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SHORT CONTRIBUTION

On the transient state of wind-driven coastal currents

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ABSTRACT

A solution to the transient state of Ekman's elementary current system is presented. The forcing consists of a suddenly imposed wind, directed along an infinite straight coast. The problem is identical to that treated by Ekman (1905) and it is shown that Ekman's solution is valid only at very great distance from the coast, or at extended times. The present solution reveals the existence of two time scales and shows that the main response is confined to a coastal region where it is characterized by the "spin up time". The width of this region is of the order of the Rossby radius of deformation. At increasing distance from the coast, the response time approaches the diffusive time scale obtained by Ekman.

1. Introduction

Ekman's elementary current system is the notation for the wind-driven currents in water of finite depth, including the top and bottom boundary layer currents and the midwater current which is essentially independent of depth. Here we consider the formation of this current system in an homogeneous ocean of constant depth in the presence of a straight coast.

Ekman (1905) made the assumption that the transport in the upper Ekman layer, normal to the coast, is at all times exactly balanced by the corresponding transport in the bottom layer. This is not correct since, during the formation, there is a component perpendicular to the coast of the midwater current that is not negligible. Taking this into account one arrives at a quite different result and since the Ekman solution does occur in modern textbooks (e.g. Neuman-Pierson, 1966) this presentation seems to be of some interest.

2. Formulation of the problem

We will use cartesian coordinates (x, y, z) with velocity components (u, v, w). The region $x > 0$ is occupied by homogeneous water of constant depth H. At time $t = 0$, a uniform and constant wind stress along the y axis is imposed. The following assumptions will be used:

(i) The pressure is hydrostatic.
(ii) The characteristic time scale of the motion is large compared to the rotation time Ω^{-1} (the quasi-geostrophic approximation).

Subject to the above restrictions, we get the following linearized set of equations for the interior motion.

$$\frac{\partial u_I}{\partial t} - 2\Omega v_I = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1}$$

$$\frac{\partial v_I}{\partial t} + 2\Omega u_I = 0 \tag{2}$$

$$p_I = \rho g z + p(x, y) \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

where $u = u_I + u_\tau + u_E$, $w = w_I + w_\tau + w_E$ and I, τ, E denotes contributions from the interior, top and bottom boundary layers, respectively.

The continuity equation (4) for the total velocity may be integrated vertically between bottom and surface, yielding

$$\frac{1}{g}\frac{\partial p}{\partial t} + \frac{\partial M}{\partial x} = 0 \tag{5}$$

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where $M = m_I + m_\tau + m_E$ denotes the total mass transport in the x-direction.

Since the response time for the upper Ekman layer is of the order Ω^{-1}, assumption (ii) is the justification for taking the values of m_τ and m_E from the steady state Ekman theory.

$$m_\tau = \frac{\tau_0}{2\Omega}\mu(t) \qquad \text{and} \qquad m_E = -\frac{H}{4\Omega}E^{1/2}\frac{\partial p}{\partial x}$$

where τ_0 is the wind stress, E is the Ekman number $(\nu/\Omega H^2)$ and $\mu(t)$ is the Heavyside unit step function. m_I is found by eliminating v_I from eqs. (1) and (2), noting that u_I is independent of z and making use of (ii).

$$m_I = \frac{H}{4\Omega^2}\frac{\partial}{\partial t}\frac{\partial p}{\partial x}$$

The following scales for x, t and p are introduced

$[x] = (gH/4\Omega^2)^{1/2}$ (Rossby radius of deformation)

$[t] = (\Omega E^{1/2})^{-1}$ (Spin up time)

$[p] = \tau_0 E^{-1/2}(g/\Omega^2 H)^{1/2}$

This puts eq. (5) into non-dimensional form

$$\frac{\partial p}{\partial t} + \frac{\partial M}{\partial x} = 0 \qquad (6)$$

$$M = \mu(t) - \frac{\partial p}{\partial x} - \frac{\partial}{\partial t}\frac{\partial p}{\partial x} \qquad (7)$$

The boundary and initial conditions are

$$M = 0 \quad \text{at} \quad x = 0 \qquad (8)$$

$$M < \infty \quad \text{as} \quad x \to \infty \qquad (9)$$

$$p = 0 \quad \text{at} \quad t = 0 \qquad (10)$$

This formulation differs from that of Ekman essentially by the presence of the last term in eq. (7), which represents the quasigeostrophic part of the mass transport perpendicular to the coast. The neglect of this term results in a pure diffusion problem with the boundary condition

$$\frac{\partial p}{\partial x} = 1 \qquad \text{at } x = 0$$

i.e. the final surface slope is reached instantly at the coast and propagates out by way of diffusion.

3. Discussion of the solution

The solution of eqs. (6) and (7) may be obtained by the use of Laplace transformation, and by applying the conditions (8)–(10). Here we shall discuss some interesting properties of the results. At the coast ($x = 0$), the solution takes the form

$$p(0, t) = te^{-t/2}(I_0(t/2) + I_2(t/2)) \qquad (11)$$

$$p_x(0, t) = 1 - e^{-t} \qquad (12)$$

where I_0, I_2 are modified Bessel functions of the first kind. Expression (12) is derived by integrating the boundary condition (8), and shows that the final slope of the surface near the coast is reached in the spin up time. The condition (11) gives the variation of the water level at the coast and from the asymptotic behaviour of this expression it is seen that for

$p(0, t) \sim t$ for $t \ll 1$

$p(0, t) \sim t^{1/2}$ for $t \gg 1$

Thus the adjustment of the water level, Fig. 1, is rapid during the spin up time before the bottom Ekman layer is fully developed.

The complete solution is found to be

$$p = x - \frac{2}{\pi}\int_0^\infty \frac{\partial\eta}{\eta^2}(1 - e^{-(\eta^2/1+\eta^2)t}\cos x\eta) \qquad (13)$$

$$\frac{\partial p}{\partial x} = 1 - \frac{2}{\pi}\int_0^\infty \frac{\partial\eta}{\eta}e^{-(\eta^2/1+\eta^2)t}\sin x\eta \qquad (14)$$

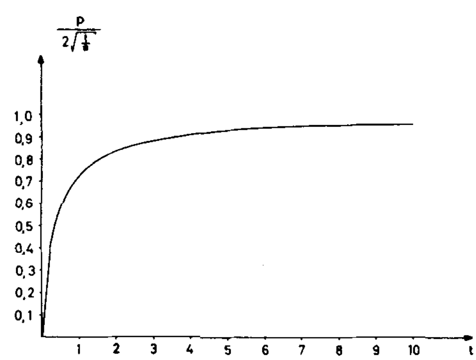

Fig. 1. Plot of $p(0, t)/2\sqrt{(t/\pi)}$ showing the rapid change of the water level at the coast during the spin up time.

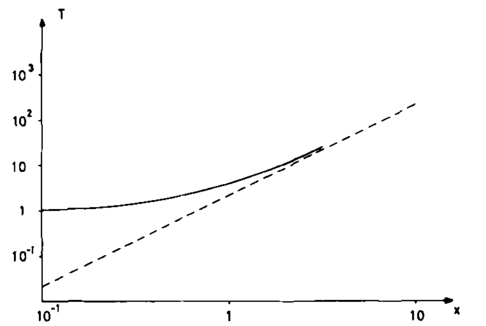

Fig. 2. Plot of the time scale as a function of distance from the coast. The dashed line represents the diffusive time scale of Ekman's solution.

It is interesting to investigate Eq. (14) for $t \ll 1$, in which case

$$\frac{\partial p}{\partial x} \approx \frac{2}{\pi} t \int_0^\infty \frac{\eta \sin \eta}{x^2 + \eta^2} d\eta$$

and evaluating the integral this becomes

$$\frac{\partial p}{\partial x} \approx t\, e^{-x}$$

showing that the initial response is restricted to a coastal region, the width of which equals the Rossby radius of deformation. It may be shown that Eq. (14) formally approaches Ekman's solution as $x \to \infty$

$$\frac{\partial p}{\partial x} = 1 - \operatorname{erf}\left(\sqrt{\frac{x^2}{4t}}\right)$$

Finally the results are summarized in Fig. 2, which gives the time scale as a function of distance from the coast. Here the time scale is chosen to be the time required for the surface inclination to attain 60% of its final value. It is seen that the departure from Ekman's result is considerable within the Rossby radius of deformation, and this regime appears to be the only one of geophysical importance, considering the great magnitude of the Rossby radius of deformation, which is 1000 km or more.

Thus we can conclude that in most real cases the response time of the geostrophic and bottom transports with respect to variations of the wind field is of the order of the spin up time.

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