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Nonlinear stationary internal and surface waves in shallow seas

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ABSTRACT

It is known that propagation of the two-dimensional stationary internal gravity waves of arbitrary amplitude can be described by a certain nonlinear equation which contains two arbitrary functions. It is shown that for the solitary waves and for infinitesimal periodic internal waves, these two unknown functions are described by the Väisälä–Brunt frequency. For nonlinear periodic waves with weak nonlinearity and dispersion approximation there is an arbitrary function of the transversal coordinate giving rise to a current. By the additional assumption that this current is reduced to zero, the phase velocity of a nonlinear periodic wave has a unique connection with the amplitude. Further, it is shown that additivity of wave motions of different modes takes place in the present approach.

Two examples of model stratification are considered. A detailed numerical analysis is performed, illustrating propagation of internal waves, based on experimental data for the Arkona basin in the Baltic Sea. The local Richardson numbers have been calculated. It is shown that Ri $< \frac{1}{4}$ can occur for solitary waves which may cause organization of some turbulent regions.

1. Introduction

Two-dimensional gravitational waves of finite amplitude in a continuously stratified fluid have been studied by a number of research workers, e.g. Ter-Krikorov (1962), Benjamin (1966), Long (1953, 1972), Magaard (1965) and Leonov and Miropolsky (1974, 1975a,b). The overall analysis of weak nonlinearity and long waves approximation carried out by Ter-Krikorov (1962) and Benjamin (1966) has shown that both the periodic and solitary internal waves can propagate when the wavelength is greater than the fluid depth. Most of the above-mentioned investigations are based on the equation derived by Dubriel-Jacotin (1932) and Long (1953) for the stationary two-dimensional flow of the stratified fluid. However, the equation in question is derived from the first two integrals of the fundamental system of stratified fluid hydrodynamic equations and it possesses two arbitrarily chosen unknown functions—all of which makes the investigation much more complicated. In the paper (Leonov and Miropolsky, 1974) on internal wave propagation in calm waters (i.e. in the absence of the given external flow) these two unknown functions were expressed through the Väisälä–Brunt frequency $N)z)$, where z is the vertical coordinate.

Leonov and Miropolsky (1975a,b) discussed the asymptotic solutions of the equation derived for long waves in shallow waters and for long waves in the intermediate "short-wave" asymptotics when the maximum wavelength is much smaller than the characteristic scale of stratification. The present paper gives the analysis of some new aspects of stationary waves in stratified shallow sea in the absence of external flow. The equation derived in the above-mentioned publications by Lenov and

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Miropolsky (1974, 1975a,b) is described in section two. It is shown that this equation is valid for both stationary solitary waves and infinitesimal waves. For nonlinear periodic waves with the weak nonlinearity and dispersion approximation there is an arbitrary function of transversal coordinate which is left in the equation. Here also some problems are discussed concerning the transformation from one moving coordinate system to another.

In section three the asymptotic solutions of section two are discussed in detail with the weak nonlinear and dispersion approximation taking into account the existence of surface waves. In contrast to infinitesimal waves and solitons, where soliton is defined by a single amplitude parameter, we have now obtained that for nonlinear periodic waves in stratified fluid the wave propagation velocity depends not only on amplitude, but also on the considerable current with vertical shear, which cannot be defined by means of stationary theory. By the additional assumption that this current is reduced to zero the phase velocity of a nonlinear periodic wave is connected with the amplitude uniquely. Linear internal waves are characterized by a large number of modes. In the work it is shown that in the approximation considered the additivity of wave motions of different modes takes place.

In section four some examples are considered. The two first concern cases of model stratification (two-layer fluid and constant Väisälä–Brunt frequency) and the influence of stratification on behaviour of surface and internal waves can be seen. In particular one can see that the "rigid lid" approximation leads to a considerable error in the analysis of internal waves if the Väisälä–Brunt frequency is considerably different from the zero on free surface. Further in section four the detailed numerical analysis is fulfilled, illustrating propagation of internal waves and based on experimental data by Kielman et al. (1973) for the Arkona basin in the Baltic Sea.

2. The basic equations of stationary nonlinear internal waves

Magaard (1965) shows that the stationary two-dimensional flow of stratified incompressible ideal fluid is described by the equation

$$\Delta\psi - \frac{1}{c_f\rho(s)}\frac{d\rho}{ds}\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial\theta}\right)^2 + \frac{1}{2}\left(\frac{\partial\psi}{\partial z} - c_f\right)^2 + gz\right] = -\frac{1}{\rho(s)c_f}\frac{d\phi}{ds} \tag{1}$$

$$p = \Phi(s) - \rho(s)\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial\theta}\right)^2 + \frac{1}{2}\left(\frac{\partial\psi}{\partial z} - c_f\right)^2 + gz\right],$$
$$s = z - \psi/c_f \tag{2}$$

where $\theta = x - c_f t$, $\Delta = \partial^2/\partial\theta^2 + \partial^2/\partial z^2$, $x =$ the longitudinal coordinate, $z =$ the transversal coordinate (axis z is directed vertically upwards), $t =$ time, $c_f =$ the wave propagation phase velocity, $\rho =$ density, $p =$ pressure, $\psi(\theta,z) =$ the stream function, and $\Phi(s) =$ arbitrary function. $-H < z < \zeta(\theta)$ is the region for the solution of the eq. (1), where $H =$ constant depth, and $\zeta =$ disturbed free surface.

The kinematic and dynamic surface condition and that at the bottom can be shown as follows

$$\psi|_{z=\zeta(\theta)} = c_f\zeta, \quad p|_{z=\zeta} = 0, \quad \left.\frac{\partial\psi}{\delta\theta}\right|_{z=-H} = 0 \tag{3}$$

Let us now consider the boundary condition according to the longitudinal coordinate:

(a) the fading condition with

$$\psi|_{|\theta|\to\infty} \to 0, \quad \zeta|_{|\theta|\to\infty} \to 0, \quad \rho|_{|\theta|\to\infty} = \rho_0(z) \tag{4}$$

Here $\rho_0(z)$ is the fluid density distribution undisturbed by the wave and it is assumed that the value $\rho_0(z)$ is known.

Equation (1) with the conditions (3), (4) considers the simplest case of the solitary waves in calm waters. It should be noted that functions $\rho(s)$ and $\Phi(s)$ besides the argument also contain some parameters that characterize the wave: the amplitude a, the length of wave λ, $\rho = \rho(s; a, \lambda)$, $\Phi = \Phi(s; a, \lambda)$, $\psi = \psi(z, \theta; a, \lambda)$, $c_f = c_f(a, \lambda)$.

For the solitary wave, when $\lambda \to \infty$ we have

$$\rho(s; a, \infty) = \rho_0(z - \psi/c_f) \tag{5}$$

which means that the function ρ does not depend on any other wave parameter; it depends only on the argument S which makes its structure quite definite. Making use of (5) for (1) in case of limit $|\theta| \to \infty$ and taking into consideration (4) we obtain

$$\frac{d\Phi}{dz} = \frac{d\rho_0}{dz}\left(\frac{c_f^2}{2} + gz\right) = -\rho_0(z)\,N^2(z)\left(z + \frac{c_f^2}{2g}\right) \tag{6}$$

For this case $\Phi(s)$ depends only on the argument S. From (1), (2) and (5), (6) for the solitary wave we obtain (Leonov, Miropolsky, 1974)

$$\Delta\psi + \left\{\frac{\psi}{c_f^2} - \frac{1}{g}\frac{\partial\psi}{\partial z} + \frac{1}{2c_f g}\left[\left(\frac{\partial\psi}{\partial\theta}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right]\right\} \times N^2\left(z - \frac{\psi}{c_f}\right) = 0, \tag{7}$$

$$p - p_0 = -g\{\rho_0(s)\,\psi/c_f + \textstyle\int_0^s \rho_0(a)\,da\} + \rho_0(s)\left\{c_f\frac{\partial\psi}{\partial z} - \frac{1}{2}\left[\left(\frac{\partial\psi}{\partial\theta}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right]\right\} \tag{8}$$

Boussinesq approximation corresponds to the elimination of the second and third terms in (7).

(b) periodic condition:

$$\psi(\theta + \lambda, z) = \psi(\theta, z), \quad \zeta(\theta + \lambda) = \zeta(\theta), \quad \rho(\theta + \lambda, z) = \rho(\theta, z) \tag{9}$$

Basically the arbitrary function $\rho(s)$ and $\Phi(s)$ in (1), (2) are unknown here. We shall consider various cases for eqs. (1)–(2) with the condition (9). For linear (infinitesimal) waves, when $a \to \infty$ we have

$$\Delta\psi - \frac{N^2(z)}{g}\frac{\partial\psi}{\partial z} + \frac{N^2(z)}{c_f^2}\psi = 0 \tag{10}$$

For weak nonlinear long waves ($\lambda/H \gg 1$) when $\psi = \lambda^{-n}\psi_s(a, z)$ the eq. (1) approximately takes the form:

$$\Delta\psi - \frac{N^2(z)}{g}\frac{\partial\psi}{\partial z} + \frac{N^2(z)}{c_f^2}\psi - \frac{[N^2(z)]'}{c_f^3}\psi^2 + \frac{N^2(z)}{2c_f g}\left[\left(\frac{\partial\psi}{\partial\theta}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] = \lambda^{-2n}\varphi(z) \tag{11}$$

where $\varphi(z)$ = arbitrary function giving rise to the current.

The arbitrary function appeared only in the second order of the disturbance theory for the long waves in calm water. For eqs. (7) and (10) the phase velocity (c_f) is different. For (7) $c_f = c_f(a, \infty)$ and for eq. (10) $c_f = c_f(0, \lambda) = \hat{c}_f(\lambda)$ where $\hat{c}_f(\lambda)$ = linear phase velocity. The change of c_f in eqs. (1)–(11) results in the change of the system of the count $\{\theta\}$. If ψ and c_f change steadily then in the coordinate system z, $\theta_* = x - c_f^* t$ (c_f = const) for $c_f \neq c_f^*$, ψ_* is found with the precision of the "phase shift"

$$\psi_* = \psi(\theta_*, z) + (c_f - c_f^*)\,z \tag{12}$$

where $\psi(\theta_*, z)$ = solution (1) with conditions (3), (4). Note that then the horizontal velocity U changes by the value $c_f - c_f^*$.

3. The theory of stationary weak nonlinear internal and surface waves in shallow water

Despite the attractive appearance of eq. (7), up to now we do not have any exact solution for it. Hence let us consider the solution for (11) with the conditions (3), (4) in approximation to the weak nonlinear theory.

To make the analysis more convenient we shall introduce dimensionless variable functions and parameters

$$\xi = \theta\lambda^{-1}, \quad \eta = zH^{-1}, \quad F = \psi(c_f H)^{-1},$$
$$\zeta_1 = \zeta H^{-1}, \quad \hat{p} = p(\rho_a c_f^2)^{-1}, \quad \hat{\rho} = \rho_0(z)\rho_a^{-1},$$
$$\Omega(\eta) = N^2(z)N_0^2, \quad K_0 = H\lambda^{-1},$$
$$\beta = H^2N_0^2 c_f^{-2}, \quad \delta = HN_0^2 g^{-1} \tag{13}$$

where $\rho_a = \rho_0(0)$. Then the eq. (11) in dimensionless variables (13) is as follows

$$K_0^2\frac{\partial^2 F}{\partial\xi^2} + \frac{\partial^2 F}{\partial\eta^2} + \Omega(\eta)\left(\beta F - \delta\frac{\partial F}{\partial\eta}\right) - \beta F^2\frac{d\Omega}{d\eta} + \frac{\delta}{2}\left[K_0^2\left(\frac{\partial F}{\partial\xi}\right)^2 + \left(\frac{\partial F}{\partial\eta}\right)^2\right]\Omega(\eta) = K_0^{2n}\varphi(\eta) \tag{14}$$

The boundary conditions for F can be given as follows

$$\left\{\frac{\beta}{\delta}F - \frac{\partial F}{\partial\eta} + \frac{1}{2}\left[\left(\frac{\partial F}{\partial\eta}\right)^2 + K_0^2\left(\frac{\partial F}{\partial\xi}\right)^2\right]\right\}_{\eta=\zeta_1} = 0;$$
$$\left.\frac{\partial F}{\partial\eta}\right|_{\eta=-1} = 0; \quad F|_{\eta=\zeta_1} = \zeta_1 \tag{15}$$

The eq. (14) with condition (15) is a nonlinear boundary-value problem, if there are conditions for fading or if there is a periodical condition in $\xi \to \infty$ coordinate which are additionally set forth.

Let us look for the solution of this equation of small parameter ($K_0 = \lambda/H \ll 1$) expansions

$$F = K_0^2 F_1 + K_0^4 F_2 + \cdots, \quad \beta = \beta_0 + K_0^2 \beta_1 + K_0^4 \beta_2 + \cdots. \tag{16}$$

In this case in the eq. (11) the parameter n in (14) is equal to 2. Substituting (16) into (14), (15) and equalizing terms with the same power of K_0 we shall get the first approximation

$$\mathscr{L}(F_1) \equiv \frac{\partial^2 F_1}{\partial \eta^2} - \delta\Omega(\eta)\frac{\partial F_1}{\partial \eta} + \beta_0\,\Omega(\eta)\,F_2 = 0;$$

$$F_1|_{\eta=-1} = 0; \quad \left[\frac{\partial F_1}{\partial \eta} - \frac{\beta_0}{\delta} F_1\right]_{\eta=0} = 0 \tag{17}$$

as well as the non-homogeneous boundary task for the second approximation.

$$\mathscr{L}(F_2) = \chi_1(\eta, \xi); \quad F_2|_{\eta=-1} = 0;$$

$$\left[\frac{\partial F_2}{\partial \eta} - \frac{\beta_0}{\delta} F_2\right]_{\eta=0} = \chi_2(\xi)$$

$$\chi_1(\eta, \xi) = -\left\{\frac{\partial^2 F_1}{\partial \xi^2} + \beta_1\,\Omega(\eta)\,F_1 - \beta_0 \frac{d\Omega}{d\eta} F_1^2 + \delta \frac{d\Omega}{d\eta} F_1 \frac{\partial F_1}{\partial \eta} + \frac{1}{2}\delta\,\Omega(\eta)\left(\frac{\partial F_1}{\partial \eta}\right)^2 - \varphi(\eta)\right\}$$

$$\chi_2(\xi) = \left[\frac{\beta_1}{\delta} F_1 + \frac{\beta_0}{\delta} F_1 \frac{\partial F_1}{\partial \eta} + \frac{1}{2}\left(\frac{\partial F_1}{\partial \eta}\right)^2\right]_{\eta=0} \tag{18}$$

The homogeneous eq. (17) admits the following solution

$$F_1(\eta, \xi) = W(\eta)\,U(\xi) \tag{19}$$

where $W(\eta)$ is obtained from the solution of the Sturm Liouville boundary-value problem

$$\mathscr{L}(W) \equiv W'' - \delta\,\Omega(\eta)\,W' + \beta_0\,\Omega(\eta)\,W = 0;$$

$$W|_{\eta=-1} = 0, \quad \left[W' - \frac{\beta_0}{\delta} W\right]_{\eta=0} = 0 \tag{20}$$

The zero mode gives the description of the surface waves and the other modes describe the internal waves.

The system of the eigenfunction $\{W_n\}$ is the orthogonality

$$\delta^{-1} W_\eta(0)\,W_j(0) + \textstyle\int_{-1}^{0} \hat{\rho}(\eta)\,\Omega(\eta)\,W_m(\eta)\,W_j(\eta)\,d\eta = 0 \quad (\eta \neq j)$$

$$\|W\|^2 \equiv \delta^{-1} W_n^2(0) + \textstyle\int_{-1}^{0} \hat{\rho}(\eta)\,\Omega(\eta)\,W_n^2(\eta)\,d\eta > 0 \tag{21}$$

Here and further on we shall leave out index n.

For the definition of the amplitude function $U(\xi)$ we shall take into account the second approximation.

$$F_2(\eta, \xi) = V(\xi)\,W(\eta) + \tilde{F}_2(\eta_1\,\xi) \tag{22}$$

Here $V(\xi)$ is the amplitude function of approximation 2 which is defined by approximation 3 and $\tilde{F}_2(\eta, \xi)$ is the solution of (18). As the eigenvalue β_0 is the same in (18) and (20) we are to observe the following condition:

$$-\chi_2(\xi)\,W(0) + \textstyle\int_{-1}^{0} \chi_1(\eta_1\,\xi)\,\hat{\rho}(\eta)\,W(\eta)\,d\eta = 0$$

Substituting the χ_1 and χ_2 from (18) to this condition we are able to find the equation for the amplitude function $U(\xi)$

$$U''(\xi) + \gamma U^2 + \alpha\beta_1 U = \tau \tag{23}$$

where

$$\alpha = \|W\|^2 (\textstyle\int_{-1}^{0} \hat{\rho}\, W^2\, d\eta)^{-1} \quad \tau = (\textstyle\int_{-1}^{0} \hat{\rho}\varphi W\, d\eta) \times (\textstyle\int_{-1}^{0} \hat{\rho} W^2 d\eta)^{-1}$$

$$\gamma = \left[\frac{3}{2}\frac{\beta_0^2}{\delta^2} W^3(0) + 3\textstyle\int_{-1}^{0} \hat{\rho}\Omega W\left(\beta_0\, WW' - \frac{\delta}{2} W'^2\right) d\eta\right] (\textstyle\int_{-1}^{0} \hat{\rho} W^2 d\eta)^{-1} \tag{24}$$

For barotropic surface waves, when $\Omega(\eta) \equiv 0$, $\hat{\rho}(\eta) = 1$, $W(\eta) = 1 + \eta$, $\beta_0 = \delta$, in case of $c_f = (gH)^{1/2}$. We have $\alpha = 3\delta^{-1}$, $\gamma = 4.5$. (Here we can see, that $\alpha > 0$ and $\gamma > 0$.) For baroclinic waves without the zero mode it is possible to accept the following simplified boundary condition: $W(0) = 0$. In this case for $n > 0$ in (24) outward terms disappear. In Boussinesq approximation (it is possible only with condition $W(0) = 0$). The coefficients α and β are as follows:

$$\alpha = \textstyle\int_{-1}^{0} \hat{\rho}\Omega W^2\, d\eta (\int_{-1}^{0} \rho W^2\, d\eta)^{-1},$$
$$\gamma = 3\beta_0 \textstyle\int_{-1}^{0} \hat{\rho}\Omega W'\, W^2\, d\eta \times (\int_{-1}^{0} \hat{\rho} W^2\, d\eta)^{-1} \tag{24a}$$

The coefficient τ is the same as in (24).

Equation (23) coincides with the stationary Korteweg–de Vries equation integrated by ξ one time. β_1 is an addition to the phase velocity c_f. While α is always positive ($\alpha > 0$) then γ may be positive or negative. If $\gamma < 0$ then $\xi \to -\xi$, $U \to -U$, $\tau \to -\tau$ and we have again (23).

Let us assume that c_f, $c_{f,s}$, $\hat{c}_f$ are the phase velocities of the nonlinear periodical, solitary and infinitesimal waves correspondingly, and $c_f^{(1)}$, $c_{f,s}^{(1)}$, $\hat{c}_f^{(1)}$ are additions to phase velocity c_f^0. Then for the phase velocity (see the appendix) we shall have

$$\hat{c}_f^{(1)} = -\frac{K^2 (c_f^0)^3}{2H^2 N_0^2 \| W \|^2} \int_{-1}^{0} \hat{\rho} W^2 \, d\eta < 0 \tag{25}$$

where K is a nondimensional wavenumber.

For the medium of negative dispersion the infinitesimal waves are prior to "sound stage" and solitary waves are "supersound". For solitary waves from (23) we find $\tau \to 0$, $\beta = \beta_{1s}$ and

$$U_s = U_0 \, ch^{-2}(\sqrt{U_0 \gamma / 6}\, \xi), \quad U_0 = -\frac{3\alpha}{2\gamma} \beta_{1s} > 0 \tag{26}$$

So $\beta_{1s} < 0$ and $c_{f,s}^{(1)} > 0$. Then $\hat{c}_f^{(1)} < c_f^{(1)} < c_{f,s}^{(1)}$ and with (13), (16) $\hat{\beta}_1 > \beta_1 > \beta_{1,s}$, moveover from (13), (16), (25) follows, that

$$\hat{\beta}_1 = -\frac{2\beta_0 \hat{c}_f^{(1)}}{K_0^2 c_f^0} = \frac{K^2}{K_0^2} \frac{1}{\alpha} \tag{27}$$

It is more appropriate to make the interval of the change in $c_f^{(1)}$ symmetrical

$$c_{f,s}^{(1)} = -\hat{c}_f^{(1)} > 0, \quad \beta_{1,s} = -\hat{\beta}_1 < 0 \tag{28}$$

Then the wavenumber of the infinitesimal wave in (25) and (27) is definable. We require that the value $\tau = 0$ (Kudomtsev, 1972) in the shifting coordinate system θ_s changing with the speed equal to the solution propagation, then eq. (23) takes the form $U'' + \gamma U^2 + \alpha \beta_{1,s} U = 0$, where $U(\xi) = U + a$, $a = \alpha\gamma^{-1}(\beta_1 - \beta_{1,s})$. Then for τ we have

$$\tau = \frac{\alpha^2}{4\gamma} (\hat{\beta}_1^2 - \beta_1^2) \tag{29}$$

Then for soliton and for infitesimal waves the parameter τ is equal to zero. For the solution of the nonlinear periodical waves it is possible to single out two constants of integration and the parameter β_1. The first unknown parameter is deduced from the periodical condition (9), the second—by using the amplitude parameter $\mathscr{s}$ which characterizes the order of the nonlinearity ($0 < \mathscr{s} < 1$) and the third—from the condition

$$<U> = \frac{1}{\tilde{\lambda}} \int_0^{\tilde{\lambda}} U(\xi) \, d\xi \tag{30}$$

Here $\tilde{\lambda} = \lambda / H$ are nondimensional wavelengths. Let us now require that $<U> = 0$. This condition for soliton and for infinitesimal waves is invariably observed.

So for periodical weak non linear waves the equations are as follows:

$$\frac{U}{U_0} = \frac{\mathscr{s}^2}{m} \mathrm{cn}^2 \left(\sqrt{\frac{\gamma U_0}{6m}} \xi; \mathscr{s} \right) + \frac{(1 - \mathscr{s}^2) K(\mathscr{s}) - E(\mathscr{s})}{m K(\mathscr{s})}$$

$$= \frac{1}{m} \left[\mathrm{dn}^2 \left(\sqrt{\frac{\gamma U_0}{6m}} \xi; \mathscr{s} \right) - \frac{E(\mathscr{s})}{K(\mathscr{s})} \right]$$

$$\tilde{\lambda} = 2K(\mathscr{s}) \left(\frac{6m}{\gamma U_0} \right)^{1/2}; \quad \frac{\tilde{U}_0}{U_0} = \frac{(2 - \mathscr{s}^2) K(\mathscr{s}) - 3E(\mathscr{s})}{m K(\mathscr{s})}$$

$$m(\mathscr{s}) = 1 - \mathscr{s}^2 + \mathscr{s}^4; \quad U_0 = \frac{3}{2\gamma} \alpha \hat{\beta}_1;$$

$$\tilde{U}_0 = -\frac{3}{2\gamma} \alpha \beta_1 \tag{31}$$

Here $\mathrm{ch}(x, \mathscr{s})$, $\mathrm{dn}(x, \mathscr{s})$, are the elliptic Jacobi functions with $\mathscr{s}(0 < \mathscr{s} < 1)$ as the modulus. $K(\mathscr{s})$, $E(\mathscr{s})$—the elliptic integrals. When $\mathscr{s} \to 1$, the periodic wave degenerates into the soliton one and when $\mathscr{s} \to 0$ into the infinitesimal one.

If $\gamma < 0$ then $U \to -U$, $\xi \to -\xi$ and the infinitesimal wave is "supersound" and soliton is "sound stage". In approximation of the weak nonlinear waves (see appendix) the principle of additivity enables us to say that

$$F_1 = \sum_{m=0}^{\infty} F_{1,m}(\eta, \xi_m) = \sum_{m=0}^{\infty} U_m(\xi_m) \, W_m(\eta)$$

$$\xi_m = \lambda^{-1}(x - c_{f,m} t), \quad c_{f,m} = \beta_m^{-1/2} H N_0,$$
$$\beta_m = \beta_{0,m} + K_0^2 \beta_{1,m} \tag{32}$$

where m is the number of the mode.

4. Concrete instances of solution

The distribution of the Väisälä–Brunt frequency has a strong influence on the structure of the internal waves. The following three examples demonstrate these effects.

4.1. *Two-layer model*

Let $0 > \eta > -l\ \tilde{\rho}(\eta) = 1, -l < \eta < -l\ \tilde{\rho}(\eta) = \tilde{\rho}_0 > 1$. Parameter $\delta = \Delta\rho/\tilde{\rho} = 2(\tilde{\rho}_0 - 1)/(1 + \tilde{\rho}_0)$. The nondimensional Väisälä–Brunt frequency $\Omega(\eta) = \delta(\eta + l)$, where $\delta(x) =$ deltafunction.

Then the solution of the eq. (20) is given as follows

$$W(\eta) = c\frac{(1 - l\beta_0/\delta)(1 + \eta)}{1 - l}\ \eta < -l;$$

$$W(\eta) = c\left(1 + \beta_0\frac{\eta}{\delta}\right)\eta > -l$$

Substituting this solution into the eq. (20)

$$W'_+\left(1 - \frac{\delta}{2}\right) - W'_-\left(1 + \frac{\delta}{2}\right) + \beta_0\, W(-l) = 0,$$

$$W'_\pm = W'(-l \pm 0)$$

we find the dispersion relation

$$\beta_0^{0,1} = \frac{1 + \delta/2}{2l(1 - l)}\left[1 \pm \sqrt{1 - \frac{4\delta l(1 - l)}{1 + \delta/2}}\right]$$

So here we have two waves—the surface wave and the wave on the surface of discontinuity of density.

The parameters α and γ are as follows

$$\alpha = \frac{3}{\delta}\frac{2 - \delta + 2\delta\varphi_l^2}{[2 + \delta(1 - 2l)]\,\varphi_l^2 + (2 - \delta)\,l(1 + \varphi_l}; \quad \varphi_l = 1 - \frac{\beta_0 l}{\delta} = \frac{W(-l)}{\|l\|};$$

$$\gamma = \frac{g}{2}c\frac{(2 - \delta)(1 - l)^2(1 - \varphi_l)^2[1 + \delta\varphi_l(\varphi_l - \frac{1}{2})] + (2 + \delta)\,l\delta\varphi_l^3[(1 - l)(1 - \varphi_l) - l/2]}{l^2(1 - l)^2\{\varphi_l^2[2 + \delta(1 - 2l)] = l(1 + \varphi_l)(2 - \delta)\}}$$

where c is defined from the condition $\|W\|^2 = 1$. The sign of the c is defined as $W_{\max} = W(-l) > 0$. If we take into consideration, that $\delta \sim 10^{-2} \ll 1$, then for surface waves we shall have $\beta_0^0 = \delta[1 + O(\delta)]$, $\alpha = 3\delta^{-1}[1 + O(\delta)]$, $\gamma = g/2\ \delta^{1/2}[1 + O(\delta)]$, $c = \delta^{1/2}$. Here the surface wave in the two-layer sea is the same as the surface wave in the barotropic fluids (see section 3). In a similar case for internal waves we shall have $\beta_0 = [l(1 - l)]^{-1}[1 + O(\delta)]$, $\alpha = 3[1 + O(\delta)]$, $c = \delta(1 - l)$, $\gamma = g/2(2l - 1)[l(1 - l)]^{-2}[1 + O(\delta)]$.

It is worth noting that if $l < \frac{1}{2}$, then $\gamma < 0$ and there appear waves with "sound stage" speed. On the other hand, if $l > \frac{1}{2}$ then $\gamma > 0$ and the waves are "supersound".

4.2. *The model of the exponential stratification*

Here $\tilde{\rho}(\eta) = \exp(-\delta\eta)$, $\Omega(\eta) = 1$, $W_m(\eta) = c_m \sin[\kappa_m(1 + \eta)]$ and the dispersion relation is as follows:

$$tg\ \kappa_m = \frac{\delta/\kappa_m}{1 - 5/8(\delta/\kappa_m)^2}, \quad \beta_{0,m} = \kappa_m^2 - \delta^2/4$$

Assuming as before, that $\delta \ll 1$ we have for eigenfunction

$$W_0(\eta) = [1 + \eta + O(\delta)]\delta^{1/2},$$

$$W_m(\eta) = \sqrt{2}\sin[\pi m(1 + \eta)] + O(\delta)$$

where index zero shows the surface waves and index m the internal waves.

The parameters α, β, γ for surface waves are as follows: $\alpha_0 = 3\delta^{-1}[1 + O(\delta)]$, $\beta_{0,0} = \delta[1 + O(\delta)]$, $\gamma_0 = g/2\ \delta^{1/2}[1 + O(\delta)]$ and for internal waves

$$\alpha_m = 1 + O(\delta); \quad \beta_{0,m} = m^2\pi^2 + \frac{3}{4}\delta^2 + O\left(\frac{\delta^3}{m\pi}\right),$$

$$\gamma_m = \frac{\delta m\pi}{\sqrt{2}}\{3(-1)^m + 2[(-1)^m - 1] + O(\delta)\}$$

$$(m = 1, 2, \cdots)$$

Again the stratification does not depend on the structure of the surface waves. But the parameters α, β, γ for internal waves essentially depend on stratification. The solitons where $N =$ const for the even mode approach the positive soliton and for odd modes $\times$ the negative one.

It is interesting to note that with condition ($W = 0$) on the surface $\gamma = \delta m\pi\sqrt{2}[(-1)^m - 1 + O(\delta)]$

(Miropolsky, Leonov, 1975a). It is noteworthy that if the Väisälä–Brunt frequency on the surface is near to zero, the condition $W = 0$ on the surface is quite effective. If the $N^2(z)$ differs radically from zero, then for the weak nonlinear waves the condition $W = 0$ on the surface does not work at all (as different from the linear internal waves).

4.3. *The model of the real stratification*

The internal wave structure calculations were made on the basis of the experimental data presented by Kielman et al. (1973) for the Arkona basin in the Baltic Sea.

The depth of this area is $H = 47$ m and the maximum value of the Väisälä–Brunt frequency is $N_0 = 5.6 \cdot 10^{-2}$ s^{-1}. The distribution of the Väisälä–Brunt frequency was taken from the same paper, smoothed and in dimensionless form $\Omega(\eta) = N^2/N_0^2$ is presented in Fig. 1 (broken line). In the same figure we find the first three eigenfunctions $W_n(\eta)$ normalized as follows $\int_{-1}^{0} W_n^2 \, d\eta = 1$. The spectral problem was solved by means of a certain QL algorithm. Along with the eigenfunction eigenvalues $\beta_{0,n}$ and parameters α_n, γ_n, were defined. The

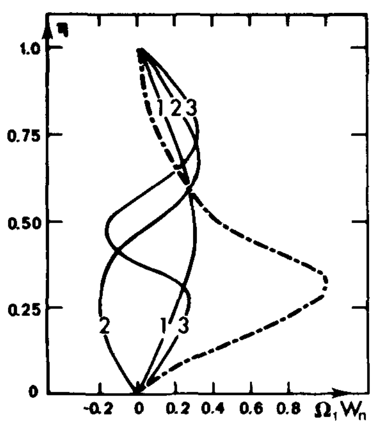

Fig. 1. Distribution of the dimensionless Väisälä–Brunt frequency $\Omega(\eta)$ (broken line) and of the first three normalized eigenfunctions $W_n(\eta)$.

phase velocity ($c_{f,n}^0 = HN_0\,\beta_{0,n}^{-1/2}$) and the addition to the phase velocity ($c_{f,n}^{(1)} = HN_0\,K_0^2/2\alpha_n\,\beta_{0,n}^{3/2}$) were calculated.

Here the nondimensional wavenumber for simplification is given as $K = K_0$.

The small parameter K_0^2 in expansions was equal to 0.05. The wavelength λ for the infinitesimal wave was equal to 1.32 km. The other parameters which characterize the weak nonlinear internal wave are given in Table 1.

From Table 1 it follows that for the third mode the value γ is negative and as a result the "negative" soliton for the third mode is to appear. The parameter $\mathscr{s}$ for cnoidal waves was chosen as equal to 0.99. In this case the wavelength of the cnoidal waves was equal to 2.14 $\bar{\lambda}$, where $\bar{\lambda}$ is the wavelength of the infinitesimal wave and phase velocity is $c_{f,n}^{(1)} \approx 0.11\ \hat{c}_{f,n}^{(1)}$. In Fig. 2 the functions $U_n(\xi)$ for the different mode numbers are shown. The amplitude functions $U(\xi)$ decrease as the modes increase. The soliton differs from the cnoidal wave only in that its minimum transforms into zero.

The streamlines of the nonlinear periodic internal waves of the horizontal structure illustrated in Fig. 2 for the first three modes are shown in Fig. 3. Judging from Fig. 3 the flow of liquid is disintegrated into cells, the number of the cells along the vertical coordinate being equal to the mode number. The vertical migration ζ of the waves is calculated by formula $\zeta = \Psi c_f^{-1} \approx H\,K_0^2\,U\,W$. Since maximum of ζ takes place for $U = U_{max}$, $W = W_{max} \approx 0.3$. The expression for the maximum of migration for solition is as follows: $\zeta = 0.3\,H\,K_0^2\,\gamma_n^{-1}$ (see Table 1).

For cnoidal waves the maximum of migration comes to 0.7 ζ_n in relation to soliton.

The pattern of streamlines for cnoidal waves of different modes ($n = 1,2,3$) is shown in Fig. 4. They have a layer cellular structure. An example of special distribution of the transverse velocity field for the first mode is given in Fig. 5c.

Table 1. *The parameters of the internal wave for the first three modes*

n	$\alpha_n\,10^2$	$\beta_{0,n}$	γ_n	$C_{f,n}\,\frac{sm}{s}$	$\hat{C}_{f,n}^{(1)}\,\frac{sm}{s}$	T_n h	ξ_n m
1	2.6	19.0	0.26	61	3.1	0.6	4.05
2	1.6	13.4	3.00	23	1.2	1.6	0.35
3	1.5	30.6	−5.00	15	0.7	2.4	0.21

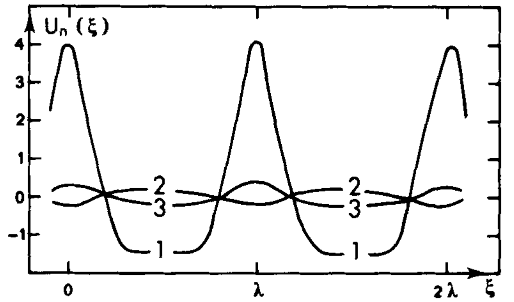

Fig. 2. Horizontal internal-wave structure $U_n(\xi)$ for various modes, $n = 1, \dots 3$.

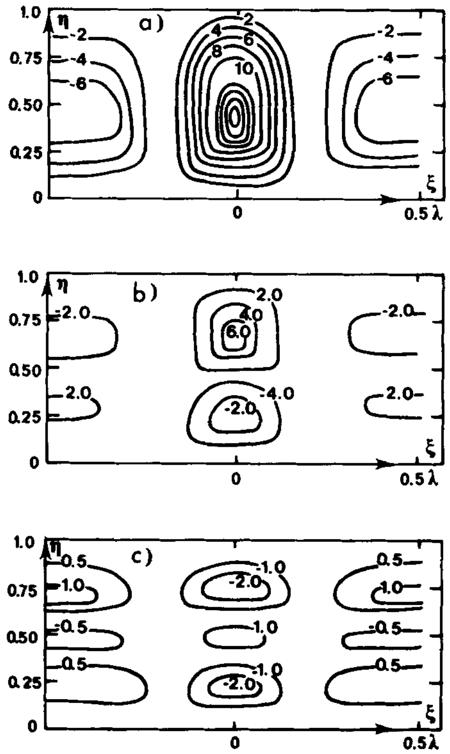

Fig. 3. Streamlines pattern for nonlinear periodic internal waves of various modes (a) $n = 1$, (b) $n = 2$, (c) $n = 3$. Isolines of the dimension values Ψ. 10^{-3} cm^2 s are marked by numbers at curves.

For solitary waves and approximately for cnoidal waves (not far from soliton) from eq. (5) we have $\rho = \rho_0(z - \Psi/c_f)$.

Using relative density definition $\sigma_T = (\hat{\rho}(\eta) - 1)$ 10^3, we obtain

$$\sigma_T = (\hat{\rho}_0(\eta) - 1) \cdot 10^3 + 10^3\, \delta K_0^2 \hat{\rho}_0(\eta)\, \Omega(\eta)\, W(\eta)\, U(\xi) \quad (33)$$

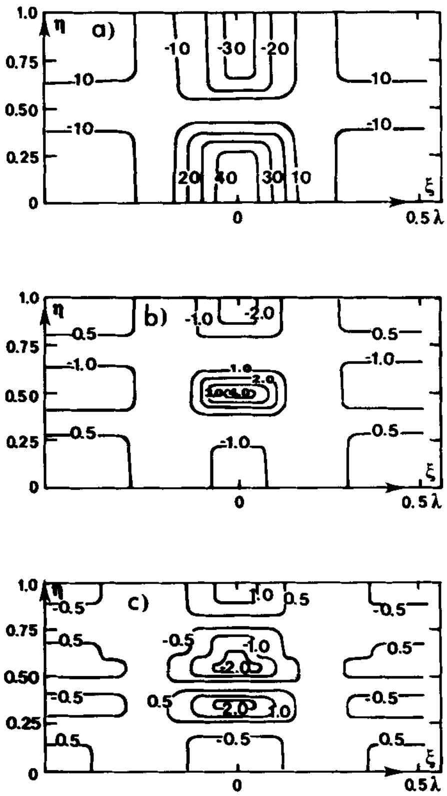

Fig. 4. Space distributions of longitudinal velocity for nonlinear periodic internal waves in the coordinate system, moving with the wave-propagation velocity v_0: (a) $n = 1$, (b) $n = 2$, (c) $n = 3$. Dimension velocity values U (cm/s) are marked by numbers at curves.

where $\hat{\rho}_0(\eta)$ is fluid density distribution undisturbed by waves. Fig. 5a shows the σ_T value distribution for the first mode of cnoidal wave.

The specific feature of the curves is the obvious concentration of isocrests in regions of the maximum Väisälä–Brunt frequency and the maximum amplitude of the functions $U(\xi)$. The secondary nonlinear effects bringing about violent mixing of stratified fluid are possible in the regions with large horizontal and vertical density gradients.

It is often supposed (Miles, 1963) that local hydrodynamic instability may appear in the regions with $R_i < \frac{1}{4}$ ($R_i = N^2(z)(du/dz)^{-2}$). The Richardson number isolines for the first mode of cnoidal wave are shown in Fig. 5b. It is obvious that the region with large density gradients has minimum value for Richardson number value (shaded parts in Fig. 5b).

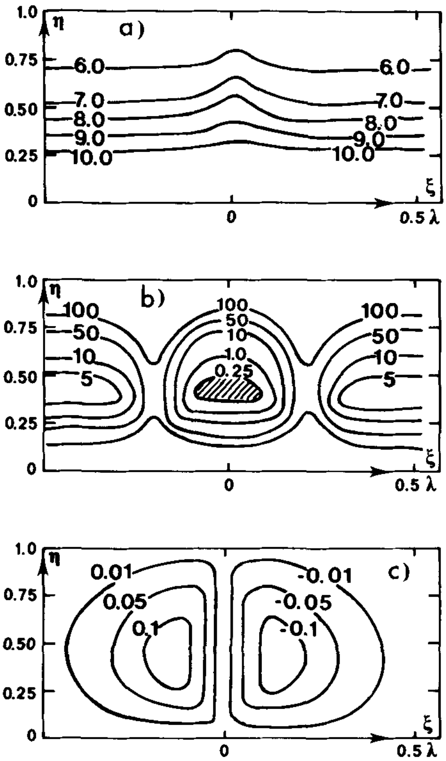

Fig. 5. (a) Space distributions of value σ_T for the first mode of nonlinear periodic wave. (b) Richardson number R_i isolines for the first mode of nonlinear periodic wave. (c) Space distribution of transverse velocity for the first mode of the nonlinear periodic wave. Dimension velocity values W (cm/s) are marked by numbers at curves.

5. Appendix

The correctness of eq. (32) will be proved below. At first the eq. (25) is investigated. The spectral problem for vertical operator describing internal waves with surface waves in nondimensional form is as follows

$$(\hat\rho_0 \varphi')' + \hat\rho_0(\Omega U^{-2} - K^2 \varphi) = 0$$

$$\varphi|_{\eta=-1} = 0, \quad \left[\varphi' - \frac{\varphi}{\delta v^2}\right]_{\eta=0} = 0 \qquad \text{(A.1)}$$

where $v = \hat c_f/HN_0$ is the nondimensional phase velocity, K is the nondimensional wave number, and φ is the eigenfunction. In case $v_0 = \lim_{K\to 0} V$ and $W = \lim_{K\to 0} \varphi$ with $K \to 0$ we have for dispersion correlation

$$v = v_0 + dK^2 + \cdots \qquad \text{(A.2)}$$

Employing the eqs. (A.1) and (A.2) on condition $\varphi = W + \tilde\varphi$ we find the equation for perturbation

$$(\hat\rho_0 \tilde\omega')' + \hat\rho_0 \frac{\Omega}{v_0^2}\tilde\varphi = \chi_1(\eta); \quad \left[\tilde\varphi' - \frac{\tilde\varphi}{\delta v_0^2}\right]_{\eta=0} = \chi_2$$

$$\chi_1(\eta) = K^2\left(1 + \frac{2d\Omega}{v_0^3}\right)W(\eta), \quad \chi_2 = -\frac{2dK^2}{v_0^3\delta}W(0) \qquad \text{(A.3)}$$

The solution for (A.3) exists in case the following condition is fulfilled:

$$-\chi_2 W(0) + \int_{-1}^{0} \chi_1(\eta)\,\hat\rho(\eta)\,W(\eta)\,d\eta = 0 \qquad \text{(A.4)}$$

Substituting (A.3) into (A.4) we find that

$$d = -\frac{v_0^3}{2\|W\|^2}\int_{-1}^{0} \hat\rho\Omega W^2\,d\eta < 0. \qquad \text{(A.5)}$$

When $dK^2 = \hat c_f^{(1)}/HN_0$ and $v_0 = c_f/HN_0$, the correctness of the equation (25) is proved. Let $c_{f,m}$ ($m = 0, 1, 2, \ldots$) be the phase velocity for long nonlinear waves satisfying the condition (28) $c_{f,m}^0 - |c_{f,m}^{(1)}| \leqslant c_{f,m} \leqslant c_{f,m}^0 + |c_{f,m}^{(1)}|$, where $c_{f,m}^0 =$ the "sound velocity" and $\hat c_{f,m} =$ correction defined by eq. (25).

It is possible to show that if $K \ll 1$ $c_{f,m} \neq c_{f,j}$ for any $m \neq j$ ($m, j = 0, 1, 2, \ldots$). In any case for $K = 0$ this condition is fulfilled. It appears that on the plane $\{c_f, K\}$ for $0 < K \leqslant K_0 \leqslant 1$ no stripes with thickness $2|\hat c_{f,m}^{(1)}$ intersect. Indeed for sufficiently large m-values we have asymptotic expansion

$$c_{f,m}^0 \sim \frac{1}{m}\,[1 + O(m^{-1}], \quad \hat c_{f,m}^{(1)} \sim K^2 (c_{f,m}^0)^3$$

$$\sim \frac{K^2}{m^3}\,[1 + O(m^{-1})]$$

Let us take $M \gg 1$ so that by $c_{f,m}^0 = 1/m[1 + O(m^{-1})]$. When $m < M$ all the stripes Δ_m do not intersect if K_0 is small enough. This relation is valid for $m > M$, because the distance between the dispersion curves is $c_{f,m}^0 - c_{f,m+1} \sim 1/m^2[1 + O(m^{-1})]$ and the semiwidth for all the neighbour stripes is equal to $|\hat c_{f,m}^{(1)}| \sim K_0^2\,m^{-3}$. Thus the inequality $c_{f,m} > c_{f,j}$ for $m < j$ has been proved. For the weak long nonlinear waves all values $\xi_m = x - c_{f,m}\,t$ ($= 0, 1, 2, \ldots$) are independent. The correctness of the eq. (32) remains to be proved when all ξ_m values are independent.

Let us use the following operator $\mathscr{L}^n (n = 0, 1, 2, \ldots)$ where n number of approximation. Let $\mathscr{L}^0 = \sum_{n=0}^{\infty} \mathscr{L}_m^0$ then we can write

$$\mathscr{L}^0 F_1 = \sum_{m=0}^{\infty} \mathscr{L}_m^{(0)} (F_{1,m}(\xi_m, \eta))$$

$$= \sum_{m=0}^{\infty} U_m(\xi_m) \mathscr{L}_m^{(0)} (W_m(\eta)) = 0$$

where $\mathscr{L}_m^0$ is a linear one-dimensional operator from boundary value problem (20). It is possible to write for the second approximation

$$\mathscr{L}^{(0)} F_2 \equiv \sum_{m=0}^{\infty} \mathscr{L}_m^{(0)} F_2(\xi_0 \xi_1 \xi_2 \ldots; \eta)$$

$$= \sum_{m=0}^{\infty} [\mathscr{L}_m^{(1)} (F_{1,m}) + \mathscr{L}_{mm}^{(1)} (F_{1,m} F_{1,m})]$$

$$+ \sum_{m \neq j}^{\infty} \mathscr{L}_{m,j}^{(1)} (F_{1,m} F_{1,j}) \tag{A.6}$$

Here $\mathscr{L}_m^{(1)}$ is a linear operator from the right hand side of (18) and $\mathscr{L}_{m,j}^{(1)}$ is the second power symmetric operator with index m, j. Considering that (A.6) we can write F_2 in the following form

$$F_2(\xi_0 \xi_1 \xi_2 \ldots; \eta) = \sum_{m=0}^{\infty} [V_m(\xi_m) W_m(\eta)]$$

$$+ F_{2,m}(\xi_m, \eta)] + \sum_{m \neq j} F_{2,m,j}(\xi_m, \xi_j; \eta) \tag{A.7}$$

where $V_m(\xi_m)$ is the amplitude function from the third approximation and $F_{2,m}$, $F_{2,m,j}$ the solution of the nonhomogeneous eq. (A.6). $F_{2,m,j}$ is the symmetric to ξ_m, ξ_j by argument and to m, j by index. Substituting (A.6) into (A.5) considering the independency of ξ_m we have

$$\mathscr{L}_m^{(0)} (F_{2,m}) = \mathscr{L}_m^{(1)} (F_{1,m}) + \mathscr{L}_{mm}^{(1)} (F_{1,m} F_{1,m})$$
$$(m = 0, 1, 2, \ldots) \tag{A.8}$$

$$\mathscr{L}_m^{(0)} (F_{2,m,j}) + \mathscr{L}_j^{(0)} (F_{2,m,j}) = \mathscr{L}_{m,j}^{(1)} (F_{1,m} F_{1,j})$$
$$(m \neq j) \tag{A.9}$$

The system (A.8) corresponds to the task (18) for the condition of solvability for (A.8) gives the Korteweg–de Vries eq. (23). The system (A.9) is solvable as the operator on the left-hand side (A.9) cannot come to the operator $\mathscr{L}_m^{(0)}$. Thus the principle of the additivity of the Eq. (32) has been proved.

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НЕЛИНЕЙНЫЕ ВНУТРЕННИЕ И ПОВЕРХНОСТНЫЕ ВОЛНЫ УСТАНОВИВШЕГОСЯ ВИДА В МЕЛКОМ МОРЕ

РЕЗЮМЕ

Распространение стационарных двумерных внутренних волн произвольной амплитуды описывается, как известно, одним нелинейным уравнением, в котором фигурируют две произвольных функции от искомой функции тока.

В работе показано, что для волновых движений солитонного типа, как и для инфинитезимальных периодических волн, обе произвольные функции однозначно определяются через невозмущенную частоту Вяйсяля-Брента. Для периодических же нелинейных волн в приближении слабой нелинейности и дисперсии в уравнении остается произвольная функция от поперечной координаты (течение). При дополнительном условии, что среднее по длине волны от амплитудной функции обращается в нуль, фазовая скорость нелинейной периодической волны оказывается однозначно связанной с амплитудой. Далее, показывается, что в рассматриваемом приближении слабой нелинейности и дисперсии сохраняется аддитивность волновых движений различных мод.

Приведены примеры, иллюстрирующие полученные общие результаты для модельных распределений частоты Вяйсяля-Брента. В конце статьи приведен детальный численный анализ для данных наблюдений в Арконском бассейне Балтийского моря. Расчёт локальных чисел Ричардсона для солитона показал, что они могут быть менее 1/4, что может приводить к образованию турбулентных пятен.