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Solitary Rossby waves in a two-layer system

By HISASHI HUKUDA, *Geophysical Institute, Tohoku University, Sendai 980, Japan*

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ABSTRACT

The solutions for solitary Rossby waves with no critical level in a two-layer system which contains a zonal, baroclinic, mean shear flow are presented. When the mean shear flow has a barotropic structure, these solutions agree with those of Redekopp (1977) in the sense that the barotropic mode obeys the Korteweg–deVries equation, whereas the baroclinic mode obeys the modified Korteweg–deVries equation. It is found that the presence of vertical shear in the mean flow generally tends to steepen the solitary Rossby waves, but the tendency depends strongly on the particular wave mode and the magnitude of the internal rotational Froude number characterizing the wave field.

1. Introduction

While the existence of solitary waves has been known to fluid dynamicists from about a century ago, it is only recently that the theory was applied to the planetary-scale wave phenomenon. Since Long (1964) derived the Korteweg–deVries (KdV) equation on a barotropic atmosphere, the theories of solitary Rossby waves (planetary solitons) have been developed by several people, e.g. Larsen (1965), Benney (1966), Clarke (1971) and Redekopp (1977), primarily in the context of the atmospheric models. Redekopp (1977) especially performed the extensive investigation on the evolution of eddies in the atmospheric models which include zonal jets with critical levels and found that long finite-amplitude Rossby waves in a homogeneous atmosphere evolve in obedience to the KdV equation, but long baroclinic waves in a uniformly stratified atmosphere (Brunt–Vaisala frequency $N(z)$ = const.) obey the modified Korteweg–deVries (mKdV) equation. Maxworthy and Redekopp (1976) showed that the Great Red Spot observed in the Jovian atmosphere may be a manifestation of these planetary solitons.

Their theory, however, assumed barotropic zonal currents and thereby was restricted to discussing only the purely barotropic and baroclinic modes. In the ocean or atmosphere, mean currents in which planetary waves are embedded have a vertical as well as a horizontal shear.

In the present paper, we study the modifications of planetary solitons in a two-layer system caused by the presence of vertical shear in the mean zonal flow. The analysis is restricted to the case where the mean baroclinic flow has a weak horizontal shear. When the limiting case of no vertical shear is examined, we review the results of Redekopp (1977), namely that the barotropic mode obeys the KdV equation, whereas the baroclinic mode obeys the mKdV equation. This means that the coefficient of non-linear term in the KdV equation happens to be zero in a special situation, e.g. a uniformly stratified (N = const.) or a two-layer system with the barotropic mean field, and then the evolution equation for the solitary Rossby wave becomes the mKdV equation. However, when we include the vertical shear in the mean flow, it is found that either wave mode obeys uniquely the KdV equation. The increase in the vertical shear tends to increase the steepness of the KdV solitons, but the tendency depends not only on the character of the wave mode but on the magnitude of the internal rotational Froude number.

2. The basic equation

The theoretical basis is found in the paper of Pedlosky (1970) treating a two-layer channel model on a beta plane. In this model the vorticity equation governing the inviscid, quasi-geostrophic fluid

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motion is, in the non-dimensional form, given by

$$(\partial_{t'} + \psi_{nx'}\,\partial_{y'} - \psi_{ny'}\,\partial_{x'}) \times \{\nabla'^2 \psi_n + \varepsilon_n F(\psi_2 - \psi_1) + \beta y'\} = 0 \quad (2.1)$$

where $n = 1$, 2 and $\varepsilon_1 = 1$, $\varepsilon_2 = -1$. In (2.1) the variables with subscripts 1, 2 refer to the quantities defined in the upper and lower layers, respectively. The ψ_n represents the geostrophic stream function in the nth layer. The zonal and meridional velocity components are then defined by $u_n = -\psi_{ny'}$ and $v_n = \psi_{nx'}$. The x' and y' are the local Cartesian coordinates pointing east and north, respectively, both scaled with L (L: the width of the channel), t' the time scaled with LU (U: the characteristic velocity scale), and ∇'^2 the horizontal Laplacian operator. The two non-dimensional parameters are defined as $F = f_0^2 L^2/(\Delta\rho g D/2\rho)$ (the internal rotational Froude number) and $\beta = \beta^* L^2/U$ (the planetary vorticity factor), where f_0 is a constant Coriolis parameter measured in the moderate position, $\Delta\rho/\rho$ the density difference anomaly between the upper and lower layers, g the gravitational acceleration, $D/2$ either depth of two layers and β^* the northward gradient of the planetary vorticity. We can specify the boundary conditions at the northern and southern walls, i.e.

$$\psi_{nx'} = 0 \quad \text{at } y' = 0, 1 \quad (2.2)$$

The reader is referred to Pedlosky (1970) for the details of derivation of eq. (2.1).

Since we anticipate the solutions resembling long Rossby waves propagating in the zonal direction, we make the transformation

$$\xi = l(x' - ct'), \quad y = y', \quad t = t'$$

Here c is the phase speed of a long Rossby wave and l the dimensionless zonal wave number assumed to be smaller than unity. Then, supposing steady motions in the new reference frame, eqs. (2.1) and (2.2) are transformed into

$$\{\psi_{n\xi}\,\partial_y - (\psi_{ny} + c)\,\partial_\xi\} \times \{l^2 \psi_{n\xi\xi} + \psi_{nyy} + \varepsilon_n F(\psi_2 - \psi_1) + \beta y\} = 0$$

$$\psi_{n\xi} = 0 \quad \text{at } y = 0, 1 \quad (2.3)$$

The subsequent analysis assumes that the total flow is composed of the mean (time-independent) shear flow plus the long wave disturbance with amplitude small but finite. The simplest of such disturbances turn out to be solitary waves described by the KdV or the mKdV equation depending on the balance between the effects of non-linearity and dispersion.

3. Expansion of solution

3.1. l^2 expansion

First, a solution to (2.3) may be sought in the form

$$\psi_n(\xi, y) = -\int_0^y U_n(y)\,dy + l^2 \psi_{n1}(\xi, y) + l^4 \psi_{n2}(\xi, y) + \cdots$$

$$C = C_0 + l^2 C_1 + \cdots \quad (3.1)$$

where $U_n(y)$ represents a zonal flow in the nth layer and C_0 is the phase speed of an infinitely long Rossby wave. We note that (3.1) satisfies exactly (2.3) when l is set to zero. Thus, substitution of (3.1) into (2.3) yields

$$\begin{aligned} &\partial_\xi[(U_n - C_0)\{\psi_{n1yy} + \varepsilon_n F(\psi_{21} - \psi_{11})\} \\ &+ \psi_{n1}\{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta\}] \\ &+ l^2(\partial_\xi[(U_n - C_0)\{\psi_{n2yy} + \varepsilon_n F(\psi_{22} - \psi_{12})\} \\ &+ \psi_{n2}\{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta\}] \\ &+ (U_n - C_0)\,\psi_{n1\xi\xi\xi} + \{\psi_{n1\xi}\,\partial_y - (\psi_{n1y} + C_1)\,\partial_\xi\} \\ &\times \{\psi_{n1yy} + \varepsilon_n F(\psi_{21} - \psi_{11})\}) + O(l^4) = 0 \end{aligned} \quad (3.2)$$

From (3.2) the problem of the lowest order is written as

$$\partial_\xi[(U_n - C_0)\{\psi_{n1yy} + \varepsilon_n F(\psi_{21} - \psi_{11})\} + \psi_{n1}\{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta\}] = 0 \quad (3.3)$$

The solution of (3.3) is

$$\psi_{n1}(\xi, y) = A(\xi)\,\phi_n(y) \quad (3.4)$$

where $\phi_n(y)$ satisfies

$$(U_n - C_0)\{\phi_{nyy} + \varepsilon_n F(\phi_2 - \phi_1)\} + \phi_n\{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta\} = 0$$

$$\phi_n = 0 \quad \text{at } y = 0, 1 \quad (3.5)$$

Equation (3.5) determines the modal structure of a long Rossby wave while the wave amplitude $A(\xi)$ is as yet undetermined to this order. Proceeding to the next order, we obtain

$$\partial_\xi[(U_n - C_0)\{\psi_{n2yy} + \varepsilon_n F(\psi_{22} - \psi_{12})\} + \psi_{n2}\{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta\}] = -(U_n - C_0)\phi_n A_{\xi\xi\xi} + \left(\frac{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta}{U_n - C_0}\right)' \phi_n^2 A A_\xi - \frac{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta}{U_n - C_0}\phi_n C_1 A_\xi \quad (3.6)$$

It is evident that the wave amplitude $A(\xi)$ must satisfy the following KdV equation for the separated solution of (3.6) to exist. This is seen by application of the solvability condition, of which the algebraic details are shielded from the reader. Thus, we have

$$e_1 A_{\xi\xi\xi} + e_2 A A_\xi + e_3 C_1 A_\xi = 0 \quad (3.7)$$

where

$$e_1 = \int_0^1 \sum_{n=1}^{2} \phi_n^2\, dy$$

$$e_2 = -\int_0^1 \sum_{n=1}^{2} \frac{\phi_n^3}{U_n - C_0} \times \left(\frac{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta}{U_n - C_0}\right)' dy$$

$$\varepsilon_3 = \int_0^1 \sum_{n=1}^{2} \frac{\phi_n^2}{(U_n - C_0)^2} \times \{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta\}\, dy \quad (3.8a, b, c)$$

In the above derivation, we have assumed that any critical level does not exist, i.e. $U_n - C_0 \neq 0$. The set of eqs. (3.5), (3.7) and (3.8) determines completely the modal structure of a long Rossby wave in a baroclinic environment with no critical level.

Although eq. (3.7) has the solitary wave solution, the coefficient of non-linear term (e_2) happens to be zero (see (4.1)). So, we pursue another possible expansion before proceeding to the solution of eq. (3.7).

3.2 *l expansion*

Another solution to (2.3) may be sought in the form

$$\psi_n(\xi, y) = -\int_0^y U_n(y)\, dy + l\psi_{n1}(\xi, y) + l^2 \psi_{n2}(\xi, y) + l^3 \psi_{n3}(\xi, y) + \cdots$$
$$C = C_0 + l^2 C_1^* + \cdots \quad (3.9)$$

Substitution of (3.9) into (3.3) yields

$$\partial_\xi[(U_n - C_0)\{\psi_{n1yy} + \varepsilon_n F(\psi_{21} - \psi_{11})\} + \psi_{n1}\{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta\}] + l(\partial_\xi[(U_n - C_0)\{\psi_{n2yy} + \varepsilon_n F(\psi_{22} - \psi_{12})\} + \psi_{n2}\{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta\}] + (\psi_{n1\xi}\partial_y - \psi_{n1y}\partial_\xi) \times (\psi_{n1yy} + \varepsilon_n F(\psi_{21} - \psi_{11})\}) + l^2(\partial_\xi[(U_n - C_0)\{\psi_{n3yy} + \varepsilon_n F(\psi_{23} - \psi_{13})\} + \psi_{n3}\{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta\}] + (U_n - C_0)\psi_{n1\xi\xi\xi} + (\psi_{n1\xi}\partial_y - \psi_{n1y}\partial_\xi) \times \{\psi_{n2yy} + \varepsilon_n F(\psi_{22} - \psi_{12})\} + \{\psi_{n2\xi}\partial_y - (\psi_{n2y} + C_1^*)\partial_\xi\} \times \{\psi_{n1yy} + \varepsilon_n F(\psi_{21} - \psi_{11})\}) + O(l^3) = 0 \quad (3.10)$$

Note that the lowest order problem agrees exactly with that of the KdV case. From the next order problem, we obtain

$$\partial_\xi[(U_n - C_0)\{\psi_{n2yy} + \varepsilon_n F(\psi_{22} - \psi_{12})\} + \psi_{n2}\{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta\}] = \left(\frac{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta}{U_n - C_0}\right)' \phi_n^2 A A_\xi \quad (3.11)$$

The solution of eq. (3.11) is

$$\psi_{n2}(\xi, y) = \tfrac{1}{2}A^2(\xi)\,\chi_n(y) \quad (3.12)$$

where $\chi_n(y)$ satisfies

$$(U_n - C_0)\{\chi_{nyy} + \varepsilon_n F(\chi_2 - \chi_1)\} + \chi_n\{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta\} = \left(\frac{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta}{U_n - C_0}\right)' \phi_n^2,$$

$$\chi_n = 0 \quad \text{at } y = 0, 1 \quad (3.13)$$

The last problem yields

$$\partial_\xi[(U_n - C_0)\{\psi_{n3yy} + \varepsilon_n F(\psi_{23} - \psi_{13})\} + \psi_{n3}\{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta\}] = -(U_n - C_0) \times \phi_n A_{\xi\xi\xi} + \left[\frac{3}{2}\left(\frac{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta}{U_n - C_0}\right)' \times \phi_n \chi_n - \frac{1}{2}\left\{\frac{1}{U_n - C_0} \times \left(\frac{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta}{U_n - C_0}\right)\right\}' \phi_n^3\right] A^2 A_\xi - \frac{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta}{U_n - C_0} \phi_n C_1^* A_\xi \quad (3.14)$$

As in (3.1), application of the solvability condition yields, this time, the mKdV equation

$$e_1^* A_{\xi\xi\xi} + e_2^* A^2 A_\xi + e_3^* C_1^* A_\xi = 0 \quad (3.15)$$

where

$$e_1^* = e_1$$

$$e_2^* = -\int_0^1 \sum_{n=1}^{2} \left[\frac{3}{2}\left(\frac{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta}{U_n - C_0}\right)' \times \frac{\phi_n^2 \chi_n}{U_n - C_0} - \frac{1}{2}\left\{\frac{1}{U_n - C_0} \times \left(\frac{-U_n'' + \varepsilon_n F(U_1 - U_2) + \beta}{U_n - C_0}\right)\right\}' \frac{\phi_n^4}{U_n - C_0}\right] dy$$

$$e_3^* = e_3 \quad (3.16\text{a, b, c})$$

The mKdV equation has also the solitary wave solution; nevertheless, the wave amplitudes described by these two evolution equations are quite different in that the mKdV soliton has the amplitude (of $O(l)$) larger than the KdV soliton (of $O(l^2)$) for the fixed l (roughly, the reciprocal of the zonal length scale of solitons).

4. Solitary solutions

In this section we discuss the solitary wave solutions of the KdV and mKdV equations. The previous studies on the solitary Rossby waves show that these waves, generally, cannot exist in a horizontally uniform current for lack of the nonlinearity to balance the effect of beta dispersion. The basic field chosen here is

$$U_1(y) = U_0 + U_s + by, \quad U_2(y) = U_0 + by \quad (4.1)$$

where assumed that $b \ll 1$, this choice being consistent with our assumption on the exclusion of critical levels and an easy one to study the effect of the baroclinicity.

4.1. *The KdV soliton*

The KdV equation (3.7) has the solitary wave solution

$$A(\xi) = \operatorname{sgn}(e_1 e_2)\, \operatorname{sech}^2(\lambda\xi)$$

$$C_1 = -\frac{|e_2|}{3e_3} \operatorname{sgn}(e_1) \quad (4.2\text{a, b})$$

where the "wave steepness" λ is defined by

$$\lambda = \left|\frac{e_2}{12e_1}\right|^{1/2} \quad (4.3)$$

In order to determine the wave structure completely, the coefficients e_1, e_2 and e_3 must be explicitly evaluated.

A solution to the eigenvalue problem, (3.5) with insertion of (4.1), may be written as follows:

$$\phi_n(y) = \sum_{i=0}^{\infty} b^i \phi_{ni}, \quad C_0 = \sum_{i=0}^{\infty} b^i C_{0i} \quad (4.4)$$

Substitution of (4.1) and (4.4) into (3.5) yields a sequence of problems for ϕ_{ni} ($i = 0, 1, \ldots$). The lowest order problem in i becomes

$$\phi_{10yy} + F(\phi_{20} - \phi_{10}) + \frac{FU_s + \beta}{U_0 + U_s - C_{00}} \phi_{10} = 0$$

$$\phi_{20yy} + F(\phi_{10} - \phi_{20}) + \frac{-FU_s + \beta}{U_0 - C_{00}} \phi_{20} = 0$$

$$\phi_{10} = \phi_{20} = 0 \quad \text{at } y = 0, 1 \quad (4.5)$$

The solutions of eq. (4.5) are

$$\phi_{10} = \sin(m\pi y)$$

$$\phi_{20} = \gamma \sin(m\pi y) \quad (4.6)$$

where $m = 1, 2, \ldots,$ with the eigenvalues

$$\gamma = \frac{(m\pi)^2 + F}{F} - \frac{FU_s + \beta}{F(U_0 + U_s - C_{00})}$$

$$C_{00} = U_0 + \frac{U_s}{2} - \frac{\beta\{(m\pi)^2 + F\}}{(m\pi)^2\{(m\pi)^2 + 2F\}}$$

$$\pm \frac{1}{2}\frac{[(m\pi)^4 U_s^2\{(m\pi)^4 - 4F^2\} + 4\beta^2 F^2]^{1/2}}{(m\pi)^2\{(m\pi)^2 + 2F\}} \tag{4.7a, b}$$

These solutions and eigenvalues correspond to the inviscid, long wave limits of those which Pedlosky (1970) obtained in the linear stability problem.

The next order problem yields

$$\phi_{11yy} + F(\phi_{21} - \phi_{11}) + \frac{FU_s + \beta}{U_0 + U_s - C_{00}} \times \phi_{11} = \frac{C_{01} - y}{U_0 + U_s - C_{00}}\{\phi_{10yy} + F(\phi_{20} - \phi_{10})\}$$

$$\phi_{21yy} + F(\phi_{11} - \phi_{21}) + \frac{-FU_s + \beta}{U_0 - C_{00}} \times \phi_{21} = \frac{C_{01} - y}{U_0 - C_{00}}\{\phi_{20yy} + F(\phi_{10} - \phi_{20})\}$$

$$\phi_{11} = \phi_{21} = 0 \quad \text{at } y = 0, 1 \tag{4.8}$$

By applying the solvability condition, C_{01} is determined to be

$$C_{01} = 1/2 \tag{4.9}$$

Thus, the inhomogeneities in eq. (4.8) are known. The particular solutions of (4.8) are obtained in the form

$$\phi_{11} = a_1(y)\sin(m\pi y) + a_2(y)\cos(m\pi y)$$

$$\phi_{21} = a_3(y)\sin(m\pi y) + a_4(y)\cos(m\pi y) \tag{4.10}$$

where

$$a_1(y) = \frac{ey}{4(m\pi)^2(\gamma^2 + 1)}, \quad a_2(y) = \frac{e(y - y^2)}{4m\pi(\gamma^2 + 1)} \tag{4.11}$$

$$a_3(y) = \frac{e\gamma y}{4(m\pi)^2(\gamma^2 + 1)}, \quad a_4(y) = \frac{e\gamma(y - y^2)}{4m\pi(\gamma^2 + 1)}$$

where

$$e = \frac{FU_s + \beta}{(U_0 + U_s - C_{00})^2} + \gamma^2\frac{-FU_s + \beta}{(U_0 - C_{00})^2}$$

Now, the coefficients in the KdV equation are explicitly evaluated to yield

$$e_1 = \tfrac{1}{2}(\gamma^2 + 1)$$

$$e_2 = \frac{4b}{3m\pi}\left\{\frac{FU_s + \beta}{(U_0 + U_s - C_{00})^3} + \gamma^3\frac{-FU_s + \beta}{(U_0 - C_{00})^3}\right\} \quad \text{for odd } m$$

$$e_2 = \frac{b^2}{m\pi}\left[-\frac{5e}{6(m\pi)^2(\gamma^2 + 1)}\left\{\frac{FU_s + \beta}{(U_0 + U_s - C_{00})^3} + \gamma^3\frac{-FU_s + \beta}{(U_0 - C_{00})^3}\right\} + 2\left\{\frac{FU_s + \beta}{(U_0 + U_s - C_{00})^4} + \gamma^3\frac{-FU_s + \beta}{(U_0 - C_{00})^4}\right\}\right] \quad \text{for even } m$$

$$e_3 = \tfrac{1}{2}e \tag{4.12a, b, c, d}$$

We find an interesting implication of (4.12b, c) by means of a little inspection. In the limiting case of no vertical shear, i.e. when $U_s = 0$, the two wave modes reduce to the purely barotropic and baroclinic modes with the eigenvalues $\gamma = 1$ and $\gamma = -1$, respectively. For the baroclinic mode (4.12b, c) tell us that $e_2 = 0$, namely that the baroclinic KdV soliton cannot exist in the absence of vertical shear in the mean flow. It follows that the missing baroclinic mode is described by the higher order evolution equation, i.e. the mKdV equation. Redekopp (1977) showed, by using the reductive perturbation method, that long Rossby waves in a homogeneous atmosphere (necessarily barotropic mode) evolve in obedience to the KdV equation, while the evolution equation for waves in a uniformly stratified ($N(z)$ = const.) atmosphere becomes the mKdV equation. The result obtained here is, therefore, consistent with his finding.

This curious fact is interpreted as follows. Mathematically, the baroclinic mode in the atmosphere with $N(z)$ constant or the two-layer ocean has a purely sinusoidal vertical structure under the vertically uniform mean field. The self-interaction of such waves vanishes for the non-linear term of a Jacobian type. Physically, the baroclinic

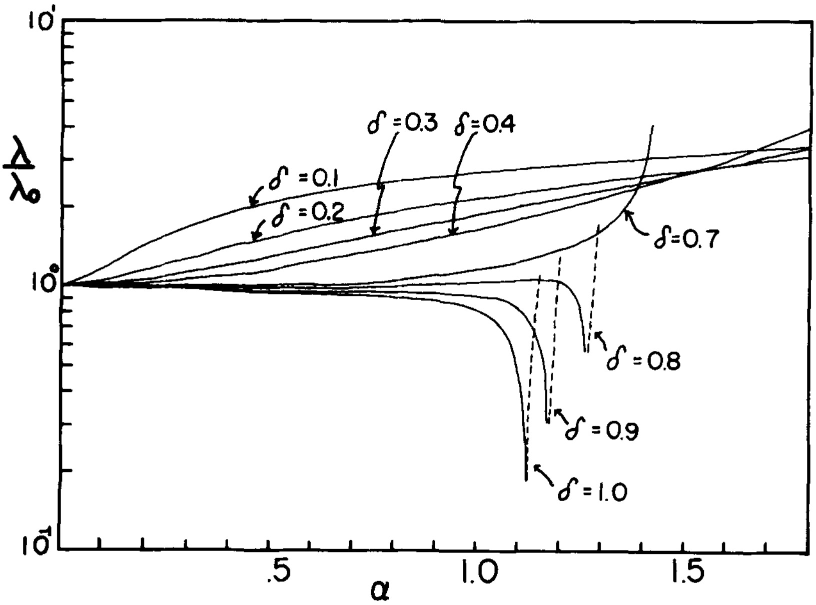

Fig. 1. The graph of the steepness of the barotropic KdV soliton ($m = 1$) as a function of the vertical shear $\alpha(=\pi U_s)$ and the internal rotational Froude number $\delta(=\pi^{-2}F)$. Other parameters are set to: $b = \pi^{-2}$, $\beta = \pi$ and $U_t = 1$. Continuous (broken) lines denote the regions $e_2 > 0 (e_2 < 0)$.

mode is accompanied with the internal divergence and, hence, with an elevation or depression of the interface. Thus, in the absence of vertical shear in the mean field, this mode cannot (or is too weak to) overcome the dispersion tendency of the gravity. This is also inferred from the fact that the increase in the vertical shear above zero produces quickly the modified baroclinic modes (see Fig. 2). The effect of the baroclinicity on the steepness of barotropic and baroclinic solitons is shown in Figs. 1 and 2. The increase in the vertical shear steepens the barotropic KdV solitons in the way inversely proportional to the magnitude of the internal rotational Froude number.

4.2. *The mKdV soliton*

As shown in (4.1), the coefficient of non-linear term in the KdV equation vanishes for a purely baroclinic mode (with mean vertical shear $U_s = 0$). In this case, the evolution equation for this mode becomes the mKdV equation. The mKdV equation (3.15) has, only if $e_2^* > 0$, the solitary wave solution

$$A(\xi) = \pm \operatorname{sech}(\lambda^* \xi)$$

$$C_1^* = -\frac{e_2^*}{6e_3^*} \tag{4.14a, b}$$

where

$$\lambda^* = \left|\frac{e_2^*}{6e_1^*}\right|^{1/2} \tag{4.15}$$

In order to determine the coefficient e_2^*, we turn to eq. (3.13). Retaining the eigenfunction to the leading order on the right-hand side and noting that $U_s = 0$, we write it as

$$\chi_{1yy} + F(\chi_2 - \chi_1) + \frac{\beta}{U_0 - C_{00}} \times \chi_1 = -\frac{b\beta}{(U_0 - C_{00})^3} \sin^2(m\pi y)$$

$$\chi_{2yy} + F(\chi_1 - \chi_2) + \frac{\beta}{U_0 - C_{00}} \times \chi_2 = -\frac{b\beta}{(U_0 - C_{00})^3} \sin^2(m\pi y)$$

$$\chi_1 = \chi_2 = 0 \quad \text{at } y = 0, 1 \tag{4.16}$$

The solutions of (4.16) are

$$\chi_1 = \frac{b}{2}\{d_1 \cos(2m\pi y) - (d_1 + d_2)\cos(m\pi y) + d_2\} \tag{4.17}$$

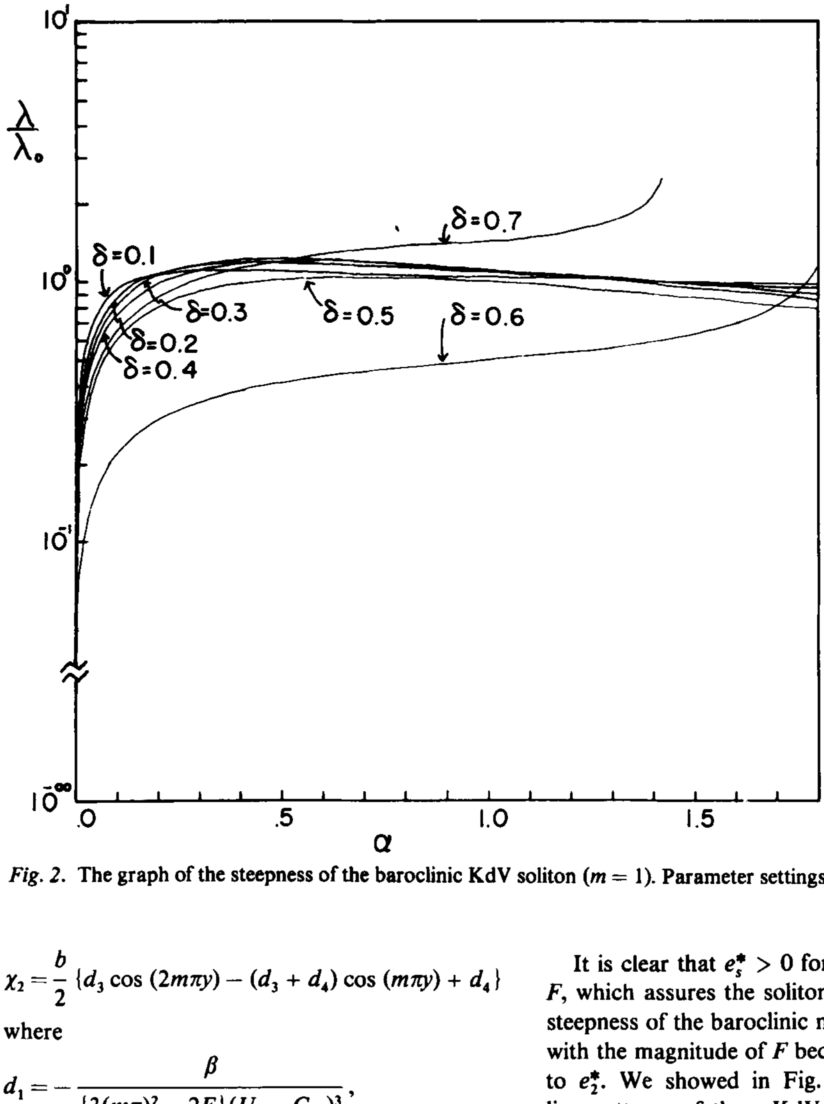

Fig. 2. The graph of the steepness of the baroclinic KdV soliton ($m = 1$). Parameter settings are same as Fig. 1.

$$\chi_2 = \frac{b}{2}\{d_3 \cos(2m\pi y) - (d_3 + d_4)\cos(m\pi y) + d_4\}$$

where

$$d_1 = -\frac{\beta}{\{3(m\pi)^2 - 2F\}(U_0 - C_{00})^3},$$

$$d_2 = -\frac{\beta}{\{(m\pi)^2 + 2F\}(U_0 - C_{00})^3}$$

$$d_3 = d_1, \quad d_4 = d_2 \tag{4.18}$$

with even m. Note that the solution with odd m, which satisfies the boundary conditions at $y = 0, 1$, does not exist. The calculation of the coefficient e_2^* yields

$$e_2^* = \frac{3b^2(m\pi)^2\{(m\pi)^2 + 2F\}^5}{2\beta^4\{3(m\pi)^2 - 2F\}}$$

for the purely baroclinic mode.

It is clear that $e_s^* > 0$ for the moderate value of F, which assures the soliton solution, and that the steepness of the baroclinic mKdV soliton increases with the magnitude of F because λ^* is proportional to e_2^*. We showed in Fig. 3 the typical streamline patterns of the mKdV solitons. One of these waves is characterized by the depression of streamlines in the centre, another one by elevation, and both are very similar to the *D*-soliton and the *E*-soliton named by Redekopp (1977). Note that the disturbance wave itself is producing a strong vertical shear.

The distinction between the mKdV soliton and the KdV soliton may be explained by the following argument. The remarkable contrast between both solitons appear to be due to the difference of nonlinear character of each soliton, maybe, of e_2^* and e_2. In spite of the absence of vertical shear in the mean flow, the baroclinic mKdV soliton can exist

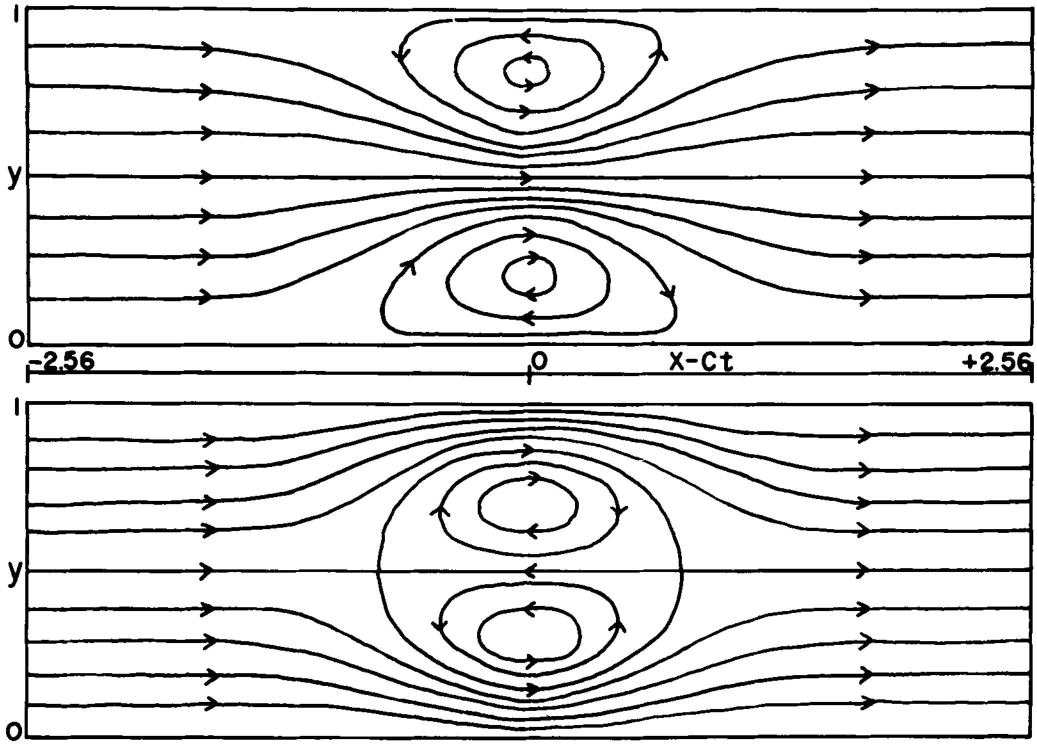

Fig. 3. Streamline patterns of the baroclinic mKdV soliton ($m = 2$): $l = 0.5$, $b = (2\pi)^{-3}$, $\beta = 2\pi$, $\delta(=(2\pi)^{-2}F) = 0.8$ and $U_0 = 1$. Two kinds of wave patterns with a depression (upper figure) and an elevation (lower figure) in the centre are possible.

because this one has a strong non-linear character and may produce, by itself, the vertical shear necessary to maintain itself as the soliton.

5. Discussion

The present analysis demonstrates that the two-layer ocean admits the planetary-scale solitary waves. These solitary waves are described by the KdV or the mKdV equation depending on the baroclinicity of the mean field. When the mean shear flow has a barotropic structure, barotropic waves obey the KdV equation, while baroclinic waves are described by the mKdV equation. However, in the general case where the mean flow has the vertical shear, both solitary waves are described uniquely by the KdV equation. The speculation was made on the effect of vertical shear on solitary waves, which suggests that the mKdV soliton has a strong non-linearity compared with the KdV soliton.

We now consider the relation between solitary waves and the meso-scale eddies observed in the mid-ocean. The recent observations, e.g. MODE, clarified the presence of meso-scale eddies which have velocity amplitudes larger than those of mean oceanic currents in the mid-ocean and which have the length scale of the order of Rossby deformation radius. Solitary Rossby waves presented here appear to have sufficient former quality, but not the latter one because of long waves. It may be suggested that the presence of both horizontal and vertical shears steepens solitary waves and admits the horizontally compact waves.

In connection with the above problem, we consider the relation between the linear theory of eddy production, e.g. the theory of the baroclinic instability, and the finite-amplitude theory of solitary Rossby waves. The baroclinic instability requires the vertical shear to grow the perturbation wave. There is then a possibility that solitary waves may play a role as the catalysts in generating meso-scale eddies. For example, imagine a subcritical mean state. The intrusion of solitary waves into this region may bring the mean state into a supercritical one and thereby accelerate the instability.

Finally, the further modifications of solitary waves will be considered; the interaction with bottom topography, the effect of viscosity and the stability of solitary waves. The present paper may be regarded as presenting the preliminaries of such problems.

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ОДИНОЧНЫЕ ВОЛНЫ РОССБИ В ДВУХСЛОЙНОЙ СИСТЕМЕ.

Представлены модальные решения для одиночных волн Россби в системе двухслойного канала без критического уровня с зональным бароклинным потоком со средним сдвигом скорости по высоте. Когда среднее сдвиговое течение имеет баротропную структуру, то найденные решения совпадают с решениями Редекоппа (1977) в том смысле, что основная баротропная мода описывается уравнением кдф, в то время как преобладающая бароклинная мода подчиняется модифицированному уравнению кдф. Найдено, что присутствие вертикального сдвига в среднем потоке приводит в общем к укручению солитонов волн Россби, однако, эта тенденция сильно зависит от конкретной моды волны и величины внутреннего вращательного числа Фруда, характеризующего волновое поле.