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Stationary flow of a quasi-geostrophic, stratified atmosphere past finite amplitude obstacles

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ABSTRACT

The stationary flow past finite amplitude bottom topography of a quasi-geostrophic atmosphere, uniform in wind and entropy stratification far from the obstacle, is studied on an f-plane. Analytic solutions for the two cases of finite slope (hemispheroid) and infinite slope (disc) mountains are discussed. The obstacles act on the flow in different ways in the two cases. In the case of finite slope the obstacle influences the model atmosphere through the lifting of isentropic surfaces near the lower boundary. The consequent shrinking of vortex tubes produces anti-cyclonic vorticity over the obstacle and, possibly, Taylor column formation, in agreement with previous results in the literature.

In the case of an infinite slope mountain a pronounced diffluence in the horizontal streamlines is produced near the obstacle. The total circulation, again associated with the lifting of isentropic surfaces, has to be prescribed in this case for the problem to be completely determined. However, the total disturbance does not vanish when the volume of the obstacle goes to zero, provided that its cross section remains finite. The implications for numerical modelling are discussed.

1. Introduction

Recent literature is rich in contributions to the understanding and modelling of the slow motion of an inviscid, stratified, rotating fluid past isolated obstacles in the bottom boundary (Huppert, 1975; Huppert and Bryan, 1976; Merkine and Kalnay-Rivas, 1976; Buzzi and Tibaldi, 1977).

Most of the theoretical work performed by mathematical analysis is, explicitly or not, based on the quasi-geostrophic approximation. This permits, in its more compact formulation (see Charney, 1973), the dynamic equations (vorticity and entropy conservation) and boundary conditions to be written in terms of a single scalar variable ψ that acts as a streamfunction for the horizontal motion and as a kind of potential function for the specific entropy fluctuations.

The quasi-geostrophic equations are non-linear but, since they express conservation of physical quantities (pseudo-potential vorticity and potential temperature) that are linear in the streamfunction ψ, they turn out to be linear in ψ for particular steady flow configurations. Linearization of the boundary condition at the bottom surface is usually performed assuming that the maximum height of the obstacle $h_{\max}$ is small with respect to a characteristic vertical dimension D of the fluid ($h_{\max}/D \gtrsim$ Rossby number) so that the boundary condition can be applied, in linearized form, at $z = 0$, instead of at the surface of the obstacle (Huppert, 1975; Buzzi and Tibaldi, 1977).

Quasi-geostrophic theory is formulated in terms of equations which only describe processes of horizontal advection. This is usually expressed by saying that the motion is "quasi-horizontal". This does not necessarily imply that the slope of the bottom boundary itself is small. The slope of the particle trajectories can be much smaller than the maximum slope of the obstacle and consequently the motion can still be quasi-horizontal.

Thus, it is possible to study the problem of quasi-horizontal flow past a finite slope obstacle (provided one knows how to handle the non-linear boundary condition along the bottom).

The proper boundary condition is obtained by

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evaluating the thermodynamic equation (conservation of potential temperature) along the bottom surface. In the first-order expansion in the Rossby number, this boundary condition is in the form of an equation of conservation of a quantity that is linear in ψ.

We have chosen to study two different cases in which complete linearization of the problem is possible. They represent two extreme geometrical configurations by which isolated synoptic scale topography can be simulated. The solution for the first case (hemispheroidal obstacle with isentropic surface) is compared with the linearized analytical solutions and with the finite amplitude numerical solution of semi-geostrophic equations (Merkine and Kalnay-Rivas, 1976) obtained by applying the same bottom boundary condition. The solution for the second case (half disc across the basic current) displays important differences from the first case. The quasi-geostrophic equations can be integrated for any given distribution of stratification along the obstacle, visualizing the influence of the lifting of the isentropic surfaces on the dynamics of the flow.

2. Formulation of the problem

A complete derivation and discussion of the quasi-geostrophic equations for a quasi-Boussinesq atmosphere is given in Charney (1973). In the simplified case of uniform Brunt-Vaisala frequency N and Coriolis parameter f, the fields of horizontal velocity $\mathbb{V}_H$, vertical velocity w and potential temperature θ are determined by the equations

$$\frac{D_H q}{Dt} = 0 \tag{2.1}$$

$$\frac{D_H}{Dt}\left(\frac{\partial\psi}{\partial z}\right) + \frac{N^2}{f} w = 0 \tag{2.2}$$

$$\mathbb{V}_H = \mathbb{K} \times \nabla_H \psi \tag{2.3}$$

$$q = \nabla_H^2 \psi + \frac{f^2}{N^2}\frac{\partial^2\psi}{\partial z^2} \tag{2.4}$$

where:

∇_H is the horizontal del-operator;

$$\frac{D_H}{Dt} = \frac{\partial}{\partial t} + \mathbb{V}_H \cdot \nabla_H;$$

g is the acceleration of gravity,

p_s, θ_s are the horizontally averaged pressure and potential temperature, the latter determining N through: $N^2 = g(\partial/\partial z)(\ln \theta_s)$; p is the pressure;

$\psi = \dfrac{p - p_s}{\rho_s f}$ is the quasi-geostrophic stream function;

$$\frac{\partial\psi}{\partial z} = \frac{g}{f}\ln\frac{\theta}{\theta_s} \approx \frac{g}{f}\frac{\theta'}{\theta_s}, \quad \text{since } \frac{\theta'}{\theta_s} \ll 1.$$

The evolution eqs. (2.1) and (2.2) are the result of a first-order expansion in the Rossby number $Ro = U/fL$ (U and L are typical values of horizontal velocity and horizontal scale), of respectively vorticity and specific entropy conservation equations.

Equation (2.1) asserts the conservation of potential vorticity in the quasi-geostrophic approximation. However q, defined in (2.4), is not conserved by a particle, as is Ertel potential vorticity, but at the horizontal projection of a particle trajectory. For this reason it is called pseudo-potential vorticity (Charney, 1973). In order to integrate eqs. (2.1)–(2.4) inside a fixed volume it is necessary to specify the normal velocity at every point along the boundary and the pseudo-potential vorticity at inflow points (Charney, Fjørtoft and von Neumann, 1950). At rigid boundaries the condition is obviously that the normal velocity component vanishes. In the case of a stationary flow ($\partial/\partial t = 0$) eqn. (2.1) reduces to

$$J_H(\psi, q) = 0 \tag{2.5}$$

where J_H is the Jacobian operator in the x, y coordinates. Condition (2.5) implies that

$$q = q(\psi, z) \tag{2.6}$$

Once the functional dependence on ψ and z of the pseudo-potential vorticity is specified, we get from (2.6) an elliptic differential equation in ψ. If the function $q(\psi, z)$ is known along a cross section of the flow cutting all streamlines, it can be extrapolated everywhere. In particular we will assume that the flow is prescribed infinitely far from an isolated obstacle, centered around the origin of a Cartesian coordinate system (see Fig. 1), such as

$$\psi_{-\infty}(y,z) = \lim_{x\to-\infty} \psi(x,y,z) = -u_{-\infty}\, y \tag{2.7}$$

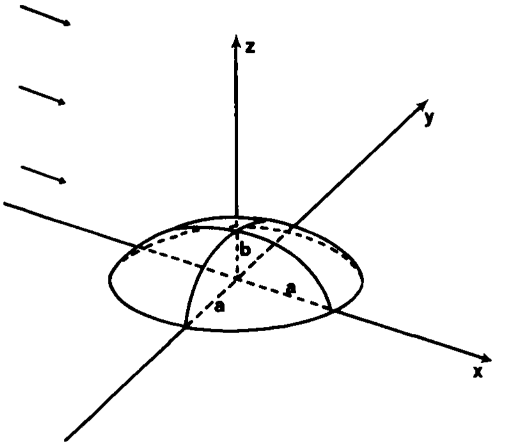

Fig. 1. Geometry of flow past a hemispheroid.

which corresponds, because of (2.3), to a uniform wind along the x-axis. It follows that the pseudo-potential vorticity is zero,

$$q=\left\{\nabla_H^2+\frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}\right\}\psi=0 \tag{2.8}$$

except, possibly, in regions of closed horizontal circulation. In order to avoid analytical difficulties arising from the presence of regions of closed streamlines, we will assume explicitly that pseudo-potential vorticity is uniform, and consequently equal to zero everywhere, including regions where streamlines do not arise far upstream from the obstacle. This is equivalent to the assumption that the stationary flows we study are the final result of initial states characterized by pseudo-potential vorticity equal to zero everywhere. In this case q, being conserved, will remain zero everywhere at any time. The equation for the ψ-field reduces to the linear form (2.8).

Equation (2.2) can be applied along non-vertical boundaries $z = h(x, y)$

$$\left[\mathbb{V}_H\cdot\nabla_H\frac{\partial\psi}{\partial z}\right]_{z=h}+\frac{N^2}{f}[w]_{z=h}=0 \tag{2.9}$$

and reduces, by means of the impenetrability condition

$$[w]_{z=h}=[\mathbb{V}_H]_{z=h}\cdot\nabla_H h \tag{2.10}$$

to the form

$$\left[\mathbb{V}_H\cdot\nabla_H\frac{\partial\psi}{\partial z}\right]_{z=h}+\frac{N^2}{f}[\mathbb{V}_H]_{z=h}\cdot\nabla_H h=0 \tag{2.11}$$

The formal expansion of advection along the boundary is:

$$\left[\mathbb{V}_H\cdot\nabla_H\frac{\partial\psi}{\partial z}\right]_{z=h}=[\mathbb{V}_H]_{z=h}\cdot\nabla_H\left[\frac{\delta\psi}{\delta z}\right]_{z=h}-\left[\frac{\partial^2\psi}{\partial z^2}\right]_{z=h}\cdot\nabla_H h \tag{2.12}$$

but, at the same order in the Rossby number, the second term on the r.h.s. of (2.12) must be neglected, because it represents (remembering (2.10)) a vertical advection of entropy fluctuations. Therefore (2.11) can be reduced to the form

$$[\mathbb{V}_H]_{z=h}\cdot\nabla_H\left[\frac{\partial\psi}{\partial z}+\frac{N^2}{f}z\right]_{z=h}=0 \tag{2.13}$$

This equation states that, in general, the streamlines have to be parallel to the horizontal projection (the gradient is calculated horizontally) of the isolines of the specific entropy function $(\partial\psi/\partial z) + (N^2/f)z$ calculated along the sloping wall. The mathematical consequence is that $(\partial\psi/\partial z) + (N^2/f)z$ is a function only of ψ calculated along the bottom surface and z. At horizontal walls, (2.13) reduces to the conservative form

$$J_H\left(\psi,\frac{\partial\psi}{\partial z}\right)=0 \tag{2.14}$$

which implies that $\delta\psi/\partial z$ along the bottom surface is a function only of ψ.

Along vertical walls the impenetrability condition cannot be written in the form (2.10). The boundary condition can be stated in terms of the horizontal streamfunction ψ, as the requirement that ψ, calculated at the wall, is a function only of the vertical coordinate z. As a consequence, because of (2.2), vertical velocity along the walls has to vanish. For both of the extreme boundary configurations (horizontal and vertical walls) it is clear that the problem we are analyzing is not uniquely determined. In fact the distribution of $\delta\psi/\partial z$ (potential temperature) along the bottom boundary can be specified in an infinite variety of ways all satisfying (2.13), in the same way as $\psi(z)$ can be arbitrarily specified in the case of vertical walls.

We will consider flows with no upper boundary. This poses the question of boundary conditions far above and downstream of the isolated obstacle. It will be assumed that the flow can be split into a far field $\psi_{-\infty}$ and a "mountain-induced disturbance" ψ', $(\psi = \psi_{-\infty} + \psi')$, that vanishes asymptotically in all directions

$$\lim_{r\to+\infty} \psi' = 0 \quad r^2 = x^2 + y^2 + z^2 \tag{2.15}$$

In synthesis we have seen that if the problem is restricted to steady flows that satisfy boundary condition (2.7) far upstream and arise from an initial condition of uniform zero potential vorticity, we are left with a mathematical problem that is linear in the differential eqn. (2.8) and the "far field" boundary condition (2.15), but non-linear in the boundary condition at the bottom surface (2.13). Moreover the problem is not uniquely determined since there are an infinity of different configurations corresponding to different initial conditions (all of uniformly zero potential vorticity) satisfying the imposed conditions.

As we have already mentioned, the quasi-geostrophic equations are the result of an expansion in which the velocity field is almost horizontal. The approximation holds as long as the ratio of vertical to horizontal velocities is

$$W/U \sim \varepsilon Ro\, \delta \quad \text{if} \quad \varepsilon \lesssim 1 \tag{2.16}$$

or

$$W/U \sim Ro\, \delta \quad \text{if} \quad \varepsilon > 1 \tag{2.17}$$

where $\varepsilon = f^2L^2/D^2N^2$ (ε^{-1} is a nondimensional radius of deformation), D is a typical vertical scale of motion. (For a vertically bounded fluid, D coincides with the fluid depth; in the present case of unbounded fluid, D is the vertical decay scale of the mountain-induced perturbation.) $\delta = D/L$ is an "aspect ratio" parameter (Charney, 1973).

Introducing into (2.16) the relation $fL \lesssim DN$ and into (2.17) the relation $fL > DN$, we obtain the same condition for all values of ε:

$$W/U \lesssim \frac{f}{N} Ro = \frac{U}{NL} \tag{2.18}$$

for quasi-geostrophic approximation to be applicable. For consistency condition (2.18) must be checked *a posteriori* on the analytical solution whenever the obstacle slope is finite. Along the obstacle surface, the vertical velocity is given by (2.10). From this expression we can derive:

$$\frac{W}{U} \sim \frac{H}{L} \cos \alpha \tag{2.19}$$

where H is a typical height of the obstacle, and α is the angle between the horizontal velocity vector and the height gradient. This angular term is essential because it describes a crucial characteristic of the flow near the obstacle, that is, the relationship between the slope of trajectories and the slope of the obstacle itself. The combination of (2.18) and (2.19) yields:

$$\cos \alpha \lesssim \frac{Lf}{NH} Ro = \frac{U}{NH} \tag{2.20}$$

If $U/NH \gtrsim 1$, or if $H/L \ll f/N$ for $Ro \ll 1$, particle trajectories may climb the obstacle along the maximum slope without violating quasi-horizontality, as in the case of an infinitesimal mountain. If, on the other hand, $U/NH < 1$, or $H/L \gtrsim f/N$, quasi-geostrophic flow must be essentially around the obstacle, possibly leading to the formation of closed streamlines. For typical atmospheric values ($U \sim 10\ \mathrm{ms^{-1}}$, $N \sim 10^{-2}\ \mathrm{s^{-1}}$) the critical number U/NH is order unity for geophysically significant values of H ($H \sim 1000$ m). The importance of this number for the determination of the nature of flow past obstacles has also been pointed out by Huppert (1975) and Buzzi and Tibaldi (1977).

We analyze in the following section two different finite amplitude obstacle geometries:

(i) flow past an isolated mountain in the form of a hemispheroid (Fig. 1) of finite slope (ratio of vertical to horizontal radius $b/a = f/N$); this case is solved in the special case of an isentropic ground;

(ii) flow past an infinite slope isolated mountain in the form of a disk (Fig. 3) (again the ratio of vertical to horizontal radius is $b/a = f/N$); this problem can be solved for an infinite class of bottom boundary conditions.

3. Flow past a gently sloping mountain

In order to keep the analytical treatment of the problem as simple as possible it is useful to simulate an isolated topographic obstacle with a hemispheroid centered in the origin of the coordinate system as shown in Fig. 1, with a ratio of minor to major semiaxes $b/a = f/N$ (slope of the order of 1% for typical atmospheric values). The bottom surface is defined as

$$h(x, y) = \begin{cases} 0 & \text{for } x^2 + y^2 > a^2 \\ b[1 - (x^2 + y^2)/a^2]^{1/2} & \text{for } x^2 + y^2 \leq a^2 \end{cases} \tag{3.1}$$

Consequently, once the coordinates and the streamfunction are non-dimensionalized as

$$\tilde{x} = x/a, \quad \tilde{y} = y/a, \quad \tilde{z} = z/b = zN/(af), \quad \tilde{\psi} = \psi/(U_{-\infty}a) \tag{3.2}$$

the differential eqn. (2.8) reduces to the Laplacian form:

$$\tilde{\nabla}^2\tilde{\psi} = 0 \tag{3.3}$$

The shape of the obstacle (3.1) in the space of non-dimensional coordinates, reduces to a sphere of unit radius

$$h(x, y) = h/b = \begin{cases} 0 & \text{for } x^2 + y^2 > 1 \\ [1 - (x^2 + y^2)]^{1/2} & \text{for } x^2 + y^2 \leq 1 \end{cases} \tag{3.4}$$

The conditions at infinity (2.7) and (2.15) give uniform potential temperature along the bottom boundary at $x = -\infty$. The conservation eqn. (2.13) permits the deduction that potential temperature has the same value everywhere along the bottom boundary only if all the streamlines are open. Potential temperature in regions where streamlines are closed cannot be deduced from boundary conditions at infinity. Huppert and Bryan (1976) give a physical interpretation of the occurrence, at the bottom surface, of regions in which the potential temperature is different from the one advected from infinity: the lower boundary becomes completely isentropic only if the kinetic energy of the upstream flow is large enough for the particles initially at $z = 0$ to reach the top of the obstacle, advecting their potential temperature from infinity. We will assume that this is the case in our model flow: specific entropy is held constant everywhere at the lower boundary. A similar assumption has been used by Merkine and Kalnay-Rivas (1976) in their numerical integration of semi-geostrophic equations for flow past isolated obstacles; their results can therefore be directly compared with ours. In non-dimensional form this boundary condition becomes (see 2.13):

$$\frac{\partial\tilde{\psi}}{\partial z} + (Ro^*)^{-1}\tilde{z} = 0 \quad \text{at } \tilde{z} = \tilde{h}(\tilde{x}, \tilde{y}) \tag{3.5}$$

It should be noted that

$$Ro^* = \frac{U_{-\infty}}{fa} = \frac{U_{-\infty}}{bN}$$

defined in terms of the far upstream velocity $U_{-\infty}$, does not necessarily coincide with the local Rossby number Ro.

Condition (2.15), written in non-dimensional form

$$\lim_{\tilde{r}\to+\infty} \tilde{\psi}' = 0 \quad \tilde{r}^2 = \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 \tag{3.6}$$

must be added to (3.3) and (3.5) for the mathematical problem to be well posed. The geometry of the obstacle obviously imposes the use of spherical coordinates

$$\left.\begin{aligned} \tilde{x} &= \tilde{r}\cos\theta\sin\phi \\ \tilde{y} &= \tilde{r}\sin\theta\sin\phi \\ \tilde{z} &= \tilde{r}\cos\phi \end{aligned}\right\} \tag{3.7}$$

in terms of which a general separated solution of the Laplace eqn. (3.3) satisfying condition (3.6) is ($m \leq n$):

$$\tilde{\psi}' = \sum_{0}^{+\infty}{}_{n,m} P_n^m(\cos\phi)\tilde{z}^{-(n+1)} \times (A_n^m \cos m\theta + B_n^m \sin m\theta) \tag{3.8}$$

where P_n^m are Legendre spherical functions. Boundary condition (3.5) applied at the spherical part of the bottom surface (3.4) reduces to

$$\frac{\partial\tilde{\psi}'}{\partial\tilde{z}} = -(Ro^*)^{-1}\cos\phi \quad \text{at } \tilde{r} = 1. \tag{3.9}$$

(3.9) and the property of spherical harmonics

$$\frac{\partial}{\partial z}(P_n^m\tilde{r}^{-(n+1)}) = -(n - m + 1)P_{n+1}^m\tilde{r}^{-(n+2)} \tag{3.10}$$

permits the deduction that

$$A_0^0 = Ro^{-1}, \quad A_n^m = B_n^m = 0 \quad \text{for } m, n \neq 0 \tag{3.11}$$

and therefore

$$\tilde{\psi}' = (Ro^*)^{-1}\tilde{r}^{-1} \tag{3.12}$$

Solution (3.12) also satisfies the boundary condition (3.5) at the flat part of the bottom surface:

$$\frac{\partial \tilde{\psi}'}{\partial \tilde{z}} = -(Ro^*)^{-1}\cos\phi\tilde{r}^{-2} = 0 \quad \text{at } \phi = \pi/2 \tag{3.13}$$

The field of horizontal streamfunction for the mountain-induced perturbation can be written in terms of non-dimensional Cartesian coordinates as

$$\tilde{\psi}' = (Ro^*)^{-1}(\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2)^{-1/2} \tag{3.14}$$

The field of horizontal velocity can be calculated from (3.14) by means of (2.3). However, the geostrophic streamfunction can equivalently be defined on the isentropic surfaces characterized by $\partial\tilde{\psi}/\partial\tilde{z} + (Ro^*)^{-1}\tilde{z} =$ constant. The so-called Montgomery streamfunction is therefore obtained from (3.14) by transforming the vertical coordinate from $\tilde{z}$ to $\partial\tilde{\psi}/\partial\tilde{z} + (Ro^*)^{-1}\tilde{z}$. Fig. 2 shows the Montgomery streamfunction along the bottom isentropic surface for $Ro^* = 0.1$; the streamlines represent the tridimensional motion along the lower boundary. The solution appears to be qualitatively very similar to those obtained analytically for the case of a small amplitude mountain, and numerically for the finite mountain, when the ground is assumed to be isentropic. Both the amplitude factor and the asymptotic behavior of the induced disturbance (decaying far from the obstacle as $\tilde{r}^{-1}$) are the same as in the solution for small isolated obstacles (see, for comparison, Huppert, 1975; Buzzi and Tibaldi, 1977). The amplitude factor, in particular, is $(Ro^*)^{-1} = fa/U_{-\infty} = bN/U_{-\infty}$. For values greater than unity, closed streamlines appear over the obstacle, forming slightly eccentrical, conical regions of separated circulation, called in the literature "stratified inertial Taylor columns". The eccentricity of order Ro provides an upslope component of velocity on the upwind side and a downslope component in the lee, also in the case of closed circulation, where particles which are at the flat part of the bottom boundary cannot reach the top of the obstacle.

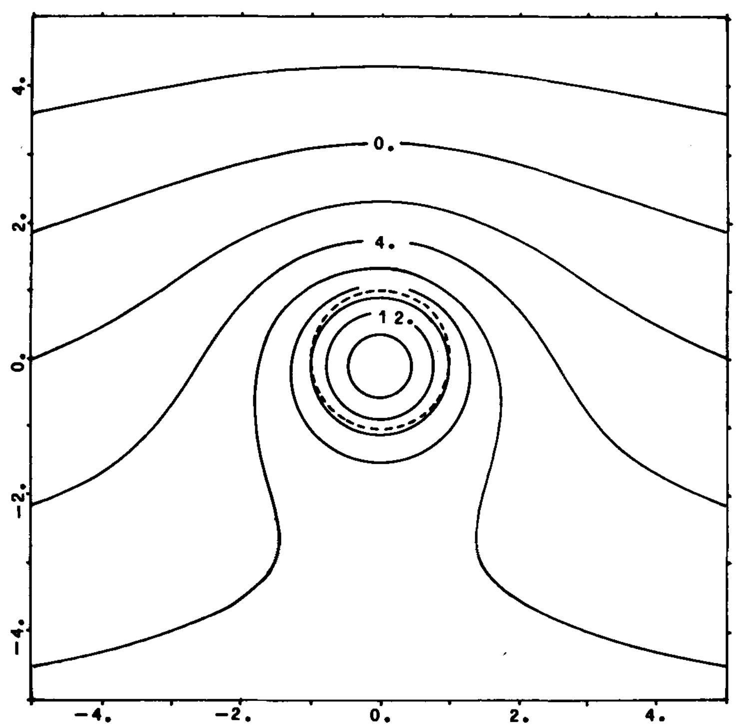

Fig. 2. Non-dimensional Montgomery streamfunction on the lower boundary surface (see text). The coordinates are non-dimensional. The dashed line indicates the contour of the obstacle. $Ro^* = 0.1$.

Anticyclonic circulation around the obstacle is present even for values of $(Ro^*)^{-1}$ that are too small for closed circulation to occur. This is obviously associated with the compression of vortex tubes over the obstacle where potential vorticity is still equal to zero, but static stability is larger than elsewhere.

Deflection of horizontal trajectories is entirely due to the superposition on the basic uniform flow of this anticyclonic vortex bounded to the mountain; no real transfer of momentum from flow "across" to flow "along" the obstacle actually takes place. As a consequence, the mountain-induced vortex is independent, in this stationary solution, of the far upstream velocity $U_{-\infty}$. Once the ground has become isentropic, the upstream velocity can decrease to zero, the anticyclonic circulation remaining unchanged.

Once the horizontal streamfunction is known, one can evaluate the vertical velocity at the surface of the obstacle:

$$w = V_H \cdot \nabla_H h = -J_H(\psi, h) \tag{3.15}$$

Both the mountain-induced part of the streamfunction field, indicated by ψ', and the mountain itself $h(x,y)$ are axially symmetric functions with respect to the z-axis. It follows that $J_H(\psi, h) = 0$ and the only contribution to the vertical velocity is given by the basic far upstream horizontal velocity $U_{-\infty}$ multiplied by $\partial h/\partial x$. Therefore the vertical velocity is the only quantity, in the mountain-induced perturbation, that depends on the far upstream velocity, and its order of magnitude is

$$W \sim U_{-\infty} b/a = f/N\, U_{-\infty} \tag{3.16}$$

Inserting (3.16) into (2.18), which assures the quasi-horizontality of the motion, we obtain:

$$U_{-\infty} f/UN \lesssim U/NL \tag{3.17}$$

where U is a typical local value of the total horizontal velocity. Remembering that $\tilde{\psi} = \psi/(U_{-\infty} a)$, it can be seen from (2.3) and (3.14), that the order of magnitude of the mountain-induced horizontal velocity is fa, or fL, if one identifies the horizontal scale of motion L with the typical width of the obstacle a. If $Ro^* \lesssim 1$, as in the example shown, we can deduce that $U \sim fL$. Criterion (3.17) becomes:

$$U_{-\infty}/fL = Ro^* \lesssim 1 \tag{3.18}$$

Since closed streamlines appear for $Ro^* \lesssim 1$, Taylor column formation is consistent with the requirement of quasi-horizontality of motion.

The limits imposed on our solution by the quasi-geostrophic approximation are discussed in the paper by Merkine and Kalnay-Rivas (1976). Their solutions take into account the advection of geostrophic momentum by the non-geostrophic velocity component. The resulting non-linear interaction between the far field velocity and the mountain-induced vortex is not described in the quasi-geostrophic model, in which a weaker u-component of velocity over the obstacle is observed compared to the semi-geostrophic model. However, the physical properties discussed above are common to both solutions, and quantitative agreement is otherwise good at non-dimensional distances from the center of the obstacle greater than 1.

It was pointed out above that the amplitude factor and the decay at large distances from the obstacle are the same for small and finite amplitude axisymmetric obstacles. In particular, it is easy to show that the mountain-induced disturbance in dimensional form is proportional, in both cases, to the volume of the obstacle. Therefore, the disturbance vanishes when the volume of the obstacle tends to zero, even if its height remains finite. This is due to the fact that the volume of an axially symmetric obstacle cannot be reduced to zero while keeping the cross section, with respect to the incident flow, finite. In the next section we study the case of an obstacle which is not axisymmetric; it will appear that, in this case, provided the cross section of the obstacle remains finite with respect to the incident flow, the induced disturbance does not vanish even for zero volume.

4. Flow past a steep mountain

The extreme case of an obstacle of small volume, but finite cross section, is that of a portion of a vertical wall of infinitesimal width. As already mentioned, the condition of impenetrability of a vertical wall by a quasi-geostrophic flow reduces to the very simple requirement that the horizontal streamfunction depends only on the vertical coordinate z along the wall itself. Consequently $\partial\psi/\partial z$ is also independent of the horizontal coordinate along the wall. Equation (2.2) for the stationary case indicates that the vertical velocity vanishes there. We assume that the obstacle is an ellipsoidal disk perpendicular to the basic flow. It corresponds to the intersection of the hemispheroid, considered in the previous section, with the plane yz (see Fig. 1). The ratio of minor (vertical) to major (horizontal) semiaxis is again $b/a = f/N$.

The same procedure used in Section 3 leads to the following boundary problem in the mountain-induced perturbation:

$$\tilde{\nabla}^2\tilde{\psi}' = 0 \tag{4.1}$$

$$\lim_{r\to+\infty} \tilde{\psi}' = 0 \tag{4.2}$$

$$\frac{\partial\tilde{\psi}'}{\partial\tilde{z}} = f(\tilde{\psi}') \quad \text{at } \tilde{z} = 0 \tag{4.3}$$

The intrinsic indeterminacy of the problem appears here explicitly in the arbitrariness of the stream-function $\psi(z)$ at the disk's surface. In order to specify this function, we will assume that stratification is constant along the disk; that is,

$$\frac{\partial\tilde{\psi}'}{\partial\tilde{z}} + s(Ro^*)^{-1}\tilde{z} = 0 \quad \text{at the disk's surface} \tag{4.4}$$

where s is a "stratification parameter" ranging between 0 and 1, and Ro^* is defined as in the previous section. The physical meaning of this condition is made clear by considering the two extreme cases $s = 0$ and $s = 1$. When the stratification parameter vanishes, the distribution of potential temperature at the surface of the disk (but not elsewhere) is that of the unperturbed atmosphere. In the opposite extreme of the stratification parameter reaching unity, the surface of the disk is isentropic, in analogy with the problem analyzed in the previous section.

The boundary problem (4.1)–(4.4) can be easily solved in the oblate spheroidal coordinate system defined by

$$\left.\begin{aligned} \tilde{x} &= \xi\eta \\ \tilde{y} &= [(1+\xi^2)(1-\eta^2)]^{1/2} \sin\phi \\ \tilde{z} &= [(1+\xi^2)(1-\eta^2)]^{1/2} \cos\phi \end{aligned}\right\} \tag{4.5}$$

in the ranges

$$-1 \le \eta \le +1, \quad 0 \le \eta \le +\infty,$$
$$-\pi/2 \le \phi \le \pi/2 \tag{4.6}$$

The ξ-const surfaces are oblate spheroids; the spheroid $\xi = 0$ is the surface of a disk of unit radius centered at the origin of the coordinate system. A general separated solution of the Laplace eqn. (4.1) in the coordinate system (4.5) has the form ($m \le n$)

$$\tilde{\psi}' = \sum_{0}^{+\infty}{}_{n,m} P_n^m(\eta) Q_n^m(i\xi)(A_n^m \sin m\phi + B_n^m \cos m\phi) \tag{4.7}$$

where P_n^m and Q_n^m are Legendre functions of the first and second kind, respectively. Boundary condition (4.4) can be integrated once and transformed into the oblate spheroidal system (4.5) to give

$$\tilde{\psi}'(0,\eta,\phi) = -0.5s(Ro^*)^{-1}(1-\eta^2)\cos^2\phi + (1-\eta^2)^{1/2}\sin\phi + k \tag{4.8}$$

where k is an integration constant.

The general solution (4.7) together with (4.8) determines the solution for the mountain-induced perturbation:

$$\begin{aligned}\tilde{\psi}' = {} & \frac{2k}{\pi}\operatorname{arctg}\frac{1}{\xi} - \frac{2}{\pi}\left[\frac{\xi}{(1+\xi^2)^{1/2}} - (1+\xi^2)^{1/2}\operatorname{arctg}\frac{1}{\xi}\right](1-\eta^2)^{1/2}\sin\phi \\ & + \frac{s}{6\pi Ro^*}\left\{-2\operatorname{arctg}\frac{1}{\xi} + \left[(1+3\xi^2)\operatorname{arctg}\frac{1}{\xi} - 3\xi\right]\cdot\right. \\ & \cdot(3\eta^2-1) - \left[3(1+\xi^2)\operatorname{arctg}\frac{1}{\xi} - \xi\frac{3\xi^2+5}{1+\xi^2}\right]\cdot \\ & \left.\cdot(1-\eta^2)\cos 2\phi\right\}\end{aligned} \tag{4.9}$$

It can be verified that $\partial\tilde{\psi}/\partial\tilde{z} = 0$ at $\tilde{z} = 0$, so that condition (4.3) is satisfied. The form of the solution (4.9) is rather involved, but it can be physically interpreted as follows:

(i) The first term, multiplied by the constant k, represents a circulation around the obstacle. This circulation is not uniquely determined by the boundary conditions, even if the stratification along the disk is specified. This indeterminacy is very similar to that appearing in the theory of two-dimensional, non-rotating flow past aerofoils. The appearance of this kind of indeterminacy is associated with discontinuities in some properties of the flow (in our case $\tilde{\psi}'$ is discontinuous along the edge of the disk). It is natural to follow the procedure, classical in applied mathematics (Morse and Feshbach, 1953) of choosing that value of k which eliminates the discontinuity. It seems reasonable to remove the singularity in $\partial\tilde{\psi}'/\partial\tilde{z}$ (potential temperature) at the top edge of the disk. This gives the value $k = 7s/6Ro^*$ that is used in the solutions plotted.

(ii) The second term in (4.9), which does not depend on s, is the one that permits satisfaction of the impenetrability condition. It gives, in fact, a v-component of velocity which cancels out the same component of the basic flow along the disk. The resulting flow is shown, at $\tilde{z} = 0$, in Fig. 3. This is also the full solution in the case $s = 0$ (with the previous choice for k), i.e. when the stratification along the obstacle is the same as far upstream. The streamlines are similar, in each horizontal plane cutting the disk, to those for the non-rotating two-dimensional flow past a flat plate, with no circulation. This is not surprising since the equation of motion and the boundary conditions are similar in the two cases. From a physical point of view, the vertical barrier acts on the flow in two ways, horizontally deflecting the trajectories as in the non-rotating case, and, possibly, producing a circulation related to the uplifting of the isentropic surfaces (see iii).

(iii) For non-zero values of s the velocity field develops asymmetries with respect to the plane $\tilde{x}\tilde{z}$. Values of s between 0 and 1 describe an uplifting of isentropic surfaces near the disk with ascending motion on the upwind side and descending in the lee, in analogy with the case described in the previous section. In the present case, on the other hand, vertical motion takes place in a region near the obstacle but not along its vertical wall, where vertical velocity vanishes. Fig. 4 shows the streamlines at $\tilde{z} = 0$ for $s/Ro^* = 1$.

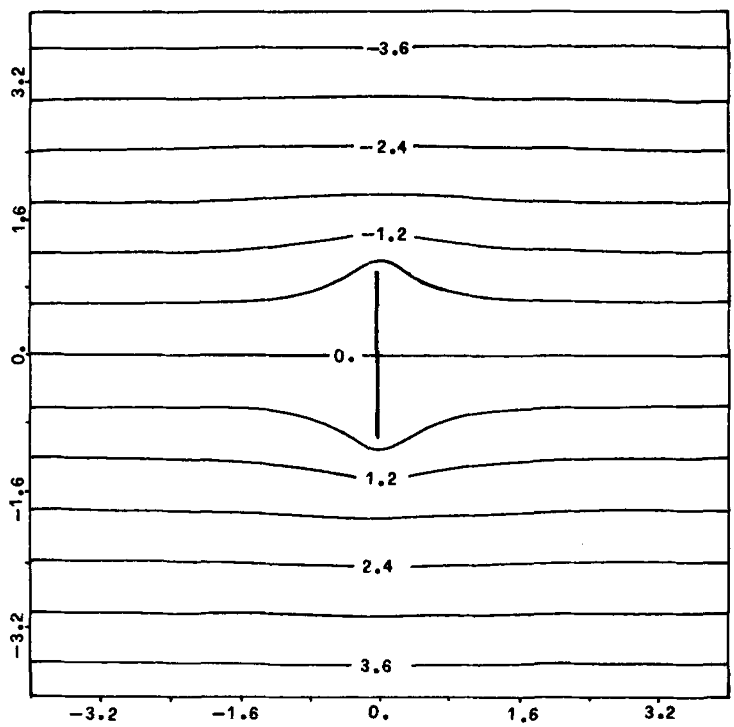

Fig. 3. Non-dimensional horizontal streamfunction at $\tilde{z} = 0$. The coordinates are non-dimensional. The thick line indicates the contour of the obstacle. $s = 0$.

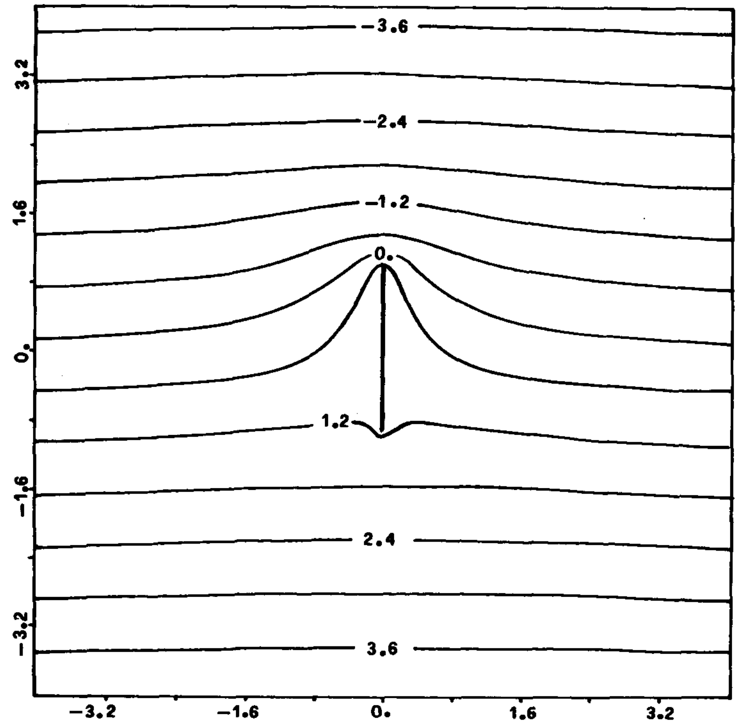

Fig. 4. Same as Fig. 3 for $s = Ro^*$.

It is now useful to compare the two solutions obtained for the two different types of mountain. The third term in (4.9) can be considered analogous to that of the solution obtained for the hemisphere, in the sense that it depends primarily on the entropy distribution along the obstacle. Unlike the case of the sphere, in the case of the disk it is possible to obtain a continuous class of solutions depending on the potential temperature distribution along the obstacle. If the value of k in the first term of (4.9) is chosen in the way previously indicated, the two terms together produce a pressure ridge over the obstacle for positive values of s, much in the same way as for the spheroid. The asymptotic behavior is also similar, since the first and third terms in (4.9) decay, far from the obstacle, as $\tilde{r}^{-1}$.

The most important difference between the two cases is the presence, for very steep reliefs, of a diffluent flow around the obstacle that is independent of the stratification along the obstacle itself. This part of the mountain-induced perturbation, described in (ii), counteracts the far upstream velocity $U_{-\infty}$ all along the obstacle. An important consequence is, in this case, the fluid flows around the obstacle without implying the occurrence of closed streamlines. Taylor column formation is therefore not necessary, in this case, to keep the flow quasi-horizontal. This component of the flow field decays as $\tilde{r}^{-2}$, more rapidly, therefore, than the other components. This means that at distances from the obstacle greater than some units in non-dimensional coordinates, the anticyclonic perturbation due to the vertical motion induced by the mountain is the dominant effect.

As mentioned previously, the total disturbance produced by a small amplitude obstacle in a rotating and stratified fluid is proportional to the volume of the obstacle itself. The same conclusion has been extended by Merkine and Kalnay-Rivas (1976) to the finite amplitude case, and is also valid for our solution of flow past hemisphere. However, as anticipated at the end of the previous section, the case analyzed here contradicts this conclusion if applied to finite height obstacles of very small volume, but of finite cross section with respect to the incident flow. In this case the disturbance remains finite for any distribution of potential temperature along the obstacle walls. Particles, in this case, cannot climb the vertical walls of the obstacle (unless they acquire an infinite vertical velocity) and must therefore be deviated around it. Practical consequences of this result are discussed in the conclusions.

There are limits to the validity of this solution. Near the edge of the disk, because of the presence of a singularity of the ξ-derivative of $\tilde{\psi}$, high horizontal velocities coupled with strong curvature produce large local Rossby numbers. This fact is directly related to the singular geometry of the obstacle (infinitesimal width); it seems nevertheless of secondary importance since the fields are well behaved on a large scale.

The solution for this second case appears realistic for atmospheric flows past steep mountain ranges of limited longitudinal extension (Alps, Pyrenees, Caucasus, etc.). In this case the flow seems to be essentially diffluent around them, with some amount of anticyclonic vorticity due to partial flow over (see, for example, the trajectories near the Alps shown in Buzzi and Tibaldi, 1978), but without closed streamline formation of the Taylor column type.

5. Conclusions

The analytical treatment of flow past finite amplitude obstacles, even if limited to some special cases, shows the limit of theories in which the lower boundary condition is applied at $z = 0$ in dealing with the perturbations induced by steep obstacles. The solution we have obtained for the problem of flow past a hemispheroid basically confirms the results of previous studies: the obstacle acts on the quasi-geostrophic flow essentially by increasing the local stratification. This results in a vortex tube shrinking, producing anticyclonic circulation and, possibly, closed streamlines. In the case of the disk-shaped obstacle across the basic current, the flow is deviated around the obstacle without necessarily inducing a local strong perturbation of the stratification field. Closed streamlines do not necessarily occur with obstacles of this shape. The analysis in this case suggests that the dynamical influence of very steep obstacles on the atmosphere goes much beyond their geometrical extent. The horizontal averaging process generally adopted in order to represent the earth topography in numerical models leads, in the case of tall and narrow mountain complexes, to a very crude simulation of the orographic forcing. The same would not happen if the disturbance were always proportional to the volume of the mountain, as seems to be the case for smooth reliefs. The simulation of a steep obstacle with a vertical wall seems more realistic in some respects. The use of this kind of representation has already been successfully attempted (Egger, 1972). Further study should be devoted to the problem of representing steep mountains, in numerical models, in terms of an "equivalent wall" and an "equivalent lifting of isentropic surfaces". It must be remembered that in both the cases of smooth and steep mountains a basic part of the physics of interaction of geophysical flows with topography is to be found in the specific entropy evolution in the proximity of the surface of the obstacle. Unfortunately this is, in general, not properly described by models of the kind discussed in this paper and, in particular, undetermined in the stationary problem. A realistic description of the evolution of the boundary layer seems, therefore, to be a necessary requisite for future studies of simulation of flow past topographic reliefs.

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СТАЦИОНАРНОЕ КВАЗИГЕОСТРОФИЧЕСКОЕ ТЕЧЕНИЕ СТРАТИФИЦИРОВАННОЙ АТМОСФЕРЫ ЗА ПРЕПЯТСТВИ ЕМ КОНЕЧНОГО РАЗМЕРА

На *f*-плоскости изучается стационарное квазигеострофическое течение атмосферы, однородной по ветру и энтропийной стратификации, вдали от препятствия конечного размера на подстилающей поверхности. Обсуждаются аналитические решения для двух случаев гор с конечным склоном (полусфера) и бесконечным склоном (диск). Препятствия действуют по-разному в этих случаях на поток. В случае конечного наклона препятствие влияет на модель атмосферы путем поднятия изэнтропических поверхностей вблизи нижней границы. Последующее сжатие вихревых трубок дает антициклоническую завихренность над препятствием и, возможно, колонну Тэйлора-в согласии с известными в литературе результатами.

В случае горы с бесконечным склоном вблизи препятствия наблюдается заметное растекание горизонтальных линий тока. Чтобы задача в этом случае была полностью определена, нужно задать полную циркуляцию, снова связанную с поднятием изэнтропических поверхностей. Однако полное возмущение не исчезает, даже если объем препятствия стремится к нулю при условии, что его поперечное сечение остается конечным. Обсуждаются следствия для численного моделирования.