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Times of rise for turbulent forced plumes

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ABSTRACT

The motion of turbulent forced plumes, issuing vertically from isolated sources into an otherwise stable environment, is examined using a Lagrangian approach. Previous Eulerian approaches have yielded only numerical solutions whereas the present approach yields partial analytic solutions from which times of rise, from the source to levels of zero buoyancy and zero momentum, are evaluated for fluid elements travelling with the mean speed of flow. Consideration of previous work describing the asymptotic behaviour of starting plumes shows that times of rise may also be found for the leading front of a starting plume rising in a neutral environment.

1. Introduction

The solutions for turbulent forced plumes found by Morton (1959) and by Morton and Middleton (1973) describe, in Eulerian terms, the turbulent flow issuing from steady sources of mass flux, momentum flux and buoyancy flux into a neutral or stably stratified environment. These solutions are valid for flows resulting from turbulent emission of fluid into an extensive and otherwise still environment and the emission of momentum is assumed vertical since inclination of the angle of emission introduces additional physical issues. The historical development, applicability and limitations of this theory of forced plumes originally made by Morton, Taylor and Turner (1956) has been discussed extensively by the above authors and by Turner (1969, 1973) and need not be repeated here.

In this paper the equations of motion for forced plumes are presented and solved in terms of a Lagrangian approach. The Lagrangian equations of motion are derived from the Eulerian equations and so the resulting solutions are equivalent, however, the Lagrangian solutions have the advantage that they may be used in a direct way to determine times of rise, from the source to specified levels, of fluid elements travelling with the mean speed of flow.

A Lagrangian representation describes the time history of elements of fluid as they proceed with the flow and, ideally, relies upon each element being able to retain its identity for all time. In a turbulent forced plume this is obviously not strictly true. Each turbulent element emitted by the source entrains ambient fluid as it rises and hence increases in volume. Further, the combined effects of lateral turbulent diffusion and mean velocity shear over a plume cross-section act to disperse the element in the axial direction and this effect is enhanced by longitudinal turbulent diffusion. The net result is that an element grows in size and, relative to its mean position, spreads its influence both ahead and behind that position. Lagrangian elements may thus be conceived as overlapping to a growing extent, but the ensemble averages of quantities at a particular height will remain approximately those averages pertaining to an equivalent Lagrangian element at the same height. The following work relies upon the assumption that, on average, fluid elements move with the mean speed of flow and explicit considerations of the effects of longitudinal dispersion within the plume are side stepped.

Some justification of a Lagrangian approach may be gained by reference to other work involving dispersion in a slowly spreading turbulent shear flow. In particular, an analysis of transport in turbulent methane wakes was made by Morton (1962) for the case of a body falling with uniform speed through an environment consisting of a layer of methane overlying a layer of air. He used a

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Lagrangian approach to determine the motion of elements of fluid travelling at mean velocity in the wake of the body and found expressions specifying the position of the "mixture front" (the level in the wake separating regions of high and low methane concentrations) as a function of time and Richardson number. Preliminary experiments were made to determine the motion of this mixture front and the results gave good general support for the theory. Morton also analysed the time variation in concentration in the attached flow behind the falling body using simple Lagrangian arguments and subsequent experimental measurements by Leach and Slack (1967) were found to be in agreement to within experimental error.

In the following a partial analytic solution of the Lagrangian equations of motion is found for forced plumes issuing into a stable environment and common types of forced plumes are readily identified by examination of the solutions.

Times of rise of turbulent fluid elements from the source to levels of zero buoyancy and zero momentum are found for the general case of emission of mass flux, momentum flux and buoyancy flux from a source of finite radius into a stably stratified environment. The analytic relation between the Lagrangian solutions and the Eulerian solutions is shown for the case of forced plumes rising in a neutral environment and times of rise of fluid elements are again found. Reference to experimental and theoretical work on starting plumes shows that times of rise of the leading front of a starting plume rising in a neutral environment may also be deduced.

2. The Lagrangian conservation equations

The following calculations are based on the same integrated equations of motion as the earlier work of Morton and Middleton (1973) and supplement the solutions found in that work. The same assumptions are made and these may be briefly restated as follows: that the fully developed turbulent flow is independent of both Reynolds and Peclét numbers, that the Boussinesq approximation holds throughout the flow and that the mean flow profiles remain similar at all plume cross-sections. It follows then that the rate of entrainment of ambient fluid into the plume must scale with the turbulent intensity which, in turn, must scale with the mean flow. Closure of the equations is then possible through use of an entrainment constant α, defined as the ratio of mean radial inflow speed at the plume edge to the mean vertical velocity at the same height.

Cylindrical polar coordinates (r, ϕ, X) are used with OX vertically upwards and the motion is assumed axisymmetric and without swirl. If ρ_0 represents the environmental density, ρ the density of the convecting fluid and ρ_1 the reference density then the mean buoyancy may be expressed as $P = g\rho_1^{-1}(\rho_0 - \rho)$. The density deficiency (or buoyancy) may be the result of a temperature excess, in which case use may be made of the relation

$$\rho_1^{-1}(\rho_0 - \rho) = \beta(T - T_0)$$

where T and T_0 refer to plume and environmental temperatures respectively and β is the coefficient of cubical expansion (for gases, $\beta \simeq T_1^{-1}$).

Top-hat profiles, representing values of mean vertical velocity and mean buoyancy which are constant across the plume section and zero outside, are used for simplicity since, in any case, only a constant of proportionality survives the lateral integration of the profile shapes and the basic axial dependencies of flow quantities remain independent of the choice of profile (see, for example, Turner, 1973). The mean vertical velocity is represented here by U, the mean buoyancy by P and the plume radius by b. The square of the Brunt–Väisälä frequency N describes the environmental stability according to $N^2 = -(g/\rho_1)d\rho_0/dX$ and this is assumed to be independent of height.

With these assumptions, the integrated equations of motion representing gross conservation of volume (or mass), momentum and buoyancy within the plume are given by

$$\frac{d}{dX}(b^2U) = 2\alpha bU, \quad \frac{d}{dX}(b^2U^2) = b^2P,$$

$$\frac{d}{dX}(b^2UP) = -b^2UN^2$$

The Lagrangian equations may be derived from these equations by utilizing a change of variable from height X to time t given by

$$\frac{d}{dX} = U^{-1}\frac{d}{dt}$$

and since U is the mean speed of flow it follows that the time variable t is an appropriate scale for evaluating the time of rise of elements travelling with this speed.

Dimensional variables are defined for the mass flux W, the buoyancy flux F and the square of the momentum flux M according to

$$W = b^2U, \quad F = b^2UP \quad \text{and} \quad M = b^2U^2 \tag{1}$$

and the Lagrangian equations of motion become

$$\frac{dW}{dt} = 2\alpha M^{3/2}W^{-1}, \quad \frac{dM}{dt} = F, \quad \frac{dF}{dt} = -N^2M$$

These equations have the general solutions

$$\begin{aligned} M &= (M_0^2 + F_0^2/N^2)^{1/2} \sin[N(t - t_0) + \delta] \\ F &= (F_0^2 + M_0^2N^2)^{1/2} \cos[N(t - t_0) + \delta] \\ W^2 &= 4\alpha \int_{t_0}^{t} M^{3/2}(t')dt' + W_0^2 \end{aligned} \tag{2}$$

Here F_0, M_0 and W_0 represent the fluxes of buoyancy, momentum (squared) and mass emitted from the source at time $t = t_0$ and the phase angle δ is given by

$$\tan \delta = \frac{NM_0}{F_0} \tag{3}$$

The solutions (2) may be simplified by defining the argument $\theta = N(t - t_0) + \delta$, and then scaling to dimensionless form by the transformations

$$M = M_0m, \quad F = F_0f, \quad W = \alpha^{1/2}M_0^{5/4}F_0^{-1/2}w$$

The dimensionless solutions are

$$\begin{aligned} m &= \operatorname{cosec} \delta \sin \theta, \quad f = \sec \delta \cos \theta \\ w^2 &= \tfrac{8}{5}(\tfrac{5}{2} \cos \delta \operatorname{cosec}^{5/2} \delta \textstyle\int_\delta^\theta \sin^{3/2} \theta' \, d\theta' + \Gamma) \end{aligned} \tag{4}$$

These represent the flow from the source level $\theta = \delta$ to the general level $\theta > \delta$. The dimensionless parameter

$$\Gamma = \frac{5F_0W_0^2}{8\alpha M_0^{5/2}} \tag{5}$$

defined originally by Morton (1959), represents the balance of flow conditions existing at the source and is a direct measure of the ratio of buoyancy effects to inertia effects at the source.

Here, the scaling has been chosen to be compatible with that used by Morton and Middleton (1973) for the case of plumes in a stable environment so the two sets of solutions are comparable. Also, the dimensionless parameter Δ defined by Morton and Middleton is related to the phase angle δ of the work by

$$\tan \delta = (2\Delta)^{1/2}.$$

The Lagrangian solutions (4) are valid for turbulent forced plumes rising in neutral or stably stratified environments, the expressions for momentum flux (squared) m and buoyancy flux f being analytic and the expression for the volume flux w involving a numerical integral of simple form. These might be compared with previous Eulerian solutions which are entirely numerical for stably stratified environments and which are concerned with heights of rise rather than times of rise. Analytic expressions for heights of rise are not available from (4) and the reader is referred to Morton (1959) and Morton and Middleton (1973) for a detailed discussion of heights of rise of turbulent forced plumes. However, the Lagrangian solutions do describe, in compact form, the full range of behaviour exhibited by forced plumes and readily provide estimates for times of rise of elements travelling with the mean speed of flow. A discussion of forced plumes in terms of the Lagrangian solutions, including expressions for the times of rise in stably stratified and neutral environments, is presented in the following sections.

3. Plumes in a stably stratified environment

The solutions (4) represent plume rise in a stably stratified environment and are shown, normalized with respect to functions of the phase angle δ, in Figs. 1 to 3. Here the common assumption is made that emission from the source is always upward and thus $M_0^{1/2} \geqslant 0$.

Some common properties of forced plumes may easily be recognized, for example, values of $\theta = \pi/2$ and $\theta = \pi$ represent levels of zero buoyancy and zero momentum respectively. The level of zero momentum is that level at which the plume achieves its maximum height of rise and this level is obtained by all forced plumes rising in a stably stratified environment. The level of zero buoyancy is reached only by those forced plumes

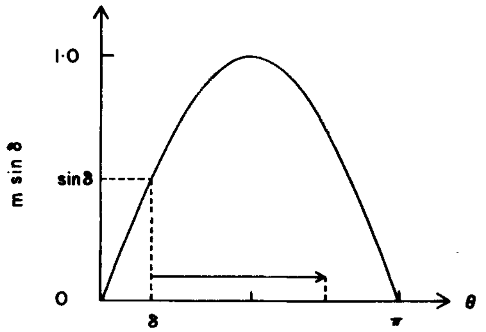

Fig. 1. The solution $m \sin \delta = \sin \theta$ specifying the dimensionless momentum flux (squared) m of a forced plume plotted against the dimensionless time scale θ. The parameter δ is defined by $\tan \delta = NM_0/F_0$ and the horizontal arrow represents the dimensionless time of rise $N(t - t_0) = \theta - \delta$ of a fluid element travelling with the mean speed of flow from the source level characterized by $\theta = \delta$ to the higher level $\delta < \theta \leq \pi$. The level of zero momentum is represented by $\theta = \pi$ and the dimensionless time of rise to this level is $N(t - t_0) = \pi - \delta$.

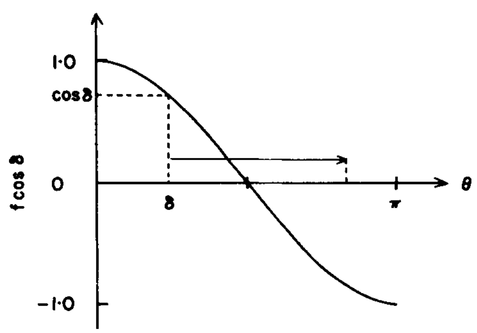

Fig. 2. The solution $f \cos \delta = \cos \theta$ specifying the dimensionless buoyancy flux f of a forced plume plotted against the dimensionless time scale θ. The parameter δ is defined by $\tan \delta = NM_0/F_0$ and the horizontal arrow represents the dimensionless time of rise as explained in the caption to Fig. 1. The level of zero buoyancy is represented by $\theta = \pi/2$ and the dimensionless time of rise to this level is given by $N(t - t_0) = \pi/2 - \delta$. Note that if $F_0 < 0$ then $\pi/2 < \delta$ and the buoyancy flux is always negative.

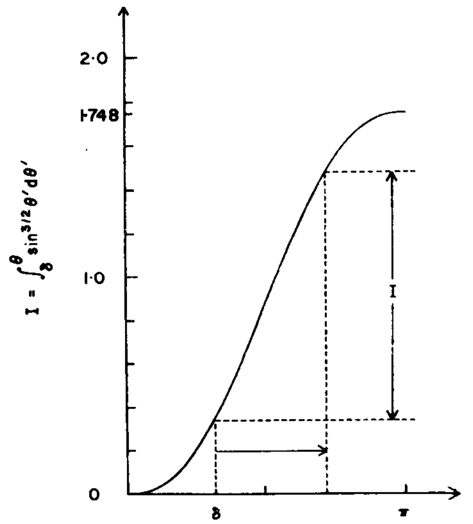

Fig. 3. The numerical solution $I = \int_\delta^\theta \sin^{3/2} \theta' \, d\theta'$ used in determining the dimensionless volume flux w according to $w^2 = \frac{8}{5}(\frac{5}{2}I \cos \delta \operatorname{cosec}^{5/2} \delta + \Gamma)$. The dimensionless time of rise is represented by the horizontal arrow as explained in the caption to Fig. 1. Since I is a monotonic function of θ (for any δ) it follows that the volume flux always increases with time.

which are emitted into a stable environment with initially positive buoyancy and the flow above this level exists only as a result of the momentum already possessed at that level.

The parameter $\Delta = \frac{1}{2}\tan^2 \delta$ may be viewed in either of two ways. It may be thought of as the environmental stability suitably made dimensionless by the source fluxes of buoyancy and momentum, as in the work of Morton and Middleton, or alternatively it may be thought of as being the ratio of source momentum flux to source buoyancy flux, suitably made dimensionless by the environmental stability. For this particular presentation, the latter is more suitable and the values of δ and Γ may be thought of as primarily describing the source, but with the restriction that the environment is stably stratified.

Sources emitting fluid which is positively buoyant at source level are specified by $\Gamma > 0$ and $0 < \delta < \pi/2$, while sources emitting fluid which is negatively buoyant at source level are specified by $\Gamma < 0$ and $\pi/2 < \delta < \pi$. The flows resulting from the latter are jet-like in behaviour. If the source momentum flux $M_0 \to 0$, then $\delta \to 0$ and $\Gamma \to \infty$, and the source tends to the asymptotically limiting case of an infinite planar source emitting fluxes of buoyancy and volume flux, but no momentum flux. For finite M_0 the source velocity is of order (source radius)$^{-1}$ and the volume flux W_0 is consequently of order (source radius). thus $W_0 = 0$ (and hence $\Gamma = 0$) characterizes a point source having zero radius, finite momentum and infinite velocity. Here the value of δ may be greater than or less than $\pi/2$ according as the buoyancy of the fluid at the source is negative or positive.

Certain information is more readily available from the Lagrangian solutions rather than from the Eulerian solutions and in particular, the Lagrangian format is more suitable for determining approximate times of rise of fluid elements from the source to various heights.

In the case of the general type of forced plume rising in a stable environment, the level of zero buoyancy is given by $N(t - t_0) + \delta = \pi/2$ and the dimensionless time of rise for each element is thus

$$N(t - t_0) = \frac{\pi}{2} - \tan^{-1}\left(\frac{M_0 N}{F_0}\right)$$

For plumes issuing from the source with positive buoyancy the time of rise to the level of zero buoyancy is larger for smaller values of source momentum and is larger for greater values of source buoyancy since then the height of the level of zero buoyancy is also greater (see Fig. 2). The dimensional time of rise is less for stronger ambient stratification.

The level of zero momentum is given by $N(t - t_0) + \delta = \pi$ and represents the level of the maximum height of rise. The dimensionless time of rise to this level is thus

$$N(t - t_0) = \pi - \tan^{-1}\left(\frac{M_0 N}{F_0}\right)$$

For plumes issuing from the source with positive buoyancy the phase angle δ lies in the first quadrant and the dimensionless time of rise is greater than $\pi/2$, while plumes issuing from the source with negative buoyancy (jets) have a dimensionless time of rise less than $\pi/2$ since, in this case, δ lies in the second quadrant (see Fig. 1). The dimensionless time of rise from the level of zero buoyancy to the level of zero momentum is given by $N(t - t_0) = \pi/2$. Plumes rising from the source with positive buoyancy have times of rise, from the source to the level of zero momentum, which are larger for smaller values of source momentum and larger for greater values of source buoyancy.

4. Forced plumes in a neutral environment

It is appropriate to show here the relation between the Lagrangian solutions and the Eulerian solutions found by Morton and Middleton for plumes rising in a neutral environment. For $N = 0$, the phase angle $\delta = 0$ and the solutions (4) reduce to

$$f = 1, \quad m = 1 + (t - t_0)F_0 M_0^{-1},$$
$$w^2 = \tfrac{8}{5}\{m^{5/2} - 1 + \Gamma\}$$

These solutions are identical with those found previously with the Eulerian approach since the dimensionless momentum flux v of Morton and Middleton is related to m by $v^2 = m$. Integration of the vertical velocity (m/w) with respect to time shows the dimensionless height of rise as a function of v to be

$$2\alpha^{1/2} F_0^{1/2} M_0^{-3/4} X = 10^{1/2} \int_1^v \frac{v^3\, dv}{(v^5 - 1 + \Gamma)^{1/2}}$$

again equivalent to the previous solution.

For positively buoyant flows $F_0 > 0$ and it follows that the momentum increases indefinitely, the height of rise is infinite and there is an infinite time of rise. For negatively buoyant flows the momentum drops to zero when $1 + (t - t_0)F_0 M_0^{-1} = 0$ and there is a maximum height of rise at which the dimensionless volume flux approaches the asymptotic value of $w = [\tfrac{8}{5}(\Gamma - 1)]^{1/2}$. The dimensional time of rise to this level is given by

$$t - t_0 = -M_0 F_0^{-1}$$

a positive value since $F_0 < 0$.

Sufficiently far from the source, any plume may be considered as being emitted from an asymptotic virtual point source of buoyancy only. The differences in height between real and asymptotic sources have been plotted by Morton and Middleton for all values of Γ between $\Gamma = -4$ and $\Gamma = 5$. These heights, in conjunction with times of rise for simple plumes, may be therefore used to approximate times of rise of elements in forced plumes issuing from various sources in cases where analytical solutions are not possible. In some cases, however, extrapolation of an observed flow profile down below the level of the real source will more easily determine the level of this asymptotic point source.

The solutions of the equations of motion for a simple plume issuing from a point source into a neutral environment are

$$F = F_0, \quad M = (t - t_0)F_0,$$
$$W = 2^{3/2} 5^{-1/2} \alpha^{1/2} (t - t_0)^{5/4} F_0^{3/4}$$

and the vertical velocity $U = M/W$ and height X are thus

$$U = 2^{-3/2}5^{1/2}\alpha^{-1/2}(t - t_0)^{-1/4}F_0^{1/4},$$
$$X = 2^{1/2}3^{-1}5^{1/2}\alpha^{-1/2}(t - t_0)^{3/4}F_0^{1/4}$$

The time of rise of elements from the asymptotic virtual point source to height X is thus given by

$$(t - t_0) = (\tfrac{9}{10}\alpha)^{2/3}\left(\frac{X^4}{F_0}\right)^{1/3} \qquad (6)$$

Times of rise from real sources are obtained from (6) by determining the height X_{avs} that the real source exists above the asymptotic virtual point source. Then the time of rise from the real source to height h above the real source is given by

$$t - t_0 = (\tfrac{9}{10}\alpha)^{2/3}F_0^{-1/3}\{(X_{avs} + h)^{4/3} - X_{avs}^{4/3}\}$$

where t_0 now represents the time of emission from the real source.

Times of rise for starting plumes may be readily deduced from times of rise for simple plumes. Following Turner (1962), such calculations are possible since the leading front (or cap) of a starting plume at any height is known to travel at a constant fraction of the mean velocity (at that same height) of an established steady plume.

If the velocity of the leading front is v_t and the mean velocity of the steady plume (evaluated at the same height) is U_t, then the proportionality of velocities may be expressed as (Middleton, 1975)

$$v_t = A_tU_t$$

where Turner's experimental results show the constant A_t to have the value

$$A_t = 0.61 \pm 0.05$$

Times of rise for starting plumes are thus larger by the constant fraction A_t^{-1} and the time of rise to height h from a real source (of height X_{avs} above the virtual source) is thus

$$t - t_0 = A_t^{-1}(\tfrac{9}{10}\alpha)^{2/3}F_0^{-1/3}\{(X_{avs} + h)^{4/3} - X_{avs}^{4/3}\}$$

5. Discussion

The Lagrangian solutions found above have some advantages over the previously found Eulerian solutions, particularly in the case of a stably stratified environment where a partial analytic solution is available for the Lagrangian equations but the Eulerian solutions may only be found numerically.

The Lagrangian solutions are fully equivalent to the Eulerian solutions, being derived from the same set of equations, but have the advantage that the partial analytic solutions readily allow evaluation of the times of rise, from the real source to levels of zero buoyancy and zero momentum, of fluid elements travelling with the mean speed of flow in an established plume. Since the leading edge of a starting plume rising within a neutral environment is known to travel at a constant fraction of the mean velocity of the steady plume behind, times of rise for starting plumes may also be determined.

This work thus supplements previous work on forced plumes and at the same time provides an alternative view to the wide range of behaviour exhibited by forced plumes.

Finally, it is evident that the solutions representing the times of rise of fluid elements may prove to be more accurate in some regimes of flow than in others and that the above work must be considered as incomplete without supporting experimental evidence. This paper is offered in the hope that it may generate such experimental interest.

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ВРЕМЕНА ПОДЪЕМА ДЛЯ ВЫНУЖДЕННЫХ ТУРБУЛЕНТНЫХ ЯЗЫКОВ

С помощью лагранжева подхода изучается движение вынужденных турбулентных языков, поднимающихся вертикально от изолированных источников в устойчиво стратифицированную среду. Прежние эйлеровы подходы приводили только к численным решениям, в то время как данный подход дает частные аналитические решения, из которых находятся времена подъема жидких элементов, распространяющихся со средней скоростью потока, от источника до уровня, где их плавучесть и вертикальный импульс равны нулю. Рассмотрение прежних работ, описывающих асимптотическое поведение языков, показывает, что времена подъема можно найти для ведущего фронта возникающего языка, поднимающегося в нейтральной среде.