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Storm surges in stratified seas

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ABSTRACT

The theory of storm surges along a straight coast in case of a continuously stratified sea of constant depth is considered.

The stratification is shown to have no appreciable effect on the surge at the coast, which therefore is mainly a barotropic response. The internal response at the coast which depends strongly on stratification is found to be extremely sensitive to how the turbulent stress depends on depth. Moreover, it is argued that in relatively deep water the internal response is substantially influenced by bottom friction, whereas the storm surge is not. A solution based on this assumption is presented.

1. Introduction

Storm surges along a straight coast are generated when longshore winds with the coast to the right (Northern Hemisphere) forces water towards the coast (Ekman, 1905). For a review see Welander (1961), Heaps (1965) and Bretschneider (1967).

The aim of the present study is to discuss the influence of stratification and bottom friction on the storm surge. The ocean model applied is one with constant depth and continuous horizontal stratification. By application of Fourier series or Fourier integrals, the response in such a model to wind and pressure forces was studied by Tomczak, Jr. (1967) and Magaard (1971) for a horizontal unbounded ocean. The method adopted in the present study is to separate off the horizontal and vertical variations by means of the standard normal modes method (Lighthill, 1969a). This method was used by Mork (1968, 1972) to study the response of a horizontally unbounded, continuously stratified sea to atmospheric forces, by Lighthill (1969b) to study the response of the Indian Ocean to the onset of the south west monsoon and by Csanady (1972) to study the response of large lakes to wind. Finally, Gill and Clarke (1974) employed the technique to discuss the coastal upwelling problem. In many respects the theory of storm surges parallels that of upwelling, which is the corresponding internal response. The presentation of the method given in Section 2, which closely follows that of Gill and Clarke (1974), is therefore only a brief outline.

An investigation of the influence of a continuous stratification on the surface displacement at the coast due to the passage of a single storm seems to be missing. This is the main theme of Section 3.

The objective of Section 4 is to study the effect of bottom friction on the surface displacement due to the passage of the storm. To this end a modified set of equations are adopted from Mork (1972). In relatively deep coastal waters it is argued that the friction terms are negligible in the barotropic mode equations but that they totally override the local accelerations (i.e. the $\partial/\partial t$ terms) in the baroclinic mode equations. A system of equations based on this argument is developed. The equations are solved to give the displacement of isopycnals at all depths *at the coast*.

The results of Sections 3 and 4 indicate that neither stratification nor bottom friction are important factors in the determination of the storm surge, except possibly for storms of unusual long duration. It should be emphasized that these conclusions are based on an ocean model where the depth is great compared to some frictional depth. The conclusions may therefore not hold for "shallow" coastal waters, where this ratio is of order one.

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2. Formulation of the problem

Consider a continuously stratified, incompressible fluid. The motion will be described relative to a Cartesian coordinate system which rotates about the vertical axes with angular velocity $\frac{1}{2}f$. The z axis points vertically upwards and the origin is in the surface of the initially undisturbed fluid. The fluid is bounded by the bottom, $z = -H(x,y)$, and the surface, $z = \eta_s(x,y,t)$, where η_s is initially zero. Moreover, the fluid is bounded by a straight coast along the x axis, y pointing seawards. The stratification is horizontally uniform so that in the equilibrium state the density $Q(z)$, and the pressure are functions of depth only. The surface is exposed to pressure and wind stress fields which are distributed over a horizontal domain with large linear dimensions compared to the depth. This means that the hydrostatic approximation is justified. The disturbances are assumed to be sufficiently small for the motion to be described by linear theory. When the Boussinesq approximation is used, the governing equations correspond to those of Gill and Clarke (1974).

The equations will be solved by division into normal modes (Lighthill, 1969a). The aim of this method is to separate off the horizontal and vertical variations in order to obtain equations in two dimensions only. To this end the variables are expanded as follows

$$\mathbf{v}(x,y,z,t) = \sum_{n=0}^{\infty} \mathbf{v}_n(x,y,t)\hat{\eta}_n'(z), \tag{2.1}$$

$$\eta(x,y,z,t) = \sum_{n=0}^{\infty} \eta_n(x,y,t)\hat{\eta}_n(z) \tag{2.2}$$

and

$$p(x,y,z,t) = p_s(x,y,t) + Q_s \sum_{n=0}^{\infty} p_n(x,y,t)\hat{\eta}_n'(z). \tag{2.3}$$

Here $\mathbf{v}$ is the horizontal part of the velocity, η the displacement of isopycnals from their equilibrium position ($\eta_s = \eta(z = 0)$), p the perturbation pressure and p_s the surface pressure anomaly. The eigenfunctions, $\hat{\eta}_n(z)$, are solutions of the eigenvalue problem

$$\hat{\eta}_n'' + \frac{N^2(z)}{c_n^2}\hat{\eta}_n = 0; \qquad -H < z < 0, \tag{2.4}$$

$$\hat{\eta}_n' - \frac{g}{c_n^2}\hat{\eta}_n = 0; \qquad z = 0, \tag{2.5}$$

$$\hat{\eta}_n = 0; \qquad z = -H. \tag{2.6}$$

Here, p_n and η_n are proportional, the factor of proportionality being the eigenvalue, c_n, squared, i.e.

$$p_n = c_n^2\eta_n. \tag{2.7}$$

$N^2(z)$ is the Väisälä–Brunt frequency defined by

$$N^2 = -\frac{g}{Q_s}\frac{dQ}{dz} \tag{2.8}$$

and g is the gravitational acceleration. Q_s is the surface value of the equilibrium density. The eigenfunctions, which are orthogonal, may conveniently be normalized as follows

$$\int_{-H}^{0} |\hat{\eta}_n'(z)|^2 dz = H. \tag{2.9}$$

With these definitions, the equations for the modes take the form (cf. Gill and Clarke, 1974),

$$\frac{\partial \mathbf{v}_n}{\partial t} + f\mathbf{k}x\mathbf{v}_n = -c_n^2\nabla\eta_n + \boldsymbol{\tau}_n, \tag{2.10}$$

$$\nabla\cdot\mathbf{v}_n + \frac{\partial\eta_n}{\partial t} = 0. \tag{2.11}$$

Here, $\boldsymbol{\tau}_n$, is given by

$$\boldsymbol{\tau}_n = -\frac{\hat{\eta}_n(0)}{Q_sH}\nabla p_s + \frac{1}{Q_sH}\int_{-H}^{0}\frac{\partial\boldsymbol{\tau}}{\partial z}\hat{\eta}_n'(z)dz, \tag{2.12}$$

and represents the forcing. $\boldsymbol{\tau}$ is the tangential stress due to turbulence acting between horizontal planes whose value at the surface, $\boldsymbol{\tau}_s$, is the prescribed wind stress.

In order to solve eqs. (2.10) and (2.11) a knowledge of the variation of $\boldsymbol{\tau}$ with depth is required, or, if the stress is put proportional to the gradient of the horizontal velocity, $\mathbf{v}$, through an eddy viscosity coefficient, the variation with depth of this coefficient is required.

Furthermore, for application purposes the eigenfunctions and eigenvalues and hence the stratification must be known. A discussion of the eigenfunctions and eigenvalues associated with the chosen stratification of Section 3 may be found in the Appendix.

The solution to eqs. (2.10) and (2.11) is subject to the condition at the coast

$$\mathbf{v}_n = 0; \qquad y = 0. \tag{2.13}$$

Far offshore the displacements have to be bounded or

$$|\eta_n| < \infty; \qquad y \to \infty. \tag{2.14}$$

The initial conditions are

$$\eta_n = 0 \quad \text{and} \quad \mathbf{v}_n = 0; \quad t = 0. \tag{2.15}$$

3. Application to the passage of a single storm

In order to be able to disclose the effects of stratification on the storm surge, it is preferable to compare directly with the solution of a homogeneous model. For this purpose the model by Gjevik and Røed (1976) (hereafter referred to as GR) will do. The forcing is therefore chosen as a moving storm with no pressure fluctuations, viz.,

$$\boldsymbol{\tau}_s = Q_s \tau_s e^{-\kappa(x-u_0 t)^2} \mathbf{i}; \qquad p_s = \text{const.} \tag{3.1}$$

The tangential stress, $\boldsymbol{\tau}(z)$, may be represented by

$$\boldsymbol{\tau} = Q_s \tau(z) e^{-\kappa(x-u_0 t)^2} \mathbf{i} \tag{3.2}$$

where

$$\tau(0) = \tau_s. \tag{3.3}$$

u is the propagation speed of the storm, and $\kappa^{-1/2}$ measures the horizontal extent of the storm, i.e. small values of κ correspond to wind fields of large extent. Invoking eq. (2.12), the forcing function, $\boldsymbol{\tau}_n$, may be written

$$\boldsymbol{\tau}_n = \tau_n e^{-\kappa(x-u_0 t)} \mathbf{i} \tag{3.4}$$

where

$$\tau_n = \frac{1}{H} \int_{-H}^{0} \frac{\partial \tau}{\partial z} \hat{\eta}'_n(z) dz. \tag{3.5}$$

With this definition eqs. (2.10) and (2.11) correspond to eq. (3.1) of GR subject to corresponding boundary and initial conditions. Thus, if the horizontal extent of the wind field, represented by $\kappa^{-1/2}$, is greater than or equal to the barotropic radius of deformation, c_0/f, the solution to eqs. (2.10) and (2.11) in terms of the displacement of each mode, η_n, *at the coast* may to a good approximation be written (cf. GR eq. (17))

$$\eta_n = \frac{1}{2} \frac{\tau_n}{c_n^2} \sqrt{\frac{\pi}{\kappa}} \frac{c_n}{c_n - u_0} [\operatorname{erf} \sqrt{\kappa}(x - u_0 t) - \operatorname{erf} \sqrt{\kappa}(x - c_n t)]; \qquad y = 0. \tag{3.6}$$

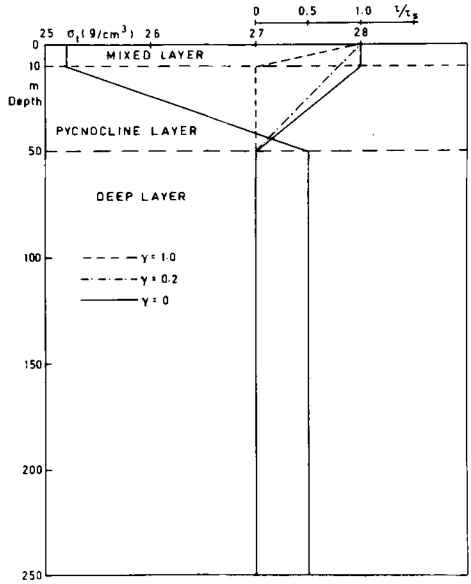

Fig. 1. The density (in terms of σ_t) and stress distribution employed in Section 3.

(If $\sqrt{\kappa}c_0/f \leq 1$, then according to appendix $\sqrt{\kappa}c_n/f \ll 1$ for $n > 1$.) In order to be able to compare with the specific homogeneous model in GR, the area of application will be the western coast of Norway during fall situations. Based on observations from several cruises along the Norwegian coast, the following stratification model is chosen (Fig. 1). A well-mixed layer of thickness $h_1 = 10$ m and with uniform density overlies a pycnocline layer of thickness $h_2 - h_1 = 40$ m, in which the density increases linearly with depth from the surface value, $Q_s = 1.0252$ g/cm³, to the value $Q_b = 1.0275$ g/cm³, which is the constant density value throughout the deep bottom layer.

The variation of the tangential stress, $\boldsymbol{\tau}(z)$, with depth is more difficult to ascertain, mostly due to the turbulent character of the motion. Three models will be considered (Fig. 1):

(1) A stress which decreases linearly from its surface value to zero at the bottom of the well-mixed layer.
(2) A stress which falls off linearly from its surface value to zero at the bottom of the pycnocline layer.
(3) A stress which is constant and equal the surface value throughout the well-mixed layer and subsequently decays linearly to zero at the bottom of the pycnocline layer.

The first model is one suggested and applied by Gill and Clarke (1974) and also by Csanady (1972) in his model of the Great Lakes. The second roughly corresponds to that applied in two-layer models such as those of Charney (1955) and Veronis and Stommel (1956). The third may be attributed to the commonplace representation of the stress,

$$\boldsymbol{\tau}(z) = \mu \frac{\partial \mathbf{v}}{\partial z}, \tag{3.7}$$

where the eddy viscosity coefficient, μ, might be but inversely proportional to the density gradient or,

$$\mu = \lambda \frac{Q^2}{dQ/dz}. \tag{3.8}$$

Such a stress model has been used by Fjeldstad (1964) in a theoretical study of the frictional damping of free internal waves and later by Mork (1968, 1972). In the well-mixed layer the density is uniform, so $\mu \to \infty$. In order to have a finite stress, the velocity according to eq. (3.7) has to be constant throughout this layer. Hence, also the stress may be chosen to be constant in this layer. Within the pycnocline the representation eq. (3.7) predicts the stress to be some function of depth and to be constant again in the deep layer, which due to the small level of turbulence may be put equal to zero, except, perhaps, near the bottom (bottom friction). The three stress models are conveniently condensed in the expression

$$\tau(z) = \tau_s \begin{cases} 1 + \gamma \dfrac{z}{h_1}; & -h_1 \le z \le 0 \\[2ex] (1-\gamma)\dfrac{z + h_2}{h_2 - h_1}; & -h_2 < z < -h_1 \\[2ex] 0; & -H \le z \le -h_2. \end{cases} \tag{3.9}$$

Model (1), (2) and (3) then corresponds to $\gamma = 1$, $\gamma = h_1/h_2$ $(= 0.2)$ and $\gamma = 0$, respectively. Substituting the expression for $\tau(z)$, eq. (3.9), in eq. (3.5) it follows by invoking expression (A.2) for the eigenfunctions that

$$\tau_n = \tau_s \frac{c_n^2}{gH} \frac{s_n(\gamma)}{\hat{\eta}_n(0)}, \tag{3.10}$$

where

$$s_n(\gamma) = \frac{g\hat{\eta}_n(0)}{c_n^2} \left\{ \gamma \hat{\eta}_n'(0) + \frac{1-\gamma}{h_2 - h_1} [\hat{\eta}_n(-h_1) - \hat{\eta}_n(-h_2)] \right\}. \tag{3.11}$$

Thus from eqs. (2.7) and (3.6) the following expression for the displacement *at the coast* is obtained:

$$\eta = \frac{1}{2} \frac{\tau_s}{gH} \sqrt{\frac{\pi}{\kappa}} \sum_{n=0}^{\infty} \frac{c_n s_n(\gamma) \hat{\eta}_n(z)}{(c_n - u_0)\hat{\eta}_n(0)} \times [\operatorname{erf}\sqrt{\kappa}(x - u_0 t) - \operatorname{erf}\sqrt{\kappa}(x - c_n t)] \tag{3.12}$$

In Table 1 are listed values of c_n and $s_n(\gamma)$ for the three stress models pertaining to the first six modes.

The solution eq. (3.12) is purely formal unless the series can be truncated after a moderate number of terms. From Appendix follows that $c_n s_n(\gamma)/(c_n - u_0)$ dies off as n^{-3}, whereas $\hat{\eta}_n(z)/\hat{\eta}_n(0)$ grows as n^2 unless z is zero. Hence, supported by the values listed in Table 1, a very good approximation to the storm surge, $\eta_s = \eta$ $(z = 0)$, is obtained with a truncation after 6–7 terms. However, the solution for the internal displacement $(z \neq 0)$ is purely formal.

Fig. 2 demonstrates both the homogeneous surge (eq. (3.12) with $c_n = \sqrt{gH}\,\delta_{no}$, $s_n(\gamma) = \delta_{no}$ and $z = 0$) and the surges in case of stratification for the three different stress models, respectively, calculated from the first 7 modes. The zeroth barotropic mode are in all three cases indistinguishable from the homogeneous surge. The baroclinic modes for which the first is the most dominant (cf. Table 1) provide a slight amplification of the maximum surge (within 5–10%). There is no time lag between the maxima. For $ft > 5$ a non-vanishing surge is left in the stratified cases, which contrasts the concomitant homogeneous surge which rapidly tends to zero.

A main shortcoming of the present model so far is the neglect of bottom friction. In spite of this the effect of stratification on the storm surge is small. Obviously the inclusion of bottom friction will tend to reduce the surge. Especially as will be argued, this is true for the baroclinic modes. Thus the conclusion is that stratification effects on storm surges do not seem to be appreciable and for prediction purposes, therefore, homogeneous models will do. At least it holds true for the present

Table 1. *($H = 250$ m, $h_1 = 10$ m, $h_2 = 50$ m) Eigenvalues c_n, and amplitudes s_n (γ). Note the peak for $s_n(\gamma)$ in the first baroclinic*

			$s_n(\gamma)$		
Mode	n	c_n (m/s)	$\gamma = 1$	$\gamma = 0.2$	$\gamma = 0$
Barotropic	0	50.0435	0.9965	0.9945	0.9975
1st baroclinic	1	0.6850	8.4223	30.5128	4.2784
2nd baroclinic	2	0.2380	5.5210	−13.6181	−3.3333
3rd baroclinic	3	0.1385	2.9825	0.8774	0.1177
4th baroclinic	4	0.0964	1.7112	−0.8754	−0.8895
5th baroclinic	5	0.0735	1.0743	0.2257	0.0126

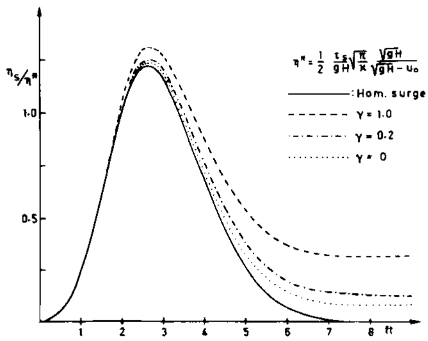

Fig. 2. The solid curve shows the surface displacement at the coast as a function of time (normalized by f) due to the passage of a single storm, as compiled from Gjevik and Røed (1976). This corresponds to the solution eq. (3.12) with $c_n = \sqrt{gH}\,\delta_{no}$, $s_n(\gamma) = \delta_{no}$ and $z = 0$. The broken curves are the corresponding surface displacements as computed from eq. (3.12) for different values of γ and visualize the effect of stratification. The values of u_0, κ, etc. correspond to those used by Gjevik and Røed (1976).

deep ocean model. This point is returned to in Section 6. The above conclusion is also supported by the remarkably good agreement between observed surges and surges estimated from homogeneous models both analytically (GR; Heaps, 1965) and numerically (Heaps 1969, 1971).

4. The effect of bottom friction

The form of the modes eqs. (2.10) and (2.11) suggests that the displacement due to each mode separately corresponds to the surge derivable from a model with a homogeneous body of water in a basin of depth

$$H_n = \frac{c_n^2}{g} \tag{4.1}$$

(cf. Csanady, 1972, Lighthill, 1969b). For the barotropic mode this depth approximates the real depth ($H_0 \simeq H$). For the first baroclinic mode this depth is 5 cm only (cf. Table 1) and for the higher baroclinics the depths are even smaller. Gammelsrød et al. (1975) have suggested a procedure whereby the time scales for the influence of the bottom friction in a homogeneous model may be ascertained. They suggest that the time scales are proportional to the depth of the basin, viz.

$$\text{frictional time scales} \simeq \frac{2\pi H}{fD},$$

where D is some frictional depth (for instance the Ekman depth). For the barotropic mode this entails $ft \sim 10$–15 depending on the ratio H_0/D. For ocean basins of depth less than D the procedure is less conclusive, but the author believes that for depths pertinent to the baroclinic modes bottom friction is important already after a short elapse of time, say $ft < 1$. Thus, in relatively deep coastal areas scaled on D the effect of bottom friction on the barotropic mode is negligible, except, perhaps, for the storms of unusually long duration, whereas for the baroclinic modes bottom friction has to be included immediately. This conclusion is supported by the remarkably large inertial displacements found by Csanady (1972), which also invalidate the linear theory assumption already after a short elapse of time.

According to Mork (1972), the mode equations (eqs. (2.10) and (2.11)) including bottom friction read

$$\left(\frac{\partial}{\partial t} + R_n\right)\mathbf{v}_n + f\mathbf{k}x\mathbf{v}_n = -c_n^2\nabla\eta_n + \boldsymbol{\tau}_n, \tag{4.2}$$

$$\nabla\cdot\mathbf{v}_n + \frac{\partial\eta_n}{\partial t} = 0, \tag{4.3}$$

where R_n is given by

$$R_n = \lambda\frac{gH}{c_n^2} \qquad (\lambda = \text{constant}). \tag{4.4}$$

Since friction is assumed to be negligible for the barotropic mode it follows from eq. (4.2) that the term $R_0\mathbf{v}_0$ should be small compared to $\partial\mathbf{v}_0/\partial t$. However, anticipating friction to be important for the baroclinic modes, one might infer that

$$R_n\mathbf{v}_n \gg \frac{\partial\mathbf{v}_n}{\partial t}; \qquad n \geq 1, \tag{4.5}$$

which change the equations governing the baroclinic modes to

$$R_n\mathbf{v}_n + f\mathbf{k}x\mathbf{v}_n = -c_n^2\nabla\eta_n + \boldsymbol{\tau}_n, \tag{4.6}$$

and

$$\nabla\cdot\mathbf{v}_n + \frac{\partial\eta_n}{\partial t} = 0. \tag{4.7}$$

The boundary conditions are given by eqs. (2.13) and (2.14) and the initial conditions are those of eq. (2.15).

In order to solve eqs. (4.6) and (4.7) note that from these equations an equation for η_n may be derived, viz.,

$$c_n^2\nabla^2\eta_n - R_n\frac{\partial\eta_n}{\partial t} = \nabla\cdot\boldsymbol{\tau}_n; \qquad n \geq 1. \tag{4.8}$$

Application of the boundary condition at the coast, eq. (2.13), yields the condition

$$c_n^2\left(f\frac{\partial\eta_n}{\partial x} - R_n\frac{\partial\eta_n}{\partial y}\right) = f\mathbf{i}\cdot\boldsymbol{\tau}_n; \qquad y = 0. \tag{4.9}$$

The baroclinic modes do have variations normal to the coast greatly in excess of the variation along the coast (Charney, 1955; Gill and Clarke, 1974 and others). Hence in the vicinity of the coast eqs. (4.8) and (4.9) may be approximated by

$$c_n^2\frac{\partial^2\eta_n}{\partial y^2} - R_n\frac{\partial\eta_n}{\partial t} = \nabla\cdot\boldsymbol{\tau}_n \tag{4.10}$$

$$c_n^2R_n\frac{\partial\eta_n}{\partial y} = -f\mathbf{i}\cdot\boldsymbol{\tau}_n; \qquad y = 0, \tag{4.11}$$

which is a diffusion equation for the slope normal to the coast. The solution to eq. (4.10) is well known. Therefore, the contribution from the baroclinic modes, with bottom friction included, to the displacement at the coast, invoking the stress distribution of Section 3, may be written

$$\eta_{\text{baroc}} = \tfrac{1}{2}\frac{\tau_s}{gH}\sqrt{\frac{\pi}{\kappa}}\,g(x,t)\sum_{n=1}^{\infty}\frac{c_n^2}{gH}s_n(y)f_n(z) \tag{4.12}$$

where

$$g(x,t) = \frac{\sqrt{gH}}{\lambda}\sqrt{\frac{\kappa}{\pi}}\left\{\frac{2f}{\sqrt{\pi\lambda}}\int_0^{\sqrt{t}} e^{-\kappa(x-u_0t+u_0\xi^2)^2}\,d\xi + \frac{\sqrt{gH}}{u_0}\left[e^{-\kappa(x-u_0t)^2} - e^{-\kappa x^2}\right]\right\}. \tag{4.13}$$

Since from eqs. (A.8) and (A.16) the product $c_n^2s_n(y)$ dies off as n^{-4} as n increases there follows that the series in eq. (4.12) may be truncated already after a few terms, 6–7 say. The total displacement, at the coast, is composed of the barotropic mode as given by eq. (3.12) with $n = 0$ and the sum of the baroclinic modes given by eq. (4.12).

In order to visualize the solution consult Fig. 3, where pycnocline displacements at the coast are plotted against the time (normalized by f) for several equilibrium depths. This particular plot has a value of λ such that $\lambda = 10^{-3}f$. Note that in consequence of eqs. (4.4) and (4.5) the admissible range for this constant is such that

$$10^{-3}\cdot f < \lambda < 10^{-1}\cdot f. \tag{4.14}$$

The other parameters such as u_0, x, etc., are the same as those of Fig. 2. The surface displacement, the storm surge, is so close to the homogeneous solution that the display is unable to resolve them, in fact the absolute error is less than $3\cdot10^{-3}$, a result already pointed out in Section 3. This is true for all admissable values of the constant λ. However, the internal response as shown by Figs. 3 and 4 varies with the depth and with the chosen value for λ. Fig. 4 shows the variations of the 50 m pycnocline with λ. The other internal isopycnals all

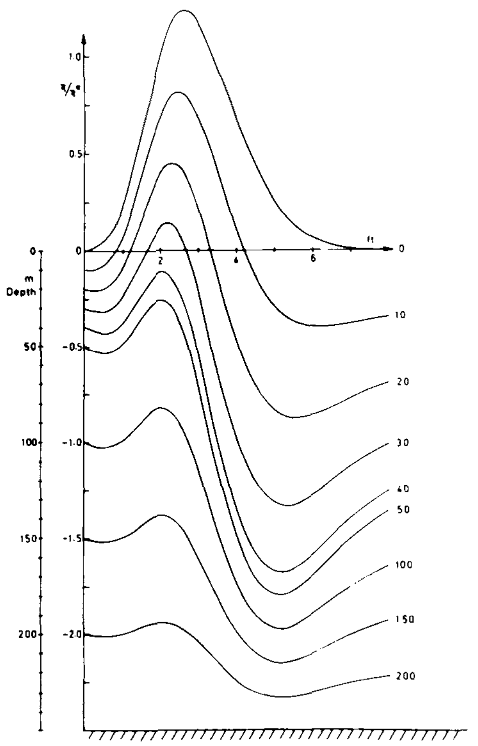

Fig. 3. The solid curves show, as function of time normalized by f^{-1}, the deviation of density surfaces from their equilibrium position at the coast. The parameter values are the same as those of Fig. 2. The frictional parameter is given the value $\lambda = 10^{-3} \cdot f$ and $\gamma = 0$. Note the remaining depression of internal isopycnals when the surge has decayed.

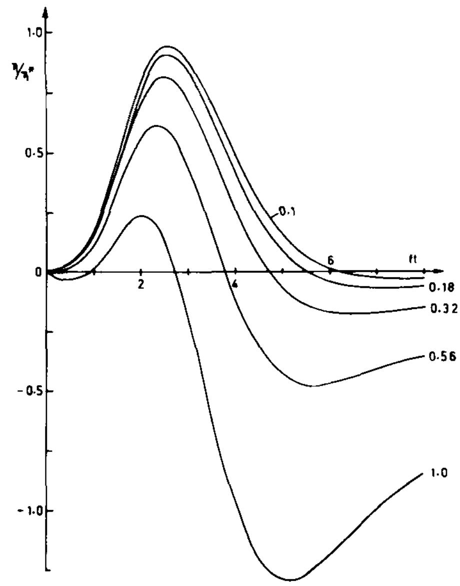

Fig. 4. Dependence of deviation of equilibrium density surfaces on the frictional parameter. Values of $(\lambda/f) \cdot 10^2$ are attached to the curves. Otherwise the parameter values are those of Fig. 3. The displayed deviation is the density surface, which in equilibrium corresponds to the 50 m isopycnal. Other density surfaces show the same behaviour.

have a similar behaviour. The variation is such that the smaller the value of λ the greater the deviation from equilibrium. The largest deviation is always found at the bottom of the pycnocline layer. It is interesting to note that when the storm has passed the internal isopycnals are depressed which entails a downwelling on a time scale much larger than the storm surge time scale, in accordance with accepted time scales for upwelling events (Gill and Clarke, 1974).

5. Summary and final remarks

The storm surge in a stratified coastal sea due to the passage of a single storm has been studied. The main conclusions arrived at are:

(1) The inclusion of the baroclinic modes has only minor effect on the storm surge.

(2) Bottom friction seems to have no appreciable effect on storm surge in relative deep seas within the lifetime of the surge due to reasonable storm durations.

(3) The internal response, however, is strongly influenced both by stratification and by bottom friction. Moreover, the internal response is sensitive to the tangential stress dependency on depth.

For prediction purposes, therefore, the storm surge is obtained with the sufficient accuracy from homogeneous models. Furthermore, if the coastal sea is relatively deep, i.e. the depth scaled on the Ekman depth is a large number, bottom friction may be neglected too, except possibly for storms of unusual long duration, i.e. storms which have not subsided locally after approximately 1 to 2 days.

However, normally homogeneous models have the shortcoming that they give no information about the vertical structure of the motion. (The vertical structure may be retained in homogeneous models by a method described in Heaps, 1971.) To this end stratification must be included and the question arises how to distribute the tangential stress. In fact, solutions of the upwelling problems hinge upon such knowledge. The storm surge might be a helpful guide in assessing the stress distribution, because it is nearly unaffected by the stratification and might, therefore, be taken as known, leaving the stress distribution as the primary unknown.

It should be noted that the above conclusions are based on an ocean model with constant depth. The possible interaction between stratification and bottom topography variations is therefore not included. There is, however, reason to believe that this interaction would mainly influence the baroclinic response (Mork, 1972). Also, in shallow water areas, the finite amplitude effect plays a part in the surge generation, as does the friction. Therefore, care should be exercised in making any inferences from the present theory in shallow coastal areas.

6. Acknowledgements

The author is indebted to Dr V. Lauvstad for helpful suggestions during the course of this work. The author has also benefited from discussions with Drs N. S. Heaps, M. Mork and B. Gjevik. The pertinent data from the Norwegian continental shelf have been made available through the Norwegian Oceanographic Data Center (NOD).

7. Appendix

On account of the density model chosen the Väisälä–Brunt frequency, eq. (2.8), is zero in the well-mixed layer and in the deep bottom layer, and constant throughout the pycnocline. In order to find the eigenfunctions, $\hat{\eta}_n(z)$, and eigenvalues, c_n, one has to solve eq. (2.4) subject to the conditions (2.5) and (2.6). At the levels $z = -h_1$ and $z = -h_2$ describing the bottom of the well-mixed layer and the pycnocline layer, respectively, the Väisälä–Brunt frequency is discontinuous, which necessitates conditions to be imposed on $\hat{\eta}_n$ at these levels. According to eq. (2.1) and (2.2) the requirement of continuous velocity and displacement entails continuity of $\hat{\eta}_n'$ and $\hat{\eta}_n$. Invoking the continuity requirements eqs. (2.4) and (2.6) yield

$$\hat{\eta}_n(z) = a_n f_n(z), \tag{A.1}$$

where

$$f_n(z) = \begin{cases} \zeta + \left(\dfrac{F_i}{\alpha_n}\right)^2; & -\delta_1 \le \zeta \le 0 \\ \dfrac{\delta_2 - \delta_1}{\alpha_n}\left[\sin\left(\alpha_n \dfrac{\zeta + \delta_1}{\delta_2 - \delta_1}\right)\right. & \\ \left. + B_n \cos\left(\alpha_n \dfrac{\zeta + \delta_1}{\delta_2 - \delta_1}\right)\right]; & -\delta_2 < \zeta < -\delta_1 \\ (\cos \alpha_n + B_n \sin \alpha_n)(1 + \zeta); & -1 \le \zeta \le -\delta_2. \end{cases} \tag{A.2}$$

Here F_i is the internal Froude number

$$F_i = \frac{NH}{\sqrt{gH}}(\delta_2 - \delta_1) \tag{A.3}$$

and

$$\delta_i = \frac{h_i}{H}; \qquad i = 1, 2, \qquad \zeta = \frac{z}{H}. \tag{A.4}$$

The α_n's, which by making use of eq. (2.5) are solutions of the transcendental equation

$$tg\,\alpha_n = \left(\frac{\delta_2 - \delta_1}{\alpha_n}\right) \times \frac{F_i^2 - (1 - \delta_2 + \delta_1)\alpha_n^2}{(\delta_2 - \delta_1)^2 + F_i^2(1 - \delta_2) - \delta_1(1 - \delta_2)\alpha_n^2} = h(\alpha), \tag{A.5}$$

are related to the eigenvalues, c_n, through the relation

$$\frac{c_n}{\sqrt{gH}} = \frac{F_i}{\alpha_n}. \tag{A.6}$$

Finally B_n is defined by the expression

$$B_n = \frac{\alpha_n}{\delta_2 - \delta_1}\left[\left(\frac{F_i}{\alpha_n}\right)^2 - \delta_1\right]. \tag{A.7}$$

The roots of eq. (A.5) may be visualized with the help of Fig. 5. Here the right-hand side of eq. (A.5), $h(\alpha)$, is drawn together with $tg\,\alpha$. For increasing values of n it is seen that α_n rapidly approaches $(n - 1)\pi$. Hence a very good approximation for c_n for moderate values of n, $n \ge 5$ say, is

$$c_n \simeq \frac{NH}{(n - 1)\pi}(\delta_2 - \delta_1) \tag{A.8}$$

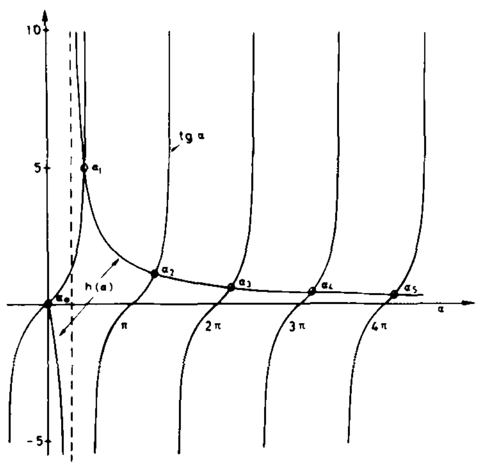

Fig. 5. Roots of the transcendental eq. (A.5). Note that $\alpha_n \to (n-1)\pi$ as n increases.

Furthermore, the zeroth mode is close to zero, which entails

$$tg\alpha_0 \simeq \alpha_0. \tag{A.9}$$

Thus from eq. (A.5) one obtains

$$\left(\frac{F_i}{\alpha_0}\right)^2 \simeq 1 \tag{A.10}$$

where use has been made of the fact that $F_i \ll 1$.

Invoking eq. (A.6) it follows that

$$c_0 \simeq \sqrt{gH} \tag{A.11}$$

in accordance with previous investigations. The amplitudes of the eigenfunctions, a_n, are determined from the normalization condition eq. (2.9) which leads to the expression

$$\left(\frac{H}{a_n}\right)^2 = 1 - \tfrac{1}{2}(\delta_2 - \delta_1)(1 - B_n^2) + \left[(1-\delta_2)B_n + \frac{\delta_2 - \delta_1}{4\alpha_n}(1 - B_n^2)\right]\sin 2\alpha_n + \left[\frac{\delta_2 - \delta_1}{\alpha_n}B_n - (1-\delta_2)(1 - B_n^2)\right]\sin^2\alpha_n. \tag{A.12}$$

The expression (3.11) for $s_n(\gamma)$ may now be written

$$s_n(\gamma) = \left(\frac{a_n}{H}\right)^2\left\{\gamma + \frac{1-\gamma}{\alpha_n} \times [B_n(1 - \cos\alpha_n) + \sin\alpha_n]\right\}. \tag{A.13}$$

For $n \geq 5$ the following approximations may be made

$$B_n \simeq -\frac{(n-1)\pi\delta_1}{\delta_2 - \delta_1} \tag{A.14}$$

$$\left(\frac{H}{a_n}\right)^2 \simeq 1 + \tfrac{1}{2}\frac{(n-1)^2\pi^2\delta_1^2}{\delta_2 - \delta_1} \tag{A.15}$$

$$s_n(\gamma) \simeq \frac{\gamma(\delta_2 - \delta_1) - (1-\gamma)\delta_1[1 + (-1)^n]}{(\delta_2 - \delta_1) + \frac{1}{2}(n-1)^2\pi^2\delta_1^2}. \tag{A.16}$$

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ШТОРМОВЫЕ ВОЛНЫ В СТРАТИФИЦИРОВАННОМ МОРЕ

Рассматривается теория штормовых волн вдоль прямолинейного берега для случая непрерывно стратифицированного моря постоянной глубины. Показано, что стратификация не оказывает заметного влияния на волны вблизи берега, которые, следовательно, возникают в основном из-за баротропных эффектов. Найдено, что внутренняя реакция моря вблизи берега, сильно зависящая от стратификации, очень чувствительна к тому, как турбулентное напряжение зависит от глубины. Более того, приводятся аргументы в пользу того, что в относительно глубоких водах внутренняя реакция существенно зависит от придонного трения, в то время как штормовые волны от него не зависят. Представлено решение, основанное на этом предположении.