Tellus (1979), *31*, 236–245

Formulas for calculating available potential energy over uneven topography

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(Manuscript received August 30; in final form December 19, 1978)

ABSTRACT

Formulas are derived that correctly account for the uneven surface topography (mountains) in calculations of atmospheric available potential energy and its rate of change. Two (barotropic) reference states for available potential energy are discussed: one is horizontally stratified, the other has uniform surface pressure. The former is a state of minimum total potential energy and the latter is a state of minimum enthalpy. Both proposed reference states lead to similar, simple expressions for generation of available potential energy. A numerical procedure is presented for calculating the horizontally stratified reference state from a given atmospheric state. For a simple case with mountains, various methods of computing available potential energy are compared, which illustrates the inherent errors in previous formulas. Potential applications of the "exact" expressions are mentioned in the concluding section.

1. Introduction

So-called "exact" formulas for calculating available potential energy and its rate of change in the atmosphere are traditionally based on the assumptions that (1) hydrostatic balance prevails, (2) the atmosphere is stably stratified everywhere, (3) latent energy does not contribute to the internal energy, and (4) surface topography can be ignored. Lorenz (1955) made these approximations when he reintroduced Margules' (1903) concept of available potential energy in connection with the general circulation of the atmosphere. Dutton and Johnson (1967) derived a "more exact" equation by eliminating the assumption of hydrostatic balance. Recently Lorenz (1978) argued that the maintenance of the general circulation could be explained more naturally by treating latent heat as a form of internal energy; this led to a newly defined quantity he called *moist available energy*. The principal purpose of the present paper is to generalize the "exact" equation for available potential energy and its rate of change, by specifically accounting for the presence of uneven surface topography. We shall retain, however, the other three assumptions given above.

Available potential energy (APE) is defined as the difference between the total potential energy (TPE) of an existing atmospheric state and some suitably defined reference state. Usually the reference state represents the minimum TPE that can be attained by (artificially) rearranging the mass of the atmosphere under thermodynamically reversible adiabatic processes.

This definition can be restated in another way using the terminology introduced by Lorenz (1978). Noting that the atmosphere could be thought of as consisting of a large number of air "parcels" of equal mass, Lorenz called two fields of mass *equivalent* if each parcel could pass from its state in one field to its state in the other by a reversible adiabatic process. For a dry atmosphere each parcel's state can be characterized by its velocity, pressure and temperature. The parcels of one mass field can be rearranged theoretically to produce the equivalent field if no restriction is placed on the mechanical forces needed to accomplish the task. Of all possible equivalent fields corresponding to a given global mass field, the one that possesses the least total potential energy was called the *reference field* by Lorenz. He defined APE as the difference between the TPE of the given field and that of the reference field.

0040-2826/79/030236-10$02.50/0

Although other reference states have been proposed (see, e.g., Pfeffer et al., 1966; Pearce, 1978), it is not clear that they offer a fundamental advantage over the one defined above. In Lorenz's reference state the atmosphere is barotropic, horizontally and stably stratified, and in exact hydrostatic equilibrium. For purposes that will become clear later, we shall initially discuss a more general class of equivalent reference states; we shall temporarily permit the equivalent reference field to have *non-horizontal* stratifications, but shall retain its barotropic and hydrostatic nature: isobars, isotherms and isopycnics continue to coincide but are not necessarily horizontal.

In the end we shall discuss two particular reference fields which are included in this general class of equivalent states. One will be horizontally stratified and will contain the minimum *total potential energy* of all equivalent fields; the other will have uniform surface pressure and will contain the minimum *enthalpy* of all equivalent fields. The former is the reference field proposed by Lorenz (1955) and is the theoretically ideal one for defining APE because in this reference state no energy is available for conversion to kinetic energy; the second proposed reference field is easier to determine from a given atmospheric field and can be related to earlier APE formulations. These two reference fields are singled out because they alone lead to a simple expression for generation of APE. If topography is disregarded (i.e., if the earth's surface is presumed to be at a uniform geopotential), these two reference states are identical and lead to the exact formula for APE given by Lorenz (1955).

It is important to realize that there is nothing wrong with Lorenz's theoretical discussion of APE or with his reference state of minimum TPE (which is one we shall indeed adopt); his "exact" formula, on the other hand, is correct only when topography can be disregarded, as he indicates. In this article we derive more general formulas that are valid even when uneven topography is present.

We shall first present a general formula for ΔTPE, the difference between the total potential energy of a given state and the TPE of an equivalent barotopic state. We shall then derive an equation for calculating the rate of change of this difference in energies. Next we shall outline how to determine each of the two proposed reference states from a given atmospheric state. An illustrative sample calculation will be performed to compare various APE formulas. Finally we shall discuss some potential applications of the formulas presented here.

2. Exact formulas and equations for APE

2.1. *A formula for ΔTPE*

We define the total potential energy of the atmosphere as

$$\mathrm{TPE} = \int_{\mathcal{M}} (I + \Phi)\, dm \tag{1}$$

where I is the internal energy per unit mass for dry air, Φ the geopotential height above sea-level, dm an element of mass, and $\mathcal{M}$ indicates that the integration covers the entire mass of the atmosphere.

If the atmosphere is an ideal gas in hydrostatic balance, the above equation can be rewritten in the following equivalent forms:

$$\begin{aligned}\mathrm{TPE} &= \int_{\mathcal{M}} (g h_s + c_p T)\, dm \\ &= \int_S \left[P_s h_s + \frac{c_p P_{00}^{-\kappa}}{(\kappa+1)g} (P_s^{\kappa+1}\theta_s \right. \\ &\quad \left. + \int_{\theta_s}^{\infty} P^{\kappa+1}\, d\theta) \right] d\sigma \end{aligned} \tag{2}$$

The symbols used here are defined below:

- h_s the surface height
- T temperature
- c_p specific heat of dry air at constant pressure
- S integration surface (roughly spherical) surrounding the globe
- P pressure
- P_{00} reference pressure (=1000 mb)
- θ potential temperature $[=(P_{00}/P)^{\kappa} T]$
- $d\sigma$ element of surface area
- g acceleration due to gravity (presumed constant)
- κ Poisson's ratio $(=R/c_p)$
- R gas constant for dry air
- s (subscript) identifying surface value of a variable

In eq. (2) we have assumed that

$P\Phi \to 0$ as $\Phi \to \infty$

and $P^{\kappa+1}\theta \to 0$ as $\theta \to \infty$ (i.e., $PT \to 0$ as $P \to 0$)

The difference in TPE between the given field and an equivalent barotropic field may be written:

$$\Delta\text{TPE} = \text{TPE} - \widehat{\text{TPE}}$$
$$= \int_S \left[(P_s - \hat{P}_s) h_s + \frac{c_p P_{00}^{-\kappa}}{(\kappa+1) g} \times (P_s^{\kappa+1} \theta_s - \hat{P}_s^{\kappa+1} \hat{\theta}_s + \int_{\theta_s}^{\infty} P^{\kappa+1} d\theta - \int_{\hat{\theta}_s}^{\infty} \hat{P}^{\kappa+1} d\theta) \right] d\sigma \quad (3)$$

where the circumflex ($\hat{}$) indicates that the variables are assigned their equivalent barotropic field values. To evaluate ΔTPE from a given state of the atmosphere, we would have to determine $\hat{P}$ as a function of θ, and $\hat{\theta}_s$ as a function of the horizontal coordinates. For suitably defined reference states, this is possible, as we shall see later.

Equation (3) can be written in a more compact form if we follow Lorenz'a formal convention and set $P = P_s$ (a function of horizontal position and time) when $\theta < \theta_s$. With this definition of P on "underground" isentropes we have

$$\Delta\text{TPE} = 4\pi a^2 \left[\overline{(P_s' - \hat{P}_s') h_s'} + \frac{c_p P_{00}^{-\kappa}}{(\kappa+1) g} \times \int_0^{\infty} (\overline{P^{\kappa+1}} - \overline{\hat{P}^{\kappa+1}}) d\theta \right]$$

where a is the earth's radius, the overbar denotes an average over an isentropic surface, and the prime denotes a deviation from such an average. Again we have used the fact that the average surface pressure is constant (since total atmospheric mass is conserved) so that $\bar{P}_s = \bar{\hat{P}}_s$. Notice that unlike Lorenz's formula, $\bar{\hat{P}}^{\kappa+1} \neq \bar{\hat{P}}^{\kappa+1}$ because in our case the isentropic surfaces of the reference state may intersect the ground.

2.2. *A formula for the rate of change of ΔTPE*

To obtain an expression for the rate of change of ΔTPE, we shall evaluate the rates of change of TPE and $\widehat{\text{TPE}}$ separately and then form their difference. The rate of change of total potential energy, when integrated over the entire mass of the atmosphere, is well known and is given here:

$$\frac{\partial}{\partial t}(\text{TPE}) = \int_{\mathcal{A}} \left(\frac{\Phi_s}{P_s} \frac{\partial P_s}{\partial t} + \alpha\omega + q \right) dm \quad (4)$$

where t is time, α specific volume, $\omega \equiv dP/dt$, and q diabatic heating rate. We assumed that the surface height $h_s = \Phi_s/g$ is independent of time. The first term on the right side is commonly neglected, but if we wish to account for the effect of topography it is essential. The second term represents conversion of TPE to kinetic energy, and the last term accounts for changes in TPE due to diabatic heating.

The equation for the rate of change of $\widehat{\text{TPE}}$, that is the total potential energy of the equivalent barotropic state, can be derived from its definition (see eq. 2).

$$\frac{\partial}{\partial t}\widehat{\text{TPE}} = \int_S \left\{ h_s \frac{\partial \hat{P}_s}{\partial t} + \frac{c_p P_{00}^{-\kappa}}{(\kappa+1) g} \left[\frac{\partial}{\partial t} (\hat{P}_s^{\kappa+1} \hat{\theta}_s) + \frac{\partial}{\partial t} \int_{\hat{\theta}_s}^{\infty} \hat{P}^{\kappa+1} d\theta \right] \right\} d\sigma \quad (5)$$

We wish to evaluate this expression in terms of variables of the *given* (observed) state. As we said earlier, it is possible to obtain $\hat{P}$ and $\hat{\theta}_s$ from the given state when the reference state is suitably defined. In the above expression, however, the *temporal derivative* of $\hat{P}$ appears which is a little more difficult to obtain because in general $\hat{P}$ cannot be written as an *explicit* function of the given-state variables.

In Appendix I we show how the above integral can be rewritten to obtain an expression for the rate of change of $\widehat{\text{TPE}}$ that is easier to evaluate. Making use of the hydrostatic approximation, the continuity equation (in isentropic coordinates), the thermodynamic equation, and implications of the kinematic boundary condition at the surface, we find (see Appendix I)

$$\frac{\partial}{\partial t}\widehat{\text{TPE}} = \int_S \left\{ \frac{\partial \hat{P}_s}{\partial t} \left[h_s + \frac{c_p P_{00}^{-\kappa}}{g} \times (\hat{\theta}_s \hat{P}_s^{\kappa} - \int_{\theta_L}^{\hat{\theta}_s} \hat{P}^{\kappa} d\theta) \right] \right\} d\sigma + \int_{\mathcal{A}} \left(\frac{\hat{P}}{P} \right)^{\kappa} q \, dm \quad (6)$$

where θ_L is the lowest potential temperature found in the given atmospheric state (and hence also in the equivalent state). The surface integral is still difficult to evaluate from the given state because it involves the *temporal derivative* of $\hat{P}_s$. As with $\partial\hat{P}/\partial t$, it is impossible to determine $\partial\hat{P}_s/\partial t$ as an *explicit* function of only the given-state variables. To eliminate this difficulty, the reference state is chosen so that the surface integral in (6) vanishes

identically. As we shall see, either of the two reference states described below eliminates the need to determine $\partial \hat{P}_s/\partial t$.

One barotropic reference state that suits our requirements has a *uniform* surface pressure. In this case $\partial \hat{P}_s/\partial t$ is zero everywhere because the mass of the atmosphere and the earth's surface area are constant. It turns out that this reference field contains the minimum enthalpy of all equivalent fields.[1]

Another reference state, however, is probably more appropriate: the equivalent state with minimum TPE. This reference state turns out to be the equivalent barotropic field that is *horizontally* stratified. In this case the first integral in (6) again vanishes as we shall now prove.

Integration of the hydrostatic equation implies that for a horizontally stratified barotropic atmosphere

$$h_s - z_L = -\frac{c_p P_{00}^{-\kappa}}{g}\left[\hat{P}_s^{\kappa}\,\hat{\theta}_s - \hat{P}_L^{\kappa}\,\theta_L - \int_{\theta_L}^{\hat{\theta}_s} \hat{P}^{\kappa}\, d\theta\right]$$

The subscript L indicates that the variable should be evaluated at θ_L, and z is the vertical height coordinate (note: $h_s \equiv z_s$). This expression can be used to rewrite the first integral in (6) as

$$\int_S \frac{\partial \hat{P}_s}{\partial t}\left[z_L + \frac{c_p P_{00}^{-\kappa}}{g}\hat{P}_L^{\kappa}\,\theta_L\right] d\sigma$$

The quantity in brackets is independent of position and may therefore be removed from inside the integral. The partial derivative operator may also be taken outside the integral. We are left with the rate of change of the integral of the surface pressure, which vanishes since the total mass of the atmosphere is conserved. With a *horizontally stratified* reference state, then, we get the same result for the rate of change of $\overline{\text{TPE}}$ as we did for a barotropic, *uniform surface pressure* reference state.

Use of either of these reference states gives the following expression for the rate of change of ΔTPE:

$$\frac{\partial}{\partial t}(\Delta\text{TPE}) = \int_{\mathcal{M}}\left\{\frac{\Phi_s}{P_s}\frac{\partial P_s}{\partial t} + \alpha\omega + \left[1 - \left(\frac{\hat{P}}{P}\right)^{\kappa}\right] q\right\} dm \qquad (7)$$

[1] We can prove this assertion using Van Mieghem's (1973) methods to show that in this reference state the rate of change of the total atmospheric enthalpy is zero for reversible adiabatic changes in state.

It should be remembered that $\hat{P}(\theta, t)$ is different in the two reference states, and hence the so-called efficiency factor $[1 - (\hat{P}/P)^{\kappa}]$ is not unique, but depends on the reference state.

In the above formula, $\partial P_s/\partial t$ can, of course, be evaluated from the given atmospheric state by integrating the continuity equation,

$$\frac{\partial P_s}{\partial t} = -\nabla \cdot \int_0^{P_s} \mathbf{v}\, dP$$

where $\mathbf{v}$ is the "horizontal" component of fluid velocity. Thus the integral in (7) is completely determined by the given state if $\hat{P}$ can be evaluated from the given field. When the reference state is the equivalent field with least TPE, then ΔTPE ≡ APE.

2.3. *Determination of the reference state*

The problem that remains, then, is to determine the reference pressure field in terms of the given atmospheric state. Specifically we need to find $\hat{P}$ as a function of θ, and to find $\hat{\theta}_s$ as a function of position. (Recall that $\hat{\theta}_s$ enters in evaluating the integral apppearing in eq. 3.)

A consequence of the definition of equivalent atmospheric states is that the total atmospheric mass between isentropic surfaces remains constant as the air parcels in the atmosphere are rearranged. This can be expressed mathematically by the relationship,

$$\bar{P}(\theta, t) \equiv \frac{1}{4\pi a^2}\left[\int_{S^*(\theta,t)} P(\lambda, \phi, \theta, t)\, d\sigma + \int_{S-S^*} P_S(\lambda, \phi, t)\, d\sigma\right]$$

$$= \frac{1}{4\pi a^2}\left[\int_{S^*(\theta,t)} \hat{P}(\theta, t)\, d\sigma + \int_{S-S^*} \hat{P}_s\, d\sigma\right] \equiv \bar{\hat{P}}(\theta, t) \qquad (8)$$

where λ is longitude, ϕ is latitude, and $S^*(\theta,t)$ and $S - S^*$ are surfaces of constant θ covering regions of S where θ is above ground and "underground", respectively. Note that the surface area above ground for a particular θ-level may be a function of time. We should also realize that $\hat{P} \neq \bar{P}$ except when $S^* = S$.

Of the two reference states already proposed, the one with *minimum enthalpy* (the one with uniform $\hat{P}_s$) is the easiest to determine. In this case none of the isentropic surfaces of the reference state intersect the earth's surface, so that $S^* = S = 4\pi a^2$ for

all θ and t. Thus $\hat{P}$ can be determined immediately from the given state:

$$\bar{\hat{P}}(\theta, t) = \hat{P}(\theta, t) = \frac{1}{4\pi a^2} \times [\int_{S^*} P d\sigma + \int_{S-S^*} P_s \, d\sigma] \tag{9}$$

This formula for calculating the reference pressure field from the given pressure field is identical to the formula given by Lorenz (1955). It turns out then that if Lorenz's formula is directly used to calculate APE when topography is important, the actual reference state is *not* one of minimum TPE. Lorenz's formula also omits the first term on the right hand side of (3) (the average correlation between surface pressure and surface height), which in general vanishes only under his assumption that $h_s = 0$. With topography, however, Lorenz's formula is actually an exact formula for calculating what might be called available enthalpy (the difference in enthalpy between a given state and an equivalent state of minimum enthalpy). Under his assumption that $h_s = 0$, the available enthalpy is, of course, equal to the available potential energy.

We turn now to a calculation of the pressure distribution of the reference field with *minimum TPE*. This field is more difficult to determine than the minimum enthalpy reference field because in this case $\hat{S}^*(\theta)$ must be deduced along with $\hat{P}(\theta)$. Toward that end we shall rewrite (8) as follows:

$$M(\theta, t) = \frac{4\pi a^2}{g} \bar{\hat{P}}(\theta, t) = -\int_{\hat{S}^*} \int_{\theta}^{\infty} \frac{\partial \hat{P}}{\partial \theta} \frac{d\theta}{g} \, d\sigma - \int_{S-\hat{S}^*} \int_{\hat{\theta}_s}^{\infty} \frac{\partial \hat{P}}{\partial \theta} \frac{d\theta}{g} \, d\sigma = -\int_{\theta}^{\infty} \hat{S}^* \frac{\partial \hat{P}}{\partial \theta} \frac{d\theta}{g} \tag{10}$$

This equation relates $\hat{P}(\theta)$ and $\hat{S}^*(\theta)$ to $M(\theta, t)$, the mass distribution of θ, which can be calculated from the given state by evaluating the first two integrals in (8). Although we do not yet know $\hat{S}^*(\theta)$, we *do* know the surface terrain height everywhere. We can therefore use ordinary topographic maps to estimate the area that lies above ground on any surface of constant z. We shall call this above-ground area $\mathscr{S}^*(z)$.

To evaluate $\hat{P}(\theta)$ in eq. (10), we need to know $\hat{S}^*(\theta)$, not $\mathscr{S}^*(z)$. We should note that the $\mathscr{S}^*$ surfaces coincide with the $\hat{S}^*$ surfaces but not necessarily with the S^* surfaces because only in the reference state are the isentropic surfaces horizontal. In the horizontally stratified reference state, then, we can write a formal relationship between $\mathscr{S}^*$ and $\hat{S}^*$:

$$\mathscr{S}^*(z) = \mathscr{S}^*(\hat{z}) = \mathscr{S}^*[\hat{z}(\theta)] = \hat{S}^*(\theta) \tag{11}$$

This expression is useless unless we can find $\hat{z}(\theta)$. In fact, at this point there are three unknowns ($\hat{P}$, $\hat{S}^*$, and $\hat{z}$), but only two equations, (10) and (11). The hydrostatic equation is needed to close the problem mathematically:

$$\frac{\partial \hat{z}}{\partial \theta} = \frac{c_p}{g} \theta \frac{\partial}{\partial \theta} \left(\frac{\hat{P}}{P_{00}} \right)^{\kappa} \tag{12}$$

$\hat{z}$ is of course independent of horizontal position because the reference state is horizontally stratified.

For given $M(\theta, t)$ and $\mathscr{S}^*(z)$, eqs. (10), (11), and (12) can now be solved simultaneously to yield $\hat{P}(\theta)$, $\hat{S}^*(\theta)$, and $\hat{z}(\theta)$. One viable method for obtaining such a solution is discussed in Appendix II.

3. A comparison of formulas for calculating APE

The purpose of this section is to illustrate the differences in various "exact" APE formulas by comparing their quantitative estimates of APE in a sample calculation. We shall treat a very simple, idealized case to minimize the computational effort required.

Suppose that nine-tenths of the earth's surface is at sea-level and the other tenth is at a uniform height of 2.43 km. Suppose that the atmosphere is in a minimum TPE, horizontally stratified, barotropic state with sea-level pressure P_0 of 1013 mb and sea-level potential temperature θ_0 of 287 K. We shall assume that the atmospheric pressure distribution is everywhere related to the potential temperature as follows:

$$P = P_0 + 1.4 \times 10^3 (\theta_0 - \theta)$$

The surface pressure on the mountain turns out to be 753 mb.

Four different formulas will be used to calculate APE for the atmospheric pressure distribution given above. In all four, the following values were assigned to the physical constants: $R = 287$ J/kg/K, $c_p = 10^3$ J/kg/K, $P_{00} = 10^3$ mb, $g = 10$ m/s^2. The first formula is the exact one described in Section 2 with the minimum TPE reference state. This formula gives APE = 0, since the *given* atmospheric state is in fact the *reference* state. The second formula is the one with a minimum enthalpy reference state, again described in Section 2. The difference between the TPE of the given state and the TPE of this reference state is -29.5×10^5 J-m^2. Lorenz's (1955) "exact" formula differs from the preceding one in that he omits the first term on the right hand side of (3), which is large and negative. His formula gives 27.3×10^5 J/m^2 and is actually a measure of the "available enthalpy", as discussed earlier. Finally, we again apply Lorenz's formula but adopt a different convention for defining pressure on "underground" isentropes. Instead of setting the pressure equal to the local surface pressure as is required by Lorenz's "exact" formula, we define the pressure on all "underground" isentropes to be 1000 mb. Dutton and Johnson (1967) adopted this procedure in a sample calculation with real data. Applying this ad hoc procedure to our idealized case, we calculate the average APE per unit area to be 0.51×10^5 J/m^2.

Although the above calculations are performed on an unrealistic given state, we might still reach some tentative conclusions as to which of the formulas would be reasonably accurate in a more realistic calculation. In our idealized calculation the atmospheric scale height, average surface pressure, average surface temperature, lapse rate, and mountain volume are similar to the earth's. Although the horizontal temperature structure is not the same as the natural atmosphere's, the magnitude of the errors in estimating APE by the different formulas for this simple case would probably be typical of those that would arise in evaluating the APE of a realistic atmospheric state. We should expect that if the errors in the formulas, as deduced by our sample calculation, are large relative to the observed APE, then the formulas would probably give equally unsatisfactory results in a more realistic calculation. We shall compare the errors from our calculation with Oort and Peixoto's (1974) estimate that in the natural atmosphere the annual average APE per unit area is about 50×10^5 J/m^2.

The results of the sample calculations are summarized in Table 1. It is clear that if we want a measure of APE as it is usually defined (i.e., the energy "available" for conversion to kinetic energy), only the first formula is precisely correct. The second formula used a minimum *enthalpy* reference state and did not provide a good measure of APE: the error in the sample calculation was of the same order as Oort and Peixoto's estimate of APE in the natural atmosphere. Lorenz's formula gave an equally poor estimate of APE because of the presence of topography. The Dutton and Johnson procedure, on the other hand, gave a good approximation with an error that had a magnitude of about 1% of the observed APE.

The reason that Dutton and Johnson's procedure gives a better estimate of APE than a strict application of Lorenz's formula is that their formula is in error only between sea-level and the top of the mountain. Lorenz's formula, on the other hand, calculates a contribution to APE for *all* underground isentropes (from 0 to about 300 K). To our knowledge no one has yet naively applied Lorenz's formula in the straightforward way that was done here, and our conclusion is that such an application should be avoided in the future, too. Instead, it would be better to adopt Dutton and Johnson's practical procedure where the underground isentropes are all at one pressure.

Table 1. *Comparison of four estimates of average APE per unit area for a simple case in which the true APE is zero*

Formula*	Definition of pressure on "underground" isentropes	Reference state	Estimate (10^5J/m^2)
$A + B$	conventional (equal to local surface pressure)	minimum TPE (horizontally stratified, see Section 2)	0.0
$A + B$	conventional	minimum enthalpy (uniform $\hat{P}_s$; see Section 2)	−29.5
A	conventional	minimum enthalpy (Lorenz, 1955)	27.3
A	equal to 1000 mb	Dutton and Johnson (1967)	0.5

* $A = \int_0^\infty (\overline{P^{\kappa+1}} - \overline{\hat{P}^{\kappa+1}})\, d\theta \qquad B = \overline{(P'_s - \hat{P}'_s)\, h'_s}$

A possibly even better procedure would be to interpolate *horizontally* the pressure in regions where the isentropic surfaces are partially underground. Although still an approximate treatment, this approach would probably reduce the errors in the lower part of the atmosphere where mountains often protrude above the nearly horizontal isentropic surfaces. When Lorenz's formula is used in conjunction with this kind of an interpolation procedure, the APE is calculated to be zero for the sample atmospheric state described above and thus, in this case, is exactly correct. For a more realistic atmosphere, we could expect that with this procedure, the error would remain quite small, although it would be prudent to conform this by using the exact formula derived in this article. One case, for example, in which Lorenz's original convention for defining pressure on underground insentropes is clearly superior to the Dutton and Johnson method or the interpolation method just discussed is when the planet's surface is "flat" (which is, of course, the only case Lorenz considered). Lorenz's formula is then free of error, as are our formulas (eq. 3); the Dutton and Johnson method and the interpolation method, however, are inexact unless the given state has uniform surface pressure and temperature.

4. Concluding remarks

When the atmospheric APE is calculated in practice, Lorenz's (1955) "exact" formula is normally replaced by one of his approximate formulas. It is unclear, however, whether this practice should continue. Boer (1975) and Dutton and Johnson (1967) have argued that the approximate formulas can lead to significant errors in estimates of the atmospheric energy budget, and perhaps these formulas should be given up in favor of more exact formulas. If their advice is heeded, the results of this paper could be of immediate practical value.

We have shown in Section 3 that in cases with mountains, Lorenz's "exact" APE formula can grossly overestimate the actual APE. It appears that Dutton and Johnson's (1967) ad hoc procedure of setting the pressure equal to 1000 mb on all "underground" isentropes might be one simple way of modifying the original formula so that reasonable estimates could be obtained even when topography is present. This possibility has yet to be demonstrated for a realistic case, however. Calculations with real data should be made to compare our exact formula (given in Section 2) to the Dutton and Johnson method, and to determine thereby whether the easier, approximate method is of sufficient accuracy to be useful in practice. Similar comparisons should be made to evaluate the interpolation method of treating "underground" isentropes, that was suggested at the end of Section 3.

Before deciding whether in practice a particular method of computing APE is superior to another, one should be well aware of the deficiencies in the observational data that are presently available. It is likely that in view of the sparse nature of the global observational network, use of the "exact" formula derived here is not yet warranted. For certain studies, however, use of the exact formulas will at some point be essential. For example, one might wish to calculate the change in APE between summer and winter, accounting, in particular, for the increase in surface pressure over land in winter (typified at that time by the semi-permanent high pressure region over the Asiatic mainland). Saltzman (1961) has already discussed the importance of this phenomenon in the total potential energy budget, but until now its effects could not be estimated in the APE budget.

A more likely immediate application of our results is in diagnostic studies of general circulation model (GCM) experiments in which the artificial data coverage is excellent. At present, reasonable, but somewhat arbitrary, interpolation or extrapolation procedures have been used to treat data points lying below the earth's surface. Perhaps justification for these could be found in the formulas presented here, especially in light of the results of Section 3. In the end, of course, it should be possible and would be desirable to make exact calculations of a GCM's available potential energy budget; this would allow one to verify that the model was formulated properly and was conserving energy exactly. The formulas derived here would be easiest to apply to GCM's that use potential temperature as the vertical coordinate.

Another possibly important, but presently premature, application of these results could arise in evaluating the atmospheric energy budget on other planets where the effects of topography are certainly not negligible. For example, the surface

pressure on Mars varies from 2 to 11 mb (Webster, 1977). The large topographic features there possess vertical scales of the same order as the atmospheric scale height and horizontal scales equal to or greater than the planetary radius. The formulas that neglect topography could lead to serious difficulties because so many grid points would lie underground. Thus, on Mars it might be essential to apply some form of the exact equations developed here. Some modifications would be required, however, to account for the seasonal change in atmospheric mass observed there (Ryan et al., 1978).

Even if no immediate applications of the exact formulas for APE are realized, it is aesthetically pleasing to have generalized the formulas without significantly complicating them. Henceforth, it is possible to decide objectively whether approximate formulas for APE are adequate, even in cases with uneven surface topography.

5. Acknowledgements

I am grateful to Professor Edward N. Lorenz and Dr Steven Ashe for reading and commenting on a preliminary version of this manuscript. This research was sponsored within the Climate Program of the Cooperative Institute for Research in Environmental Sciences (CIRES) through a Visiting Fellowship.

6. Appendix I

In this appendix we shall present a derivation of eq. (6), the rate of change of the total potential energy of an equivalent barotropic state. The initial equation was given in the text, eq. (5). Applying Liebnitz's rule for differentiating an integral with variable limits leads to

$$\frac{\partial}{\partial t}\widehat{\mathrm{TPE}} = \int_S \left[h_s \frac{\partial \hat{P}_s}{\partial t} + \frac{c_p P_{00}^{-\kappa}}{g}\left(\hat{\theta}_s \hat{P}_s^{\kappa} \frac{\partial \hat{P}_S}{\partial t} + \int_{\hat{\theta}_s}^{\infty} \hat{P}^{\kappa} \frac{\partial \hat{P}}{\partial t} d\theta \right) \right] d\sigma \qquad \text{(I-1)}$$

We are looking for an equivalent expression for the right hand side that can be evaluated in terms of the *given* state. This obviously entails finding an alternate expression for $\partial \hat{P}/\partial t$. This expression can be obtained by differentiating eq. (8), again moving the derivative inside the integral:

$$\int_{S^*} \frac{\partial \hat{P}}{\partial t} d\sigma + \int_{S-S^*} \frac{\partial \hat{P}_s}{\partial t} d\sigma = \int_{S^*} \frac{\partial P}{\partial t} d\sigma + \int_{S-S^*} \frac{\partial P_s}{\partial t} d\sigma \qquad \text{(I-2)}$$

This time the two boundary terms that arise on each side of the equation (because $\hat{S}^*$ and S^* are functions of time) cancel one another identically.

The right hand side of (I-2) may be rewritten as follows:

$$\int_{S^*} \frac{\partial P}{\partial t} d\sigma + \int_{S-S^*} \frac{\partial P_s}{\partial t} d\sigma = -\int_{S^*} \int_{\theta}^{\infty} \frac{\partial}{\partial t}\left(\frac{\partial P}{\partial \theta}\right) d\sigma - \int_{S-S^*} \int_{\theta_s}^{\infty} \frac{\partial}{\partial t}\left(\frac{\partial P}{\partial \theta}\right) d\sigma + \int_{S-S^*} \left.\frac{\partial P}{\partial \theta}\right|_s \frac{\partial \theta_s}{\partial t} d\sigma$$

Substituting the continuity equation for a hydrostatic fluid (in isentropic coordinates),

$$\frac{\partial}{\partial t}\left(\frac{\partial P}{\partial \theta}\right) + \nabla \cdot \left(\frac{\partial P}{\partial \theta}\mathbf{v}\right) + \frac{\partial}{\partial \theta}\left(\frac{\partial P}{\partial \theta}\frac{d\theta}{dt}\right) = 0$$

and applying the relationship

$$\left.\frac{d\theta}{dt}\right|_s = \frac{\partial \theta_s}{\partial t} + \mathbf{v}_s \cdot \nabla \theta_s$$

which is a consequence of the kinematic boundary condition at the surface, yields

$$\int_{S^*} \frac{\partial P}{\partial t} d\sigma + \int_{S-S^*} \frac{\partial P_s}{\partial t} d\sigma = \int_{S^*} \nabla \cdot \int_{\theta}^{\infty} \frac{\partial P}{\partial \theta} \mathbf{v}\, d\theta d\sigma + \int_{S-S^*} \nabla \cdot \int_{\theta_s}^{\infty} \frac{\partial P}{\partial \theta} \mathbf{v}\, d\theta d\sigma - \int_{S^*} \frac{\partial P}{\partial \theta}\frac{d\theta}{dt} d\sigma \qquad \text{(I-3)}$$

The sum of the two integrals involving the divergence can be shown to vanish identically according to Gauss' Theorem.

Equation I-2 then becomes

$$S^* \frac{\partial \hat{P}}{\partial t} + \int_{S-S^*} \frac{\partial \hat{P}_s}{\partial t} d\sigma$$

$$= -\int_{S^*} \frac{\partial P}{\partial \theta} \frac{d\theta}{dt} d\sigma$$

The first term could be integrated immediately because $\partial \hat{P}/\partial t$ is independent of horizontal position. The above equation when substituted into the last integral in (I-1) eliminates the need to evaluate $\partial \hat{P}/\partial t$ (except at the surface):

$$\int_S \int_{\hat{\theta}_s}^{\infty} \hat{p}^\kappa \frac{\partial \hat{P}}{\partial t} d\theta d\sigma = \int_{\theta_L}^{\infty} \hat{p}^\kappa S^* \frac{\partial \hat{P}}{\partial t} d\theta$$

$$= -\int_{\theta_L}^{\infty} \int_{S-S^*} \hat{p}^\kappa \frac{\partial \hat{P}_s}{\partial t} d\sigma d\theta - \int_{\theta_L}^{\infty} \int_{S^*} \hat{p}^\kappa \frac{\partial P}{\partial \theta} \frac{d\theta}{dt} d\sigma d\theta$$

$$= -\int_S \int_{\theta_L}^{\hat{\theta}_s} \hat{p}^\kappa \frac{\partial \hat{P}_s}{\partial t} d\theta\, d\sigma - \int_S \int_{\theta_s}^{\infty} \hat{p}^\kappa \frac{\partial P}{\partial \theta} \frac{d\theta}{dt} d\theta\, d\sigma$$

The equation for the rate of change of $\widehat{\text{TPE}}$ may now be rewritten in the desired form:

$$\frac{\partial}{\partial t} \widehat{\text{TPE}} = \int_S \left[h_s \frac{\partial \hat{P}_s}{\partial t} + \frac{c_p P_{00}^{-\kappa}}{g} \left(\hat{\theta}_s \hat{p}_s^\kappa \frac{\partial \hat{P}_s}{\partial t} \right.\right.$$

$$- \int_{\theta_L}^{\hat{\theta}_s} \hat{p}^\kappa \frac{\partial \hat{P}_s}{\partial t} d\theta$$

$$\left.\left. - \int_{\theta_s}^{\infty} \hat{p}^\kappa \frac{\partial P}{\partial \theta} \frac{d\theta}{dt} d\theta \right) \right] d\sigma$$

$$= \int_S \frac{\partial \hat{P}_s}{\partial t} \left[h_s + \frac{c_p P_{00}^{-\kappa}}{g} \right) \hat{\theta}_s \hat{p}_s^\kappa$$

$$\left. - \int_{\theta_L}^{\hat{\theta}_s} \hat{p}^\kappa d\theta \right) \Bigg] d\sigma$$

$$+ \int_{\mathcal{A}} \left(\frac{\hat{p}}{P} \right)^\kappa q\, dm$$

We have used the first law of thermodynamics,

$$\frac{d\theta}{dt} = \left(\frac{P_{00}}{P} \right)^\kappa \frac{q}{c_p}$$

to obtain the last integral.

7. Appendix II

Here we shall give one numerical procedure for calculating the reference state of minimum total potential energy, that is the reference state for APE. From the given state we know $M(\theta)$ (the total atmospheric mass above a θ-level), and we know $\mathcal{S}^*(z)$ (the area above ground at a particular level z). We can use eqs. (10), (11) and (12) to determine $\hat{P}(\theta)$, $\hat{S}^*(\theta)$, and $\hat{z}(\theta)$, thereby uniquely determining the reference state.

For numerical reasons it is convenient to differentiate eq. (10) producing:

$$g \frac{\partial M}{\partial \theta} = \hat{S}^* \frac{\partial \hat{P}}{\partial \theta} \tag{II-1}$$

To retain the full information of the original equation we also require that

$$gM(\theta_L) = \int_S \hat{P}_s\, d\sigma \tag{II-2}$$

These two equations along with the hydrostatic relationship (eq. 12) will be cast in finite difference form in order to apply the numerical method.

The atmosphere can be divided into I isentropic levels, each level labeled by a different value of θ or, alternatively, a different subscript value for θ_i. The lowest value of $\theta(=\theta_L)$ corresponds to θ_1 and the highest value corresponds to θ_I. At each level, the variables $\hat{P}$, $\hat{z}$, M, and $\hat{S}^*$ will carry the same subscript value as θ at that level. The equations (II-1, 12, and II-2) for determining the reference state can then be written in the following forms:

$$M_{i+1} - M_i - \tfrac{1}{2}(\hat{S}^*_{i+1} + \hat{S}^*_i)(\hat{P}_{i+1} - \hat{P}_i) = 0$$
$$\text{for } 1 \le i \le I-1 \tag{II-3}$$

$$\hat{z}_{i+1} - \hat{z}_i + \frac{R}{2g} P_{00}^{-\kappa} (\hat{P}^\kappa_{i+1} - \hat{P}^\kappa_i)(\theta_{i+1} + \theta_i) = 0$$
$$\text{for } 1 \le i \le I-1 \tag{II-4}$$

$$M(\theta_1) - \tfrac{1}{2} P_1 (\hat{S}^*_1 + \hat{S}^*_2)$$
$$- \tfrac{1}{2} \sum_{k=2}^{K} \hat{P}_k (\hat{S}^*_{k+1} - \hat{S}^*_{k-1}) = 0 \tag{II-5}$$

where the summation is allowed to stop at level K, defined such that $\hat{S}^*_k = S$ for $k \ge K$.

To mathematically close the system of equations we must also relate $\hat{S}^*(\theta)$ to $\mathcal{S}^*(z)$. If we presume

that $\mathscr{S}^*(z)$ is given at discrete values of z labeled by a subscript j, we can linearly interpolate from those j-levels to the i-levels through the formulas

$$\hat{S}_i^* = \frac{z_u \mathscr{S}_l^* - z_l \mathscr{S}_u^*}{z_u - z_l} + \frac{\hat{z}_i}{z_u - z_l}(\mathscr{S}_u^* - \mathscr{S}_l^*) \quad \text{for } i \le K$$

$$\hat{S}_i^* = S = 4\pi a^2 \quad \text{for } i \ge K \tag{II-6}$$

where in the reference state, l is the j-level at or just below the i-level, and u is the j-level at or just above the i-level; more concisely, $z_l \le \hat{z}_i \le z_u$ and $u = l + 1$.

Now we can use an iterative process to compute $\hat{P}_i$, $\hat{z}_i$, and $\hat{S}_i$ for a given field of $M(\theta_i)$ and for given values of $\mathscr{S}^*(z_j)$. First we make an initial guess of the reference pressure field at grid points i. One reasonable guess might be to let $\hat{P}_i$ be equal to the average pressure on the isentropic surface θ_i in the *given* (observed) state:

$$\hat{P}_i = \frac{1}{S^*(\theta_i)} \int_{S^*} P \, d\theta$$

We can compute a corresponding initial field of $\hat{z}_i$ with (II-4). The initial guess for $\hat{S}_i^*$ is then determined by (II-6). Several iterative schemes for correcting the initial guess are available. The Newton-Raphson method is not necessarily the most economical but has the advantage that it usually converges and is not difficult to apply to the above set of equations (II-3, II-4, II-5, and II-6).

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ФОРМУЛЫ ДЛЯ ВЫЧИСЛЕНИЯ ДОСТУПНОЙ ПОТЕНЦИАЛЬНОЙ ЭНЕРГИИ НАД НЕРОВНОЙ ТОПОГРАФИЕЙ

Выводятся формулы, которые корректно описывают неровную топографию поверхности (гóры) при вычислении атмосферной доступной потенциальной энергии и скорости ее изменений. Обсуждаются два баротропных состояния отсчета для доступной потенциальной энергии: одно с горизонтальной стратификацией, другое с постоянным давлением на поверхности. Первое является состоянием минимума полной потенциальной энергии, а второе-состоянием миниймума энтальпии. Оба предложенных состояния отсчета приводят к похожим простым выражениям для генерации доступной потенциальной энергии. Представлена численная процедура для вычисления горизонтально стратифицированиого отсчетного состояния по заданному состоянию атмосферы. Для простого случая с горами сравниваются разные методы вычисления доступной потенциальной энергии, что иллюстрирует ошибки, присущие предыдущим формулам. В заключительном разделе говорится о возможных применениях "точных" выражений.