditional case. Again in a manner analogous to Lindzen's (1974) interpretation of a conditional heating due to a monochromatic oscillatory source in terms of its Fourier-decomposition, we can examine the Bessel-decomposition of, for instance, a monochromatic zero order Bessel function of the first kind that is defined in some limited radial domain. In this decomposition we also find that the original function from which the monochromatic source was derived has a substantial amplitude (~50% of the synthesis). Noteworthily this reduction in the effective amplitude of the CISK heating will influence the relative growth rates of the modes as indicated in the previous section. Thus we can infer merely that in the initial stages of the CISK instability the unconditional case will be a useful qualitative indicator of the behaviour of the corresponding conditional situation.

For $l \geqslant 1$ our results complement earlier studies of quasi-balanced CISK models. In this range the variation of the growth rate in general resembles that obtained by Koss (1976) and the remarks of Shukla (1978). Indeed the results obtained in section 3 indicating the role of the higher order vertical modes of the diabatic heating could perhaps be useful in interpreting some of the erratic variations noted in Koss' study. For $l \ll 1$ our analytical model with its continuous vertical structure also reveals unequivocally the pronounced changes in the growth rate, the preferred horizontal length scale, and the vertical structure of unstable modes for profiles with a low-level diabatic heating maximum. All these changes are in the sense to produce a more reasonable comparison with the observed structure of incipient tropical cyclones.

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О ПРЕДПОЧТИТЕЛЬНОЙ МОДЕ В СЛОЕ ЭКМАНА С УЧЕТОМ КИСК

Исследуется соотношение между вертикальным распределением неадиабатического нагревания и предпочтительной модой движения в слое Экмана с учетом КИСК (условной неустойчивости второго рода). В начале используется нереалистическое, но эвристически удобное синусоидальное распределение нагревания по вертикали для установления связи между скоростью нарастания неустойчивых мод, радиусом деформации Россби и профилем неадиабатического нагревания. Результаты анализа, дополненные результатами, полученными с более реалистическим распределением нагревания, указывают, что быстро растущие предпочтительные моды, имеющие масштаб тропических циклонов, возникают из профилей неадиабатического нагревания, имеющих максимум на нижнем уровне. Вертикальная структура такой моды обладает некоторыми реалистическими чертами. В частности она имеет больой вертикальный градиент относительного вихря по сравнению с модами, связанными максимумом неадиабатического нагревания в средней тропосфере.