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Steady states and stability properties of a low-order barotropic system with forcing and dissipation

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ABSTRACT

A low-order barotropic system with Newtonian forcing and dissipation is investigated. The system has one component in the zonal flow characterized by a Legendre function and two components in the eddy flow having the same longitudinal wave number, but different meridional indices and described by two associated Legendre functions. Five real amplitudes characterizing the system and a system of five ordinary, non-linear differential equations describe the behaviour of the system.

The steady states of the system can be determined by a numerical solution of a 9th degree equation for a given intensity of the forcing. Section 3 contains a description of the determination of the steady states and shows that up to five steady states may exist for a given forcing. The most complicated of the steady states have energy in all components of the system and the non-linear exchange of energy among the components is essential for maintaining the state. Other steady states are either zonal or contain energy in one component only.

A general procedure for the determination of the stability of a given steady state is outlined in Section 4 together with a determination of the stability of all steady states within a given parameter range in the forcing. It turns out that among the steady states which have energy in the wave components we find both stable and unstable configurations. It would thus appear that this result is in general agreement with the synoptic experience that some wave configurations can exist in a relatively unchanged form for a long time while others, presumably corresponding to the unstable configurations, will change rapidly. A close correspondence with real atmospheric conditions is impossible in a low-order system.

The final part of Section 4 gives the result of some numerical experiments with the simple system to illustrate that the type of initial disturbance given to a steady, unstable state will determine the asymptotic steady state in a case where two or more such states exist.

1. Introduction

The present paper reports on an investigation of a low-order, three-component, non-linear system based on the barotropic vorticity with forcing and dissipation. The main purposes of the investigation are to determine the non-linear steady states of the system and to explore the stability of these steady states. In addition, it will be possible to investigate numerically the behaviour of the system if it has an initial state close to an unstable steady state.

Most stability investigations of atmospheric flow have been limited to cases in which the basic flow is zonal with vertical and/or horizontal gradients. Many very useful results have been obtained from these studies. It is, however, of interest to conduct stability studies based on the more general situation where the basic state would have not only a zonal current but also a wave motion superimposed on this current. If basic states of this kind can be found it will be possible to investigate their stability properties in the usual way by superimposing infinitesimally small perturbations on the basic state and determining if these disturbances will grow or decline. The main difficulty with this approach is that it is normally very difficult to determine the basic steady states because such a state by definition has to satisfy the non-linear equations with all time derivatives equal to zero.

It has been instructive in many investigations to start with a relatively simple system which in some ways can be treated from an analytical point of

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view. The low-order systems proposed by Lorenz (1960) have been particularly useful. With maximum simplification it is even possible to find analytical solutions in terms of elliptic functions. A very detailed investigation of these solutions has been carried out by Pitcher (1974). These solutions are based on trigonometric representations on a plane but they can be carried over to equally simple systems on the sphere as shown by Baer (1970).

We shall adopt a so-called three-component system on the sphere in this investigation. Legendre and associated Legendre functions will be used to represent the various meteorological components. The three components will be (o, n), the zonal component, and (l, n_1), (l, n_2), the wave components having the same longitudinal wave number l and the meridional scales characterized by n_1 and n_2.

The system can be considered with and without forcing and dissipation. Most of this investigation is devoted to the former case because it permits the determination of a number of steady states which can be investigated for stability. It is thus possible to isolate the stable steady states which may be asymptotic states.

After formulation of the model we shall first determine the possible steady state. Each of these will be investigated for stability. We shall finally show some examples of long term integrations illustrating how the system will approach an asymptotic steady state.

2. The model

As a first model to explore steady states and their stability we shall adopt an equivalent barotropic model with forcing and dissipation. While it is relatively easy to formulate a dissipation mechanism based on the classical incorporation of the effect of the planetary boundary layer it is much more difficult to prescribe a realistic forcing because the main forcing in the atmosphere is through heat sources and sinks to be specified in the thermodynamic equation. Since thermodynamic effects cannot be incorporated directly in an equivalent barotropic model it is unavoidable that the forcing will be of an artificial nature. As we shall see later the adopted forcing will be of a Newtonian nature. It is known that a heating function of this kind depicts the gross aspects of atmospheric heating and since the mean temperature in the troposphere is closely related to the 500 mb height it is felt that the forcing should be a reasonable first approximation.

The second aspect of the model to be used here is its low order in the sense that only a few components will be included. If more components were permitted it would be difficult or impossible to determine the steady states. The non-linear nature of the model will, however, be preserved in the determination of the steady states.

It is convenient to work with non-dimensional quantities. With a as the radius of the earth and Ω its angular velocity we may write

$$\psi_* = \Omega a^2 \psi, \quad t_* = \Omega^{-1} t \tag{2.1}$$

where the asterisk denotes the dimensional quantity. With $\mu = \sin \phi$ where ϕ is latitude we may write the governing equation in the form

$$\frac{\partial \zeta}{\partial t} = J(\zeta, \psi) - 2\frac{\partial \psi}{\partial \lambda} + \beta(\psi - \psi_E) - \varepsilon\zeta \tag{2.2}$$

where $\zeta = \nabla^2 \psi$ is the vorticity, J the Jacobian with respect to λ, the longitude, and μ, ψ is the non-dimensional streamfunction, and ψ_E is a specified streamfunction in the forcing. The two coefficients β and ε are non-dimensional empirical factors measuring the intensity of the forcing and the dissipation, respectively. They are related to the dimensional quantities by the expressions

$$\beta_* = \Omega a^{-2} \beta, \quad \varepsilon_* = \Omega\varepsilon \tag{2.3}$$

The last term in (2.2) measures the dissipation which has been selected proportional to $\zeta = \nabla^2 \psi$. This formulation has been chosen because it depicts the way in which the influence of the planetary boundary layer normally is introduced in an equivalent barotropic model. It would have been possible to select a dissipation proportional to $\nabla^2 \zeta$ but it is believed that horizontal diffusion in the atmosphere is of less importance than the dissipation in the boundary layer.

The numerical values of β and ε will be discussed later. Equation (2.2) can be converted into a set of equations for the amplitudes of the associated Legendre polynomials when ψ, and thus ζ, is written as a series of these functions. The general aspects of this problem were considered in detail by Platzman (1960). In our case we shall include one zonal component described by the Legendre function $P_n(\mu)$. One longitudinal wave number l will be permitted in the wave components, but two meridional indices, n_1 and n_2, will be included. The wave components are thus described by the two associated Legendre functions $P^l_{n1}(\mu)$ and $P^l_{n2}(\mu)$. With these preliminary remarks we may write the streamfunction in the form

$$\begin{aligned}\psi(\lambda, \mu, t) = {} & \psi(n, t)\, P_n(\mu) + \psi_c(n_1, t)\, P^l_{n1}(\mu) \cos(l\lambda) \\ & + \psi_s(n_1, t)\, P^l_{n1}(\mu) \sin(l\lambda) \\ & + \psi_c(n_2, t)\, P^l_{n2}(\mu) \cos(l\lambda) \\ & + \psi_s(n_2, t)\, P^l_{n2}(\mu) \sin(l\lambda)\end{aligned} \tag{2.4}$$

The expression (2.4) is inserted in (2.2) and we derive 5 equations for the rate of change of the coefficients. These equations are:

$$\begin{aligned}
\frac{d\psi(n)}{dt} = {} & \alpha(\psi_c(n_1)\,\psi_s(n_2) - \psi_s(n_1)\,\psi_c(n_2)) \\
& - \left(\varepsilon + \frac{\beta}{c}\right)\psi(n) + \frac{\beta}{c}\,\psi_s(n_1) \\
\frac{d\psi_c(n_1)}{dt} = {} & \gamma_1\,\psi(n)\,\psi_s(n_2) + \frac{l}{c_1}\,\psi_s(n_2) \\
& - \left(\varepsilon + \frac{\beta}{c_1}\right)\psi_c(n_1) + \frac{\beta}{c_1}\,\psi_{EC}(n_1) \\
\frac{d\psi_s(n_1)}{dt} = {} & -\gamma_1\,\psi(n)\,\psi_c(n_2) - \frac{l}{c_1}\,\psi_c(n_1) \\
& - \left(\varepsilon + \frac{\beta}{c_1}\right)\psi_s(n_1) + \frac{\beta}{c_1}\,\psi_{ES}(n_1) \\
\frac{d\psi_c(n_2)}{dt} = {} & \gamma_2\,\psi(n)\,\psi_s(n_1) + \frac{l}{c_2}\,\psi_s(n_2) \\
& - \left(\varepsilon + \frac{\beta}{c_2}\right)\psi_c(n_2) + \frac{\beta}{c_2}\,\psi_{EC}(n_2) \\
\frac{d\psi_s(n_2)}{dt} = {} & -\gamma_2\,\psi(n)\,\psi_c(n_1) - \frac{l}{c_2}\,\psi_c(n_2) \\
& - \left(\varepsilon + \frac{\beta}{c_2}\right)\psi_s(n_2) + \frac{\beta}{c_2}\,\psi_{ES}(n_2)
\end{aligned} \tag{2.5}$$

The following notations have been introduced in (2.5)

$$c = n(n+1), \quad c_1 = n_1(n_1+1), \quad c_2 = n_2(n_2+1)$$

$$I = I(n_1, n_2, n) = \int_{-1}^{-1} P^l_{n1} \cdot P^l_{n2} \frac{dP_n}{d\mu}\, d\mu \tag{2.6}$$

$$\alpha = \tfrac{1}{4} l \frac{c_1 - c_2}{c} I, \quad \gamma_1 = \tfrac{1}{4} l \frac{c_2 - c}{c_1} I,$$

$$\gamma_2 = \tfrac{1}{4} l \frac{c_1 - c}{c_2} I$$

Among these quantities we note that I is the interaction integral among the three components in the system. The quantities α, γ_1 and γ_2 are important for the energy and enstrophy properties of the system. These properties have been investigated in detail by Platzman (1960), by Baer and Platzman (1961) and by Baer (1970) where the last investigation deals with the highly truncated spectral vorticity equation. A similar analysis has been carried out by Galin and Kurbatkin (1975).

The system (2.5) can naturally be integrated numerically. In such integrations it may be convenient to keep track of the energetics of the system. The kinetic energy of the three components of the system is

$$\begin{aligned}K(n) &= \tfrac{1}{2} c\, \psi(n)^2 \\ K(n_1) &= \tfrac{1}{2} c_1 (\psi_c(n_1)^2 + \psi_s(n_1)^2) \\ K(n_2) &= \tfrac{1}{2} c_2 (\psi_c(n_2)^2 + \psi_s(n_2)^2)\end{aligned} \tag{2.7}$$

Using these definitions it is possible to derive the three energy equations which may be written in the following symbolic form

$$\begin{aligned}\frac{dK(n)}{dt} &= KEX(n) + G(n) - D(n) \\ \frac{dK(n_1)}{dt} &= KEX(n_1) + G(n_1) - D(n_1) \\ \frac{dK(n_2)}{\mathrm{d}t} &= KEX(n_2) + G(n_2) - D(n_2)\end{aligned} \tag{2.8}$$

where KEX is the symbol for the kinetic energy exchange, G the generation due to the forcing and D the dissipation. For the sake of completeness we list all of these quantities:

$$G(n) = \beta\psi(n)\,(\psi_E(n) - \psi(n))$$

$$\begin{aligned}G(n_1) = {} & \beta(\psi_c(n_1)\,(\psi_{EC}(n_1) - \psi_C(n_1)) \\ & + \psi_S(n_1)\,(\psi_{ES}(n_1) - \psi_S(n_1)))\end{aligned}$$

$$\begin{aligned}G(n_2) = {} & \beta(\psi_c(n_2)\,(\psi_{EC}(n_2) - \psi_C(n_2)) \\ & + \psi_S(n_2)\,(\psi_{ES}(n_2) - \psi_S(n_2)))\end{aligned}$$

$$D(n) = \varepsilon c \psi(n)^2$$

$$D(n_1) = \varepsilon c_1 (\psi_c(n_1)^2 + \psi_s(n_1)^2) \tag{2.9}$$

$$D(n_2) = \varepsilon c_2 (\psi_c(n_2)^2 + \psi_s(n_2)^2)$$

$$KEX(n) = \alpha c \psi(n) \cdot B$$

$$KEX(n_1) = \gamma_1 c_1 \psi(n) \cdot B; \quad B = \psi_c(n_1) \psi_s(n_2) - \psi_s(n_1) \psi_c(n_2)$$

$$KEX(n_2) = -\gamma_2 c_2 \psi(n) \cdot B$$

We note that the sum of the three values of KEX vanishes because

$$\alpha c + \gamma_1 c_1 - \gamma_2 c_2 = 0 \tag{2.10}$$

The above statement expresses the well known fact that the total kinetic energy of the three components is constant if there is no generation ($\beta = 0$) and no dissipation ($\varepsilon = 0$). It is equally straightforward to derive the three equations for the rate of change of the enstrophy of the three components. The definitions are:

$$E(n) = \tfrac{1}{2} c^2 \psi(n)^2$$

$$E(n_1) = \tfrac{1}{2} c_1^2 (\psi_c(n_1)^2 + \psi_s(n_1)^2) \tag{2.11}$$

$$E(n_2) = \tfrac{1}{2} c_2^2 (\psi_c(n_2)^2 + \psi_s(n_2)^2)$$

and it is found that

$$\frac{dE(n)}{dt} = cKEX(n) + cG(n) - cD(n) \tag{2.12}$$

with corresponding expressions for the other two components (n_1 and n_2). In the absence of forcing and dissipation we find that the total enstrophy is conserved because

$$c^2 \alpha + c_1^2 \gamma_1 - c_2^2 \gamma_2 = 0 \tag{2.13}$$

3. Steady states

In this section we shall do some preparatory work to permit a stability analysis to be carried out in a later section. Most stability analyses of the barotropic type have been carried out using a basic state which is a zonal current. Important criteria for the instability of the westerlies have been derived in this way, notably by Kuo (1949).

The system described in Section 2 has both forcing and dissipation. It is therefore likely that it will possess one or several steady states. The forcing is determined by the five components of ψ_E, i.e. $\psi_E(n)$, $\psi_{EC}(n_1)$, $\psi_{ES}(n_1)$, $\psi_{EC}(n_2)$ and $\psi_{ES}(n_2)$. In addition to these values which imply a choice of l, n, n_1 and n_2 we must also select values of β and ε. Provided all steady states can be determined by some procedure we are therefore in a position to investigate the stability of each of the steady states. Since some of these steady states will have both zonal current and a wave characterized by l, n_1 and n_2 we can investigate if some wave configurations are stable or unstable to small perturbations.

A steady state is defined as a solution to the system (2.5) if all time derivatives are zero, i.e. $d/dt = 0$ for all five variables. Under these conditions we find five equations from (2.5) and the unknowns are $\psi(n)$, $\psi_c(n_1)$, $\psi_s(n_1)$, $\psi_c(n_2)$ and $\psi_s(n_2)$. The last four equations forming a set of four linear inhomogeneous equations in the last four unknowns can be solved for $\psi_c(n_1)$, $\psi_s(n_1)$, $\psi_c(n_2)$ and $\psi_s(n_2)$ in terms of $\psi(n)$ and the other parameters in the problem. When these solutions are substituted in the first equation we obtain a single equation in the unknown $\psi(n)$. It turns out that this final equation is of the 9th degree and it must thus be solved by numerical methods. Only the solutions resulting in a real value of $\psi(n)$ have a physical meaning. To express the results of these lengthy calculations it is necessary to introduce some notations. They are:

$$p = \varepsilon + \frac{\beta}{c}, \quad p_1 = \varepsilon + \frac{\beta}{c_1}, \quad p_2 = \varepsilon + \frac{\beta}{c_2},$$

$$q_1 = \frac{l}{c_1}, \quad q_2 = \frac{l}{c_2}$$

$$R = p_1 p_2 - q_1 q_2, \quad S = \gamma_1 \gamma_2, \quad Q = p_1 q_2 + p_1 q_1$$

$$A_1 = \frac{\beta}{c_1} (p_2 \psi_{EC}(n_1) - q_2 \psi_{ES}(n_1))$$

$$A_2 = \frac{\beta}{c_2} (p_1 \psi_{EC}(n_2) - q_1 \psi_{ES}(n_2))$$

$$B_1 = \gamma_1 \frac{\beta}{c_2} \psi_{ES}(n_2) \tag{3.1}$$

$$B_2 = \gamma_2 \frac{\beta}{c_1} \psi_{ES}(n_1)$$

$$D_1 = \frac{\beta}{c_1}(p_2\, \psi_{ES}(n_1) + q_2\, \psi_{EC}(n_1))$$

$$D_2 = \frac{\beta}{c_2}(p_1\, \psi_{ES}(n_2) + q_1\, \psi_{EC}(n_2))$$

$$E_1 = \gamma_1 \frac{\beta}{c_2}\, \psi_{EC}(n_2)$$

$$E_2 = \gamma_2 \frac{\beta}{c_1}\, \psi_{EC}(n_1)$$

The steady state solutions will in the following be denoted by a bar and we will in particular set $\overline{\psi(n)} = X$. The final equation for X is:

$$\begin{aligned}
&-pS^4 X^9 + \frac{\beta}{c}\psi_E(n)\, S^4 X^8 - p4RS^3 X^7 \\
&+\left(\frac{\beta}{c}\psi_E(n)\, 4RS^3 + \alpha S^2(B_2 E_1 - B_1 E_2)\right) X^6 \\
&+ (\alpha S^2(B_1 D_2 - B_2 D_1 + A_2 E_1 - A_1 E_2) \\
&- p2S^2(3R^2 + Q^2))\, X^5 + \left(\alpha(2RS(B_2 E_1\right. \\
&- B_1 E_2) + S^2(A_1 D_2 - A_2 D_1)) \\
&\left.+ \frac{\beta}{c}\psi_E(n)\, 2S^2(3R^2 + Q^2)\right) X^4 + (\alpha(2RS(B_1 D_2 \\
&- B_2 D_1 + A_2 E_1 - A_1 E_2) - p4RS(R^2 + Q^2))\, X^3 \\
&+\left(\alpha(2RS(A_1 D_2 - A_2 D_1)\right. \\
&+ (Q^2 + R^2)(B_2 E_1 - B_1 E_2) \\
&\left.+ \frac{\beta}{c}\psi_E(n)\, 4RS(R^2 + Q^2)\right) X^2 \\
&+ (\alpha(R^2 + Q^2)(B_1 D_2 - B_2 D_1 \\
&+ A_2 E_1 - A_1 E_2) - p(R^2 + Q^2)^2)\, X \\
&+\left(\alpha(R^2 + Q^2)(A_1 D_2 - A_2 D_1)\right. \\
&\left.+ \frac{\beta}{c}\psi_E(n)\, (R^2 + Q^2)^2\right) = 0
\end{aligned} \tag{3.2}$$

When (3.2) has been solved by numerical methods for $X = \overline{\psi(n)}$ we can for each value of X obtain the corresponding values of the remaining steady state parameters. With the notation

$$N = S^2 X^4 + 2RSX^2 + R^2 + Q^2 \tag{3.3}$$

it is found that the remaining values are:

$$\begin{aligned}
\overline{\psi_c(n_1)} &= N^{-1}((A_1 + B_1 X)(R + SX^2) \\
&\quad + Q(D_1 - E_1 X)) \\
\overline{\psi_s(n_1)} &= N^{-1}((D_1 - E_1 X)(R + SX^2) \\
&\quad - Q(A_1 + B_1 X)) \\
\overline{\psi_c(n_2)} &= N^{-1}((A_2 + B_2 X)(R + SX^2) \\
&\quad + Q(D_2 - E_2 X)) \\
\overline{\psi_s(n_2)} &= N^{-1}((D_2 - E_2 X)(R + SX^2) \\
&\quad - Q(A_2 + B_2 X))
\end{aligned} \tag{3.4}$$

Equations (3.2) and (3.4) are the general equations for the steady states for arbitrary forcing. It is interesting to consider a number of special cases. The most simple case is the one where the forcing is entirely in the zonal motion. If so we have $\psi_E(n) \neq 0$, but $A_1 = B_1 = D_1 = E_1 = A_2 = B_2 = D_2 = E_2 = 0$. Under these conditions we find that (3.2) becomes much simpler. It turns out that we may write (3.2) in the form

$$\left(pX - \frac{\beta}{c}\psi_E(n)\right) \cdot (S^2 X^4 + 2RSX^2 + R^2 + Q^2)^2 = 0 \tag{3.5}$$

It is easily seen that this equation has one and only one real root, i.e.

$$X = \overline{\psi(n)} = \frac{1}{p}\frac{\beta}{c}\psi_E(n) = \frac{\beta}{\varepsilon c + \beta}\psi_E(n) \tag{3.6}$$

Equation (3.4) shows that all the wave amplitudes are zero in this case, and (3.6) expresses therefore the complete steady state in the example.

In our next example we shall assume that the forcing is on the component (l, n_1). We may assume that $\psi_{EC}(n_1) \neq 0$ while all the other components of the forcing are zero. In this case we find from the definitions in (3.1) that

$$\begin{aligned}
A_1 &= \frac{\beta}{c_1} p_2\, \psi_{EC}(n_1) & A_2 &= 0 \\
B_1 &= 0 & B_2 &= 0 \\
D_1 &= \frac{\beta}{c_1} q_2\, \psi_{EC}(n_1) & D_2 &= 0 \\
E_1 &= 0 & E_2 &= \frac{\beta}{c_1}\gamma_2\, \psi_{EC}(n_1)
\end{aligned} \tag{3.7}$$

The general equation for X (3.2) can in this case be written in the form

$$X(S^2X^4 + 2RSX^2 + R^2 + Q^2)\left(p(S^2X^4 + 2RSX^2 + R^2 + Q^2) + \alpha\left(\frac{\beta}{c_1}\right)^2 \gamma_2 p_2 \psi_{EC}(n_1)^2\right) = 0 \tag{3.8}$$

The only roots which may be real are

$$X = 0; \quad X = \pm\left(-\frac{R}{S} \pm \frac{1}{S}\left\{-\alpha\gamma_2\frac{p_2}{p}\left(\frac{\beta}{c_1}\right)^2 \psi_{EC}(n_1) - Q^2\right\}^{1/2}\right)^{1/2} \tag{3.9}$$

Considering first $X = \overline{\psi(n)} = 0$ we may calculate from (3.4) that

$$\overline{\psi_c(n_1)} = \frac{p_1}{p_1^2 + q_1^2}\frac{\beta}{c_1}\psi_{EC}(n_1);$$
$$\overline{\psi_s(n_1)} = -\frac{q_1}{p_1^2 + q_1^2}\frac{\beta}{c_1}\psi_{EC}(n_1) \tag{3.10}$$

while (3.4) combined with (3.7) also shows that $\overline{\psi_c(n_2)} = \overline{\psi_s(n_2)} = 0$. $X = 0$ corresponds therefore to a situation in which the response is on the same scale as the forcing, i.e. (l, n_1). Equation (3.10) shows that the ridge in the response is less than a quarter of wave length to the west of the ridge in $\psi_{EC}(n_1)\cos(l\lambda)$. On the other hand, since the forcing is proportional to the difference between the given field and the response field we find that the forcing is

$$\frac{\beta}{c_1}\psi_{EC}(n_1)\left\{\left(1 - \frac{p_1}{p_1^2 + q_1^2}\frac{\beta}{c_1}\right)\cos(l\lambda) + \frac{q_1}{p_1^2 + q_1^2}\frac{\beta}{c_1}\sin(l\lambda)\right\} \tag{3.11}$$

which shows that the ridge in the forcing is less than a quarter of a wavelength to the east of the ridge in $\psi_{EC}(n_1)\cos(l\lambda)$. The final result is therefore that the ridge in the response field is to the west of the ridge in the forcing field or, in other words, that a trough in the response field is created to the east of the maximum forcing.

We consider next the second possibility in (3.9). These roots will represent a steady state only if they are real. We may say immediately that X can be real only if $\alpha\gamma_2 < 0$. It is easy to see that this condition is equivalent to $c < c_1 < c_2$ or $n < n_1 < n_2$. For these steady states to exist it is thus necessary that the forcing takes place on the middle scale among the three scales included in the model. This condition is, however, not sufficient because we must also require that $\psi_{EC}(n_1)$ is sufficiently large so that the radicand in the two square roots in (3.9) will be positive. It is seen that a necessary condition is

$$\psi_{EC}(n_1)^2 > \frac{Q^2}{(-\alpha\gamma_2)\left(\frac{\beta}{c_1}\right)^2\frac{p_2}{p}} \tag{3.12}$$

If $R > 0$, it is required that

$$\psi_{EC}(n_1)^2 > \frac{R^2 + Q^2}{(-\alpha\gamma_2)\left(\frac{\beta}{c_1}\right)^2\frac{p_2}{p}} \tag{3.13}$$

in which case there will be a total of three steady states. If $R < 0$ there will be five steady states if $\psi_{EC}(n_1)^2$ is between the values given in (3.12) and (3.13).

It is of interest to discuss the energetics of the steady states. We note first of all that if the forcing is solely on the zonal component we find that only $G(n)$ and $K(n)$ are different from zero. Correspondingly, if the forcing is in the component (l, n_1), and if we consider the steady state for which $X = \overline{\psi(n)} = 0$ we find that only $G(n_1)$ and $D(n_1)$ are different from zero. However, if we consider any other steady state with forcing on the component (l, n_1) we will get a more complicated energy diagram where all quantities are different from zero. To illustrate this point we have calculated the energetics in an example. We have selected a case where $l = 3$, $n = 2$, $n_1 = 3$ and $n_2 = 4$. In addition, $\varepsilon = 0.04$ and $\beta = 0.5$. One of the steady states will then have the following values:

$$\begin{aligned}
\overline{\psi(n)} &= -0.0666\\
\overline{\psi_c(n_1)} &= +0.0671\\
\overline{\psi_s(n_1)} &= -0.0822\\
\overline{\psi_c(n_2)} &= 0.0386\\
\overline{\psi_s(n_2)} &= -0.0200
\end{aligned} \tag{3.14}$$

when $\psi_{EC}(n_1) = 0.5$. The energetics were then computed using the formulas given in (2.9). The results are shown in the following diagram which shows that $G(n_1)$ provides the energy for the

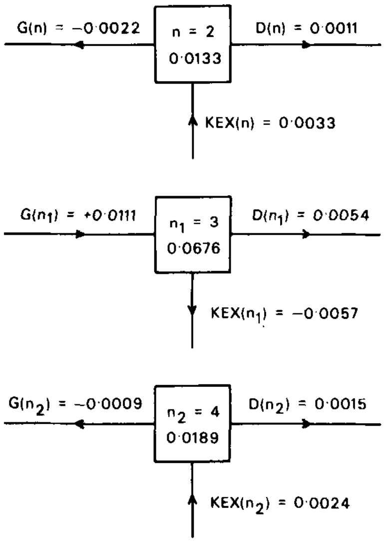

Fig. 1. The energy diagram for a steady state forced by $\psi_{EC}(3) = 0.5$. The components in the system are P_2^0, P_3^3 and P_4^3. Parameters: $\beta = 0.5$, $\varepsilon = 0.04$. It is seen that the energies on the components P_2^0 and P_4^3 are maintained by a non-linear transfer of energy from the component P_3^3, while the components P_2^0 and P_4^3 dissipate energy through the Newtonian forcing and the frictional dissipation.

system. A part of the energy is dissipated through $D(n_1)$ but the other part leaves the component (l, n_1) through the non-linear exchange process $KEX(n_1)$. The greater part of $KEX(n_1)$ is received by the component (o, n) through $KEX(n)$ which in the steady state provides the only energy input to (o, n) which is depleted by both $G(n)$ and $D(n)$. The component (l, n_2) receives energy through the non-linear exchange $KEX(n_2)$ but is depleted by $G(n_2)$ and $D(n_2)$. The diagram shows therefore an example of a three-component system which is in a steady state and when the non-linear exchange processes play an important role in the maintenance of the steady state.

The example above was calculated for $\psi_{EC}(n_1) = 0.5$. We shall next explore the number of steady states as a function of $\psi_{EC}(n_1)$. In an example we had $n = 2$, $l = 3$, $n_1 = 3$ and $n_2 = 4$. Using again $\varepsilon = 0.04$ and $\beta = 0.5$ we find that $R < 0$. According to the analysis resulting in (3.12) and (3.13) we have then the result that one steady state will exist if $\psi_{EC}(n_1) < 0.44$, five steady states are present for $0.44 < \psi_{EC}(n_1) < 0.67$ and three steady states exist when $\psi_{EC}(n_1) > 0.67$. These results have been confirmed by a calculation of the steady state for selected values of $\psi_{EC}(n_1)$ as we shall see in the next section.

4. Stability analysis

The steady states of the system derived in the last section may be stable, neutral or unstable. In this section we shall investigate the stability of the steady states to small perturbations. We shall first consider the general problem. In this case we have as the basic state any steady state computed by the procedures given in Section 3. In addition to the parameters $(\varepsilon, \beta, l, n, n_1$ and $n_2)$ we have the amplitudes $\overline{\psi(n)}$, $\overline{\psi_c(n_1)}$, $\overline{\psi_s(n_1)}$, $\overline{\psi_c(n_2)}$ and $\overline{\psi_s(n_2)}$ of the components included in the system. The perturbation equations are obtained from (2.5). Denoting the perturbation quantities by a prime we get:

$$\frac{d\psi^1(n)}{dt} = \alpha(\overline{\psi_c(n_1)}\,\psi_s^1(n_2) + \overline{\psi_s(n_2)}\,\psi_c^1(n_1) - \overline{\psi_s(n_1)}\,\psi_c^1(n_2) - \overline{\psi_c(n_2)}\,\psi_s^1(n_1)) - p\psi^1(n) \tag{4.1}$$

$$\frac{d\psi_c^1(n_1)}{dt} = \gamma_1(\overline{\psi(n)}\,\psi_s^1(n_2) + \overline{\psi_s(n_2)}\,\psi^1(n)) + \frac{l}{c_1}\psi_s^1(n_1) - p_1\psi_c^1(n_1)$$

$$\frac{d\psi s^1(n_1)}{dt} = -\gamma_1(\overline{\psi(n)}\,\psi_c^1(n_2) + \overline{\psi_c(n_2)}\,\psi^1(n)) - \frac{l}{c_1}\psi_c^1(n_1) - p_1\psi_s^1(n_1)$$

$$\frac{d\psi_c^1(n_2)}{dt} = \gamma_2(\overline{\psi(n)}\,\psi_s^1(n_1) + \overline{\psi_s(n_1)}\,\psi^1(n)) + \frac{l}{c_2}\psi_s^1(n_2) - p_2\psi_c^1(n_2)$$

$$\frac{d\psi_s^1(n_2)}{dt} = -\gamma_2(\overline{\psi(n)}\,\psi_c^1(n_1) + \overline{\psi_c(n_1)}\,\psi^1(n)) - \frac{l}{c_2}\psi_c^1(n_2) - p_2\psi_s^1(n_2)$$

To perform the stability analysis we look for solutions to (4.1) of the type $\hat{\psi}\exp(\omega t)$. A given steady state will thus be unstable if the real part of ω is positive and stable if it is negative. It is seen that this procedure leads to a normal eigenvalue problem of the form

$$\{A\} - \omega\{I\} = 0 \tag{4.2}$$

where $\{I\}$ is the identity matrix and $\{A\}$ is the following matrix

$$\{A\} = \left\{\begin{array}{ccccc} -p & \alpha\overline{\psi_s(n_2)} & -\alpha\overline{\psi_c(n_2)} & -\alpha\overline{\psi_s(n_1)} & \alpha\overline{\psi_c(n_1)} \\ \gamma_1\overline{\psi_s(n_2)} & -p_1 & l/c_1 & 0 & \gamma_1\overline{\psi(n)} \\ -\gamma_1\overline{\psi_c(n_2)} & -l/c_1 & -p_1 & -\gamma_1\overline{\psi(n)} & 0 \\ \gamma_2\overline{\psi_s(n_1)} & 0 & \gamma_2\overline{\psi(n)} & -p_2 & l/c_2 \\ -\gamma_2\overline{\psi_c(n_1)} & -\gamma_2\overline{\psi(n)} & 0 & -l/c_2 & -p_2 \end{array}\right\} \tag{4.3}$$

In the general case where all of the steady state amplitudes are different from zero it is most convenient to solve the eigenvalue problem by numerical methods. It is for this reason that it has been preferred to express (4.1) in the real domain. Another procedure which can be used in special cases is to use the complex domain by defining

$$\psi^1(n_1) = \psi_c^1(n_1) + i\psi_s^1(n_1) \qquad \overline{\psi(n_1)} = \overline{\psi_c(n_1)} + i\overline{\psi_s(n)}$$
$$\tag{4.4}$$
$$\psi^1(n_2) = \psi_c^1(n_2) + i\psi_s^1(n_2) \qquad \overline{\psi(n_2)} = \overline{\psi_c(n_2)} + i\overline{\psi_s(n_2)}$$

Using (4.4) we may write the last four equations in (4.1) in the form:

$$\frac{d\psi^1(n_1)}{dt} = -i\gamma_1\overline{\psi(n)}\,\psi^1(n_2) - i\gamma_1\overline{\psi(n_2)}\,\psi^1(n)$$
$$-i\frac{l}{c_1}\psi^1(n_1) - p_1\psi^1(n_1)$$
$$\tag{4.5}$$
$$\frac{d\psi^1(n_2)}{dt} = -i\gamma_2\overline{\psi(n)}\,\psi^1(n_1) - i\gamma_2\overline{\psi(n_1)}\,\psi^1(n)$$
$$-i\frac{l}{c_2}\psi^1(n_2) - p_2\psi^1(n_2)$$

Equation (4.5) is used to consider the important special case where the steady state is characterized by $\overline{\psi(n)} \neq 0$ while all the other amplitudes in the steady state are zero. In this case we see from the first equation in (4.1) that one value of ω is $-p$ while the remaining values are determined from (4.5). Using perturbations of the same type as before we get from (4.5):

$$(p_1 + iq_1 + \omega)\,\hat{\psi}(n_1) + i\gamma_1\,\bar{\psi}(n)\,\hat{\psi}(n_2) = 0$$
$$\tag{4.6}$$
$$i\gamma_2\,\bar{\psi}(n)\,\hat{\psi}(n_1) + (p_2 + iq_2 + \omega)\,\hat{\psi}(n_2) = 0$$

The frequency equation is obtained from (4.6) by noting that the determinant must vanish in order to get non-trivial solutions. It is seen that this procedure leads to a quadratic equation in ω with complex coefficients. The steady state $\overline{\psi(n)} \neq 0$ will be unstable if the equation has a solution in which the real part of ω is positive. The solution may be written in the form

$$\omega = \tfrac{1}{2}\left(-(p_1 + p_2) - i(q_1 + q_2) \pm D^{1/2}\right) \tag{4.7}$$

where

$$D = (p_1 - p_2)^2 - (q_1 - q_2)^2 - 4\gamma_1\gamma_2\overline{\psi(n)}^2$$
$$+ i2(p_1 - p_2)(q_1 - q_2) \tag{4.8}$$

Let us write the discreminant in the form

$$D = \rho(\cos\theta + i\sin\theta) \tag{4.9}$$

leading to

$$D^{1/2} = \rho^{1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \tag{4.10}$$

It is then seen that the steady state will be unstable if

$$\rho\cos^2\left(\frac{\theta}{2}\right) > (p_1 + p_2)^2 \tag{4.11}$$

Equation (4.11) can be evaluated using the definition (4.9) which together with (4.8) determines ρ and $\cos\theta$. After considerable reductions it turns out that (4.11) leads to the inequality

$$\overline{\psi(n)}^2 > \frac{p_1 p_2}{(-\gamma_1\gamma_2)}\left(1 + \frac{l^2}{c_1^2 c_2^2}\frac{(c_2 - c_1)^2}{(p_2 + p_1)^2}\right) \tag{4.12}$$

where it has been assumed that $\gamma_1\gamma_2 < 0$. It is easy to show that if $\gamma_1\gamma_2 > 0$ then the basic state is always stable. Equation (4.12) shows that the basic

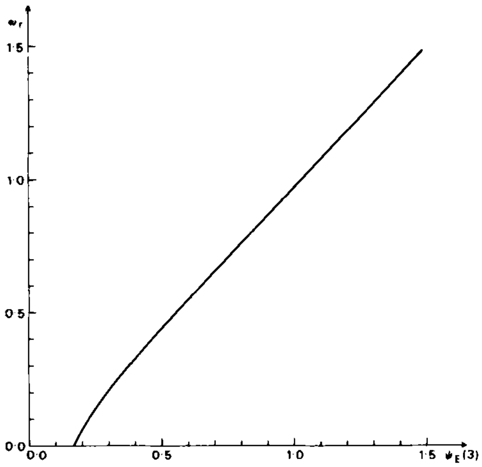

Fig. 2. The degree of instability, measured by ω_r, as a function of the forcing $\psi_{EC}(3)$, i.e. zonal forcing. The components are P_2^2, P_3^0 and P_4^2. $\beta = 0.5$, $\varepsilon = 0.04$.

state is unstable for small perturbations if $|\bar{\psi}(n)|$ is sufficiently large. Since $\overline{\psi(n)}$ is closely related to $\psi_E(n)$ according to (3.6) we find that a sufficiently strong forcing will create instability. Fig. 2 shows ω_r as a function of $\psi_E(n)$ for the case where the three components are (0, 3), (2, 2) and (2, 4), $\beta = 0.5$, $\varepsilon = 0.04$ and $\psi_E(3) \neq 0$, while $\psi_{EC}(2) = \psi_{ES}(2) = \psi_{EC}(4) = \psi_{ES}(4) = 0$. It is seen that ω_r increases almost linearly with $\psi_E(3)$.

The next example has again $\beta = 0.5$ and $\varepsilon = 0.04$. The components are (2, 2), (2, 3) and (0, 4), and it is assumed that $\psi_{EC}(3) \neq 0$ while $\psi_E(4) = \psi_{ES}(3) = \psi_{EC}(2) = \psi_{ES}(2) = 0$. In this case, which is analogous to the case considered in Section 3, there are either 1, 3 or 5 steady states depending on the value of $\psi_{EC}(3)$. It is found that 1 steady state is present when $0 < \psi_{EC}(3) < 0.6$, 5 steady states exist when $0.7 < \psi_{EC}(3) < 0.9$ and 3 steady states will be possible when $1.0 < \psi_{EC}(3) < 1.5$. In the first case (1 steady state) it is found from the stability analysis that this state is always stable. The next case having 5 steady states is characterized by 2 unstable and 3 stable steady states. Finally, in the third case with 3 steady states we find that one state is unstable while 2 of them are stable.

Fig. 3 shows ω_r as a function of $\psi_{EC}(3)$. The minimum between 0.9 and 1.0 in ω_r coincides with the transition from 5 to 3 steady states. It is to be expected that the instability mechanisms are different in the two regimes. This can be demonstrated by considering the unstable steady states in the two cases. In the first case (5 steady states of which 2 are unstable) we find that the unstable steady states are composed of all three components included in the systems while the second case (3 steady states of which 1 is unstable) has energy on the middle scale only. These cases are illustrated in Fig. 4 which shows the energetics for $\psi_{EC}(3) = 0.8$ and $\psi_{EC}(3) = 1.5$.

The arrows for each box have the following interpretations. $G(n)$ is on the left side, $D(n)$ on the right side, while $KEX(n)$ is below the box. In the

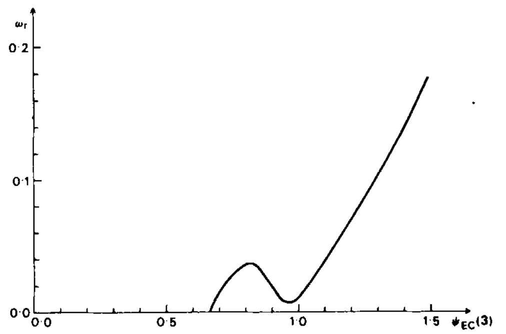

Fig. 3. The degree of instability, measured by ω_r, as a function of the forcing $\psi_{EC}(3)$, i.e. wave forcing. The components are P_2^2, P_3^2 and P_4^0.

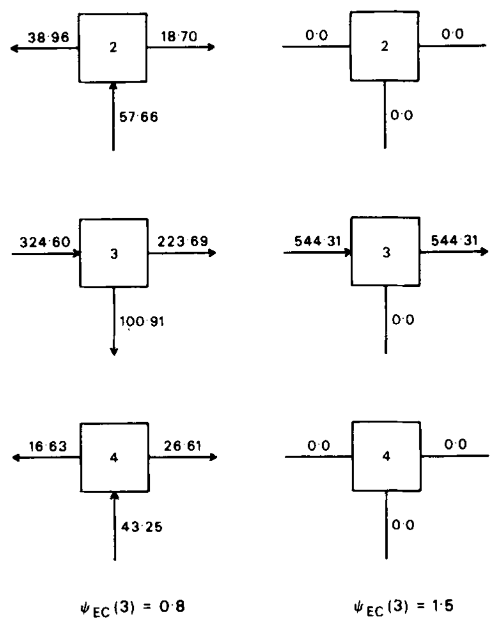

Fig. 4. Energy diagrams for two different unstable steady states. The components are in both cases P_2^2, P_3^2 and P_4^0. The forcing is $\psi_{EC}(3) = 0.8$ on the left side and $\psi_{EC}(3) = 1.5$ on the right side of the diagram.

first case we find that the unstable steady state is a truly non-linear state while in the second case the energetics is restricted to the middle scale. The numbers in Fig. 4 are non-dimensional.

If we start a numerical integration of the model equations in the immediate vicinity of an unstable steady state we know from the stability analysis that the model will move away from the steady state in phase space, i.e. in this model a five-dimensional space with the amplitudes $\psi(n)$, $\psi_c(n_1)$, $\psi_s(n_1)$, $\psi_c(n_2)$ and $\psi_s(n_2)$ along the axes. It is possible that the asymptotic state for such an integration is one of the other steady states unless the model goes into an oscillatory mode in phase-space. Some test calculations have been made. In each case it was found that the asymptotic state was another stable steady state. In our examples we have found that 2 or 3 stable steady states may be present. It would be important to know the conditions that determine which of the possible steady states is the asymptotic state from the given initial conditions. So far it has not been possible to solve this problem.

We shall illustrate the asymptotic cases by some example. Referring to Fig. 3 we noticed that the stability properties are divided in two separate regions. Similar properties are found in other cases. We consider a case in which the components are (0, 2), (3, 3) and (3, 4). With $\beta = 0.5$, $\varepsilon = 0.04$ and the forcing specified by $\psi_{EC}(3)$ while all other components of the forcing were zero the steady states were determined using the procedures outlined in Section 3. Based upon these results the stability analysis of Section 4 was carried out. The results of these calculations are summarized in Table 1. For $0.1 \leq \psi_{EC}(3) \leq 0.4$ there is only one steady state consisting of the component (3, 3). This steady state is stable. When $0.5 \leq \psi_{EC}(3) \leq 0.6$ we have 5 steady states. The first consists of (3, 3) only and is stable. The remaining four steady

Table 1. *Steady states and stability for a system consisting of (0,2), (3,3) and (3,4) with $\beta = 0.5$, $\varepsilon = 0.04$ and $\lambda_{EC}(3) \neq 0$. All other components of the forcing vanish*

$\psi_{EC}(3)$	$\overline{\psi(2)}$	$\overline{\psi_c(3)}$	$\overline{\psi_s(3)}$	$\overline{\psi_c(4)}$	$\overline{\psi_s(4)}$	Stability
0.1	0	0.0049	−0.0151	0	0	Yes
0.2	0	0.0098	−0.0301	0	0	Yes
0.3	0	0.0148	−0.0452	0	0	Yes
0.4	0	0.0197	−0.0602	0	0	Yes
0.5	0	0.0246	−0.0753	0	0	Yes
	−0.0666	0.0671	−0.0822	0.0386	−0.0200	No
	0.0666	0.0671	−0.0822	−0.0386	0.0200	No
	−0.1092	0.1059	0.0072	0.0634	0.0328	Yes
	0.1092	0.1059	0.0072	−0.634	−0.0328	Yes
0.6	0	0.0295	−0.0904	0	0	Yes
	−0.0400	0.0437	−0.0967	0.0194	−0.0176	No
	0.0400	0.0437	−0.0967	−0.0194	0.0176	No
	−0.1214	0.1005	0.0342	0.0587	0.0534	Yes
	0.1214	0.1005	0.0342	−0.0587	−0.0534	yes
0.7	0	0.0344	−0.1054	0	0	No
	−0.1304	0.0944	0.0485	0.0540	0.0658	Yes
	0.1304	0.0944	0.0485	−0.0540	−0.0658	Yes
0.8	0	0.0394	−0.1205	0	0	No
	0.1379	0.0892	0.0576	−0.0500	−0.0749	Yes
	−0.1379	0.0892	0.0576	0.0500	0.0749	Yes
0.9	0	0.0443	−0.1355	0	0	No
	0.1447	0.0848	0.0639	−0.0467	−0.0822	Yes
	−0.1447	0.0848	0.0639	0.0467	0.0822	Yes

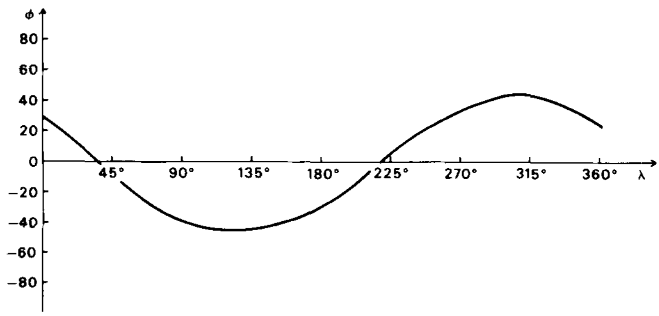

Fig. 5. The result of a numerical experiment in which initial states were selected on a small sphere in phase-space with a steady, unstable state as the centre. The spherical surface was covered by a grid with a grid distance of 10° similar to longitude (λ) and latitude (ψ). The curve on the figure separates the points on the sphere in two classes. All points above the curve represent initial states for which one asymptotic steady state exists, while the points below the curve approach another asymptotic steady state.

states have energy on all three components, but two of these steady states are unstable. A numerical integration of the model equations was carried out starting from an initial state very close to one of the unstable steady states for which we selected the second steady state listed under $\psi_{EC}(3) = 0.6$. The integration was continued until a steady state was reached. It was found that this asymptotic state was identical to the first steady state listed under $\psi_{EC}(3) = 0.6$.

When $0.7 \le \psi_{EC}(3) \le 0.9$ there are three steady states. In this case we find that the first one has energy on the component (3, 3) only and is unstable while the remaining two steady states have energy on all three components, but these states are stable. It is likely that the asymptotic state will depend on the initial condition. To illustrate this problem it was decided to perform a number of numerical experiments. We consider first the steady state characterized by $\overline{\psi_c(3)} = 0.0443$, $\overline{\psi_s(3)} = -0.1355$, while $\overline{\psi(2)} = \overline{\psi_c(4)} = \overline{\psi_s(4)} = 0$ for $\psi_{EC}(3) = 0.9$. In the phase space we consider next a sphere with the point $(\overline{\psi_c(3)}, \overline{\psi_s(3)})$ as centre. Points on the sphere are characterized by the expressions:

$$\psi(2) = A \cos \phi \cos \lambda$$

$$\psi_c(4) = A \cos \phi \sin \lambda$$

$$\psi_s(4) = A \sin \phi$$

where A is the radius of the sphere and $-\pi/2 \le \phi \le \pi/2$, $0 \le \lambda < 2\pi$. For the experiments we selected $A = 10^{-4}$ and used a 10° interval for λ and ϕ. By noting in each integration the asymptotic state we can divide the surface of the sphere into two regions where all points in a region have the same asymptotic state which is either the second or the third state listed for $\psi_{EC}(3) = 0.9$ in Table 1. The result of these experiments is shown in Fig. 5 where λ and ϕ are used as the linear co-ordinates. All points above the curve in Fig. 5 will approach the steady state with $\overline{\psi(2)} = -0.1447$ while those below the curve will have $\overline{\psi(2)} = +0.1447$. Identical results were obtained for $A = 10^{-3}$ and $A = 10^{-5}$. It is therefore conceivable that the region around the stationary unstable point in the phase space can be divided in sub-regions which have a particular asymptotic state. It is beyond the scope of this investigation to explore what happens if the initial conditions are on the interface between these sub-regions.

5. Conclusions

It has been shown that a barotropic, low-order system with forcing and dissipation may possess several steady states. In addition to the purely zonal flow as a steady state it is also possible to determine steady states which contain waves. Among these some states have energy on one wave component only, but others have energy on all components in the system, and the steady states are maintained by a non-linear exchange of energy among the components.

Each steady state may be used as the basic state for a linear stability analysis. If the basic state is a zonal flow we find that the flow is unstable for a sufficiently strong forcing. In the more interesting case where the basic state contains waves it is found that there may be 1, 3 or 5 steady states depending on the intensity of the forcing. In the examples treated in the paper it is shown that if only one steady state exists it is stable. For the case of 5 steady states it is found that 2 are unstable and 3 are stable. With 3 steady states the result is that one is unstable and 2 are stable. It has thus been demonstrated for the low-order system that for a given intensity of the forcing it is possible to find stable, steady states which contain energy on all wave components. This result corresponds to the general synoptic experience that certain wave configurations in the atmosphere may persist for a relatively long time while others, presumably corresponding to the unstable cases, will break down at a fast rate.

In the case where at least two stable steady states exist it is of interest to determine when the system from given initial conditions will approach one or the other symptotic steady states. No general solution to this problem has been found but it has been demonstrated by numerical experiments that the type of initial disturbance applied to a given unstable steady state will determine the final asymptotic state.

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СТАЦИОНАРНЫЕ СОСТОЯНИЯ И ИХ УСТОЙЧИВОСТЬ В БАРОТРОПНОЙ СИСТЕМЕ НИЗКОГО ПОРЯДКА С ВОЗБУЖДЕНИЕМ И ДИССИПАЦИЕЙ

Исследуется баротропная система низкого порядка с возбуждением по Ньютону и диссипацией. Система имеет одну компоненту в зональном потоке, характеризуемую функцией Лежандра, и две компоненты в вихревом течении, имеющие одно и то же долготное волновое число, но разные меридиональные индексы, и описываемые двумя присоединенными функциями Лежандра. Систему характеризуют пять вещественных амплитуд, и пять обыкновенных нелинейных дифференциальных уравнений описывают поведение системы.

Стационарные состояния системы могут быть определены численным решением уравнения 9-й степени при заданной интенсивности возбуждения. Раздел 3 содержит описание процедуры нахождения стационарных состояний; показано, что при заданном возбуждении в системе может существовать до пяти стационарных состояний. Наиболее сложное из стационарных состояний содержит энергию во всех компонентах системы, и нелинейный обмен энергией между компонентами является существенным для поддержания этого состояния. Другие стационарные состояния являются или зональными, или содержат энергию только в одной компоненте.

В разделе 4 описывается общая процедура определения устойчивости заданного стационарного состояния вместе с определением устойчивости всех стационарных состояний внутри интервала параметров возбуждения. Найдено, что среди стационарных состояний, обладающих энергией в волновых компонентах, мы находим как устойчивые, так и неустойчивые конфигурации. Таким образом, представляется, что этот результат находится в общем согласии с синоптическими представлениями, что некоторые волновые конфигурации могут существовать в сравнительно неизменной форме в течение долгого времени, тогда как другие, по-видимому, соответствующие неустойчивым конфигурациям, будут быстро меняться. В системе низкого порядка близкое соответствие с реальными атмосферными условиями невозможно.

В конце раздела 4 даны результаты численных экспериментов с простой системой, иллюстрирующие, что определенный тип начального возмущения, наложенный на стационарное неустойчивое состояние, будет определять асимптотическое стационарное состояние в случае, когда существуют два или более таких состояний.