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The effect of the Helmholtz term on a two-layer model baroclinic atmosphere

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ABSTRACT

The effects of the inclusion of a Helmholtz term in a quasi-geostrophic model, formulated as part of a lower boundary condition, are investigated using a two-layer model. The Helmholtz term has two main effects, both of which appear desirable in a quasi-geostrophic model. As well as reducing the phase speed of all waves, and thereby controlling the otherwise large retrograde motion of ultra-long Rossby waves, it reduces growth rates of unstable waves. The Helmholtz term reduces the westward slope with height of unstable waves, more so for shorter wave lengths.

The Helmholtz term is shown to be important for all quasi-geostrophic models. However, its use with an enhanced factor of about 10, although desirable for "best" forecasts, has little physical basis and merely serves to offset shortcomings of the model.

1. Introduction

Errors in forecasting ultra-long waves have been a problem ever since the introduction of numerical weather prediction techniques. In quasi-geostrophic models, part of the source of poor performance can be traced to the scaling assumptions, since these waves are governed by a different set of equations (Burger, 1958; Phillips, 1963). In early barotropic models, the problem was corrected by introducing "barotropic divergence" (Cressman, 1958) and changing the formulation to that of the equivalent barotropic model. This introduces an extra term, the so-called Helmholtz term, which changes the equation to be solved from one of Poisson type to Helmholtz type. (For a review, see Haltiner, 1971; Section 6.1). The effect of this term is to substantially reduce the phase speed of the ultra-long waves.

Results from numerical weather prediction using a 6-level quasi-geostrophic model in New Zealand (Trenberth, 1973; Trenberth and Neale, 1977) revealed problems with the very long waves similar to those encountered in the barotropic model. A Helmholtz term was included in the model's lowest layer, formulated as part of the lower boundary condition, and the model showed extreme sensitivity to variations in its amplitude. It has been used to control errors of several decameters occurring at all levels and covering most of the grid (about one-third of the Southern Hemisphere). Early prediction experiments also showed a marked tendency for the model to overdevelop systems (Trenberth, 1973). This was initially controlled by introducing surface friction and diffusion into the model (Trenberth, 1973) and smoothing the analyses (Trenberth, 1977), as well as incorporating the Helmholtz term. However, further experiments have shown that a considerably enhanced Helmholtz term—by a factor of about 10—is desirable. It is interesting that this is also approximately the factor required to produce optimum forecasts for the equivalent barotropic model (Leith, 1974).

The Helmholtz term arises by changing the lower boundary condition from one of vanishing material derivative of pressure ($\omega = 0$) to one of zero vertical velocity ($w = 0$) on a constant pressure surface. It can be easily introduced into a baroclinic model by making an equivalent barotropic approximation for the lowest layer. The purpose of this paper is to investigate the effects of inserting this term with an enhanced coefficient, and we show, using a two-layer model, that it reduces the phase speed of all waves and also reduces growth rates of unstable waves. Both features appear to be desirable in a quasi-geostrophic model.

The Helmholtz term has been shown to be very important in tidal theory (Lindzen, 1968; Geisler and Dickinson, 1975; White, 1978) and the latter paper refers to it as the "stratification term". Lindzen showed that its presence gives rise to a "non-Doppler" effect on the phase-speed of waves. Earlier, Wiin-Nielsen (1971) investigated the effects of the same change in lower boundary condition on baroclinic instability. All these studies emphasize that without the more correct lower boundary condition there is a considerable over-estimate of the retrograde phase speed of long Rossby waves.

The approach we use here to investigate the effects of the Helmholtz term on a baroclinic atmosphere is to analyse the baroclinic instability problem of the two-layer model but incorporating the Helmholtz term. A method along the lines presented in standard texts (Holton, 1972, pp. 133 and 186; Haltiner, 1971, p. 131) will be used, and in order to shorten and simplify the discussion, for the most part the notation of Holton will be used.

In practice, surface friction, other boundary layer effects, orography, and diabatic heating should be included in the equations since they are mainly responsible for forcing the ultra-long stationary waves. However, these effects are included to some degree in the 6-level New Zealand model, and for simplicity we omit them here in order to determine the sensitivity of the quasi-geostrophic model to the changed lower boundary condition.

2. The two-layer model atmosphere

The thermodynamic equation applied at level 2 (500 mb) is combined with the vorticity equations applied at levels 3 (750 mb) and 1 (250 mb) to give to potential vorticity equations

$$\frac{\partial q_1}{\partial t} = -J(\psi_1, q_1) \tag{1}$$

$$\frac{\partial q_3^*}{\partial t} = -J(\psi_3, q_3^*) + \frac{f_0\omega_4}{\Delta p} \tag{2}$$

where

$$q_1 = \nabla^2\psi_1 + f - \lambda^2(\psi_1 - \psi_3) \tag{3}$$

$$q_3^* = \nabla^2\psi_3 + f + \lambda^2(\psi_1 - \psi_3) \tag{4}$$

are the potential vorticities, J is the Jacobian and $\lambda^2 = f_0^2/\sigma\Delta p^2$.

In order to eliminate ω_4 without introducing further unknowns, the equivalent barotropic approximation applied only to the lower layer is introduced, and we put $\psi_4 = \alpha'\psi_3$, where α' is a constant. Then at the lower boundary

$$\omega_4 = f_0\rho_4\alpha' \frac{\partial\psi_3}{\partial t} \tag{5}$$

When substituted into (2), this becomes part of the Helmholtz term in the prognostic equations, in analogy to the equivalent barotropic model (Trenberth, 1973). We may then rewrite (2) and (4) as

$$\frac{\partial q_3}{\partial t} = -J(\psi_3, q_3) \tag{6}$$

where

$$q_3 = \nabla^2\psi_3 + f + \lambda^2(\psi_1 - \psi_3) - 2\alpha^2\psi_3 \tag{7}$$

are the re-defined potential vorticity equations for 750 mg, and $\alpha^2 = f_0^2\rho_4\alpha'/2\Delta p$.

We then perform a perturbation analysis on a β-plane of (1), (3), (6) and (7), following Holton (p. 187), by putting

$$\begin{aligned} \psi_1 &= -U_1 y + \psi_1'(x, t) \\ \psi_3 &= -U_3 y + \psi_3'(x, t) \end{aligned} \tag{8}$$

where U_1 and U_3 are the zonal velocities at levels 1 and 3 respectively, and assume sinusoidal solutions for the perturbations

$$\psi_1' = A\, e^{ik(x-ct)}, \quad \psi_3' = B\, e^{ik(x-ct)} \tag{9}$$

Adding and subtracting the resulting equations and putting

$$\begin{aligned} &\Psi = \tfrac{1}{2}(A + B), \quad \tau = \tfrac{1}{2}(A - B) \\ &\text{and} \\ &U = \tfrac{1}{2}(U_1 + U_3), \quad W = \tfrac{1}{2}(U_1 - U_3) \end{aligned} \tag{10}$$

so that U and W are the vertically averaged zonal wind and the basic state thermal wind for the interval $\Delta p/2$, we get

$$[(U-c)(k^2+\alpha^2)-\beta-\alpha^2 U]\Psi - [(U-c)\alpha^2 - U\alpha^2 - k^2 W]\tau = 0 \tag{11}$$

$$[(U-c)(k^2+2\lambda^2+\alpha^2 U]\tau - [(U-c)\alpha^2 - U\alpha^2 - (k^2-2\lambda^2)W]\Psi = 0 \tag{12}$$

Non-trivial solutions exist only if the determinant of the coefficients of Ψ and τ is zero, so that $(U-c)$ must satisfy

$$\begin{aligned}(U-c)^2&[k^2(k^2+2\lambda^2)+2\alpha^2(k^2+\lambda^2)] \\ &-2(U-c)[\beta(k^2+\lambda^2)+\alpha^2(\beta-(k^2-\lambda^2)W \\ &+[k^2+\lambda^2)U)]+[\beta^2-k^2(k^2-2\lambda^2)W^2 \\ &+2\alpha^2(\beta-(k^2-\lambda^2)W)U]=0\end{aligned} \tag{13}$$

This has a solution

$$c = U - \frac{\beta(k^2+\lambda^2)+\alpha^2[\beta-(k^2-\lambda^2)W+(k^2+\lambda^2)U]\pm\delta^{1/2}}{[k^2(k^2+2\lambda^2)+2\alpha^2(k^2+\lambda^2)]} \tag{14}$$

where, after some manipulation,

$$\begin{aligned}\delta \equiv &\{k^4 W + \alpha^2[(k^2+\lambda^2)U+(k^2-\lambda^2)W-\beta]\}^2 \\ &+\lambda^4[\beta+2k^2W+2\alpha^2(W+U)][\beta-2k^2W]\end{aligned} \tag{15}$$

If the Helmholtz term is omitted, we set $\alpha = 0$ in (14) and (15) and the solution reduces to (10.19) of Holton (p. 188).

Instability occurs if δ is negative, so that we put $c = c_r + ic_i$ where c_r, c_i are the real and imaginary parts of c. The phase velocity is given by c_r, or the Doppler-shifted phase velocity by $c_r - U$, and the growth rates of the disturbance given by kc_i. Clearly a necessary, but not sufficient, condition for instability, which also applies when $\alpha = 0$, is $W > \beta/2k^2$.

The addition of the Helmholtz term has complicated the analytic solution and results are therefore presented for typical mid-latitude cases. Note, however, that whereas $c_r - U$ and c_i are independent of U for $\alpha = 0$, this is not the case with the Helmholtz term included.

3. Results

Solutions will be presented for the following set of parameters, typical of mid-latitudes.

$$\begin{aligned}f_0 &= 1.03 \times 10^{-4}\ \mathrm{s^{-1}} \\ \beta &= 1.618 \times 10^{-11}\ \mathrm{m^{-1}\,s^{-1}} \\ \sigma &= 3.05\ \mathrm{m^3\,mb^{-1}\,kg^{-1}} \\ \lambda^2 &= 1.39 \times 10^{-12}\ \mathrm{m^{-2}} \\ \alpha^2 &= 1.282 \times 10^{-13}\ \alpha'\ \mathrm{m^{-2}}\end{aligned}$$

The parameter α' is of order unity in this formulation but will be varied to illustrate the effects of changes in the Helmholtz term. Results will be presented for different values of U and W. For reference purposes the solution where there is no Helmholtz term, $\alpha' = 0$, is presented in Fig. 1. We also present a solution for $\alpha' = 10$. This is approximately the optimum value for the equivalent barotropic formulation (Leith, 1974) and for the 6-level quasi-geostrophic model.

Fig. 1a shows the growth rates for $\alpha' = 0$. A formula for the neutral stability curve is given by Holton (p. 191) who also shows that no growth occurs for $L < 2\pi\sqrt{2}\lambda$ (~3768 km) and the minimum W for instability is $W = \beta/2\lambda^2$(~5.82 m s^{-1}). The most unstable wavelength for a given W is also indicated.

Figs. 1b and 1c show the two solutions for $(U - c_r)$. Only one solution exists for unstable waves, and this is not a function of W. Fig. 1b reveals the large retrograde movement of the long waves associated with the Rossby wave formula.

When α' is non-zero, the solutions become a function of U and, rather than presenting results for U = constant, in which case the vertical wind profile would be unrealistic for some values of U and W (recall $U_3 = U - W$, $U_1 = U + W$), we present a solution for constant $U_3 = 5$ m s^{-1}. Then $U = 5 + W$, and westerlies are present throughout the model atmosphere.

Fig. 2a shows the growth rates for $\alpha' = 10$, and comparison with Fig. 1a reveals a reduction everywhere, but most pronounced for shorter waves. This shifts the most unstable wave to longer wavelengths. Figs. 2b and 2c, to be compared with Figs. 1b and 1c, show the solutions for $U - c_r$ for $\alpha' = 10$. Most noticeable is the substantial reduction in the retrograde speed of the very long waves in Fig. 2b. Also the phase speed

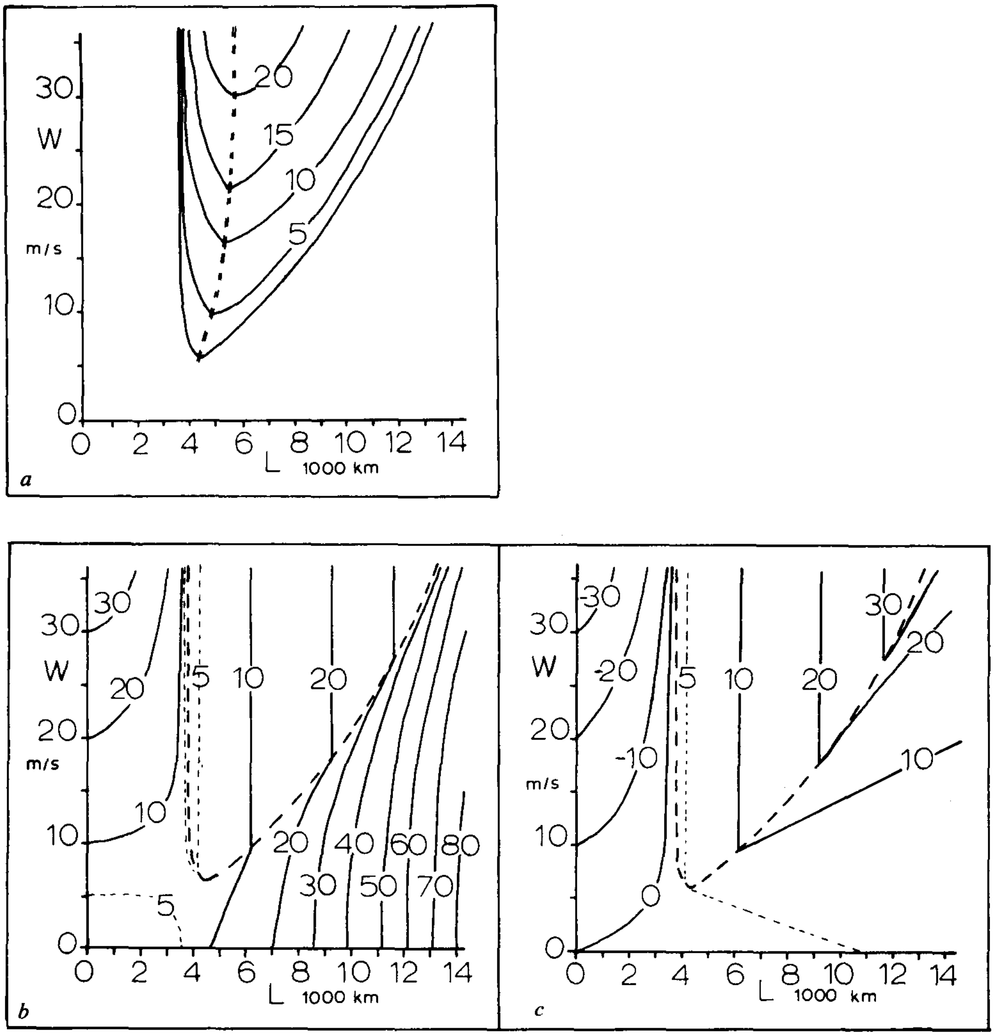

Fig. 1. Solution for $\alpha' = 0$. (a) Growth rates (kc_i) in 10^{-6} s^{-1}. Maximum growth is shown by the double dashed line. (b) and (c) Two solutions for $(U - c_r)$ in m s^{-1} as a function of wavelength L and thermal wind W. The neutral stability curve is given by heavy dashed line.

of the unstable waves now depends on both U and W.

The changes in phase speeds through the Helmholtz term can be better seen by presenting this information in a different fashion since, in Fig. 2, $(U - c_r)$ varies because U varies with W as well as from changes in c_r. Fig. 3 presents the same information as in Figs. 1b, 1c and 2b, 2c but showing c_r, assuming $U = 5 + W$. Immediately it becomes clear for $\alpha' > 0$ that c_r is reduced for all wavelengths in both solutions. The points where $c_r = 0$ (Figs. 3a, 3c) are independent of α' and occur where

$$U^2 k^2(k^2 + 2\lambda^2) - 4U\beta(k^2 + \lambda^2) + \beta^2 - W^2k^2(k^2 - 2\lambda^2) = 0 \tag{16}$$

The reduction in phase speed is monotonic and growth rates are also generally reduced with increasing α', but greatest for an increase in α' from 0 to 1. Table 1 shows this for $W = 20$ m s^{-1}, $U = 25$ m s^{-1} for wavelengths of 6000 and 14,000 km. There can be a very small increase in

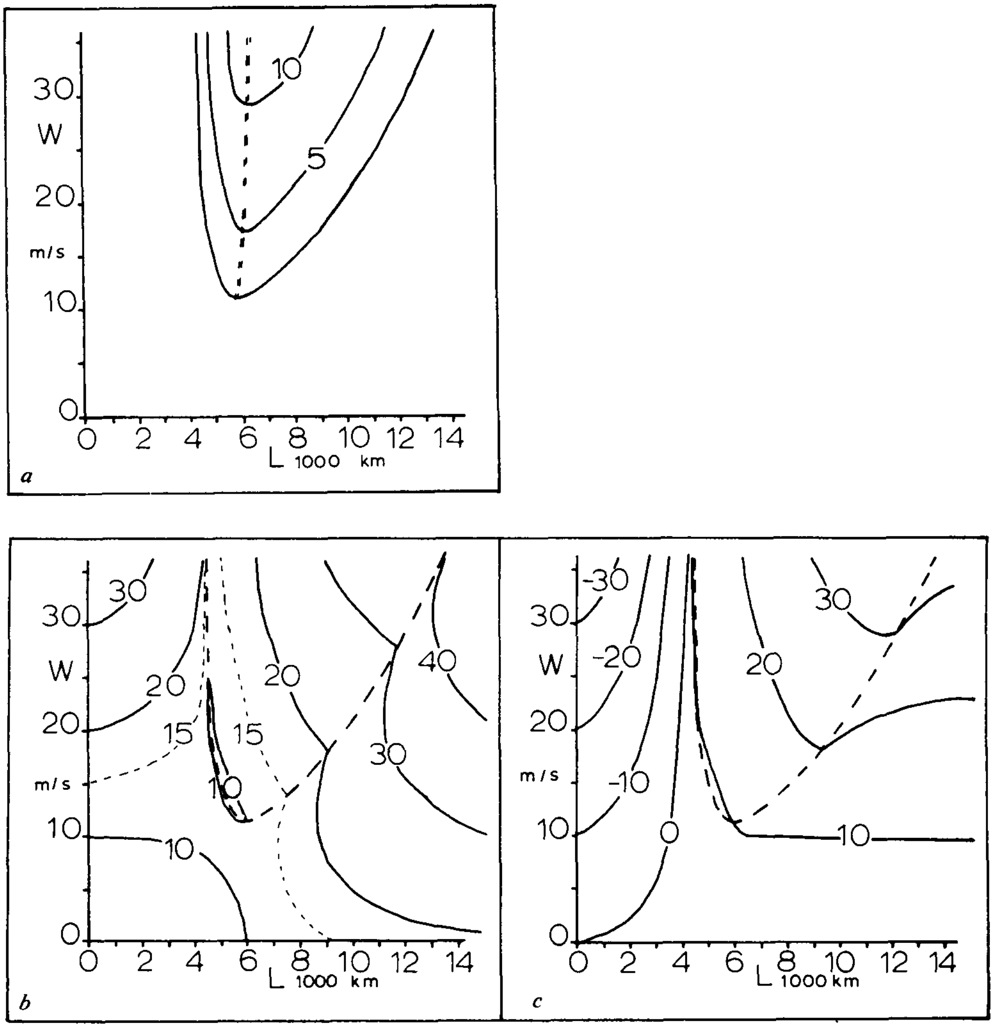

Fig. 2. As for Fig. 1, but for $\alpha' = 10$ and using a basic state where $U = 5 + W$.

growth rate for $\alpha' = 1$ (Wiin-Nielsen, 1971) but it affects only a very limited range of the longest unstable waves and is insignificant in the two-layer model.

It is of interest to consider the change in growth rates for a given α' as a function of U. Since vertical motions at the lower boundary are related to the surface tendency [see eq. (5)] a change in the speed of advection of the entire system will change ω_4. For instance, a change in c_r from westward to eastward movement can be induced by increasing U, but would cause a reversal in ω_4 so that upward motion changes from west to east of the trough. Intuitively therefore, we might expect greatest growth rates to occur for that value of U which reduces $c_r \approx 0$, and therefore approximately minimizes the effect of the Helmholtz term. This proves to be the case. Greatest growth rates as a function of U occur where

$$c_r = [Wk^2(\lambda^2 - k^2)(2\lambda^2 + k^2) - \lambda^4\beta)F(\alpha^2)$$

and $F(\alpha^2)$ is a function of α^2.

For unstable waves $\bar{k} \sim \lambda(L = 5330$ km) and $c_r \sim 0$. For $W = 20$ m s^{-1} and $\alpha' = 10$, maximum growth occurs for $U = 8.1$ m s^{-1} and $c_r = -0.8$ m s^{-1}.

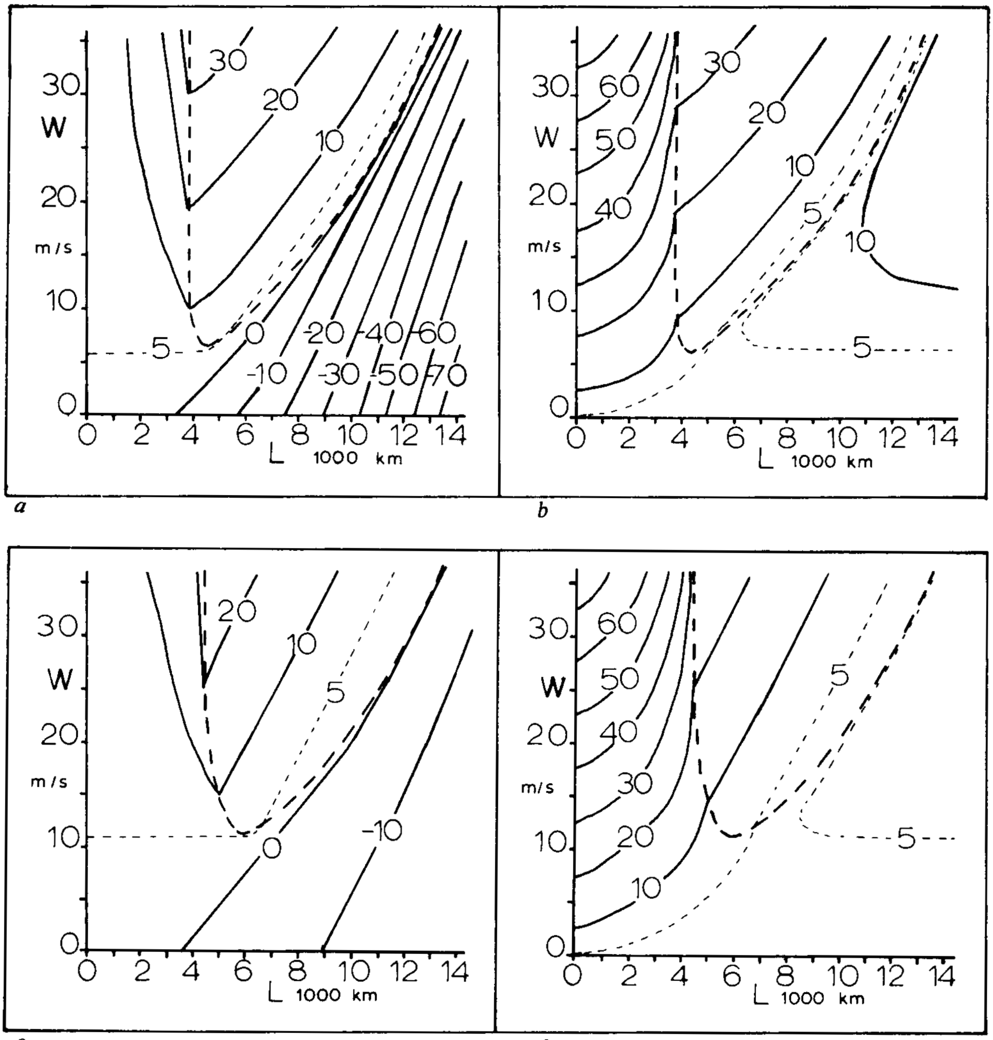

Fig. 3. Actual phase speed c_r (m s^{-1}) for a basic state where $U = 5 + W$. Both solutions are for $\alpha' = 0$ in (a), (b); and $\alpha' = 10$ in (c), (d).

Table 1. *Growth rates (10^{-6} s^{-1}) and/or phase speed (m s^{-1}) for 6000 and 14,000 km wavelengths with $W = 20$ m s^{-1} and $U = 25$ m s^{-1}, as a function of α'*

	α'				
	0	1	2	5	10
6000 km, c_r	15.5	14.6	13.8	12.3	11.0
kc_i	12.6	11.7	10.8	8.7	6.0
14,000 km, c_{r_1}	−50.4	−37.2	−26.9	−17.7	−12.3
c_{r_2}	14.6	12.8	11.6	9.5	7.7

The structure of the unstable waves also changes substantially with the Helmholtz term. For reference purposes the structure of unstable waves for $\alpha' = 0$ is outlined in Table 2, showing the amplitude and phase of the thermal wave and the phase of the 750 mb stream function and the vertical motion field relative to the 500 mb stream function. Thus the thermal wave lags by 72° to 36° and upward motion occurs from 8° to 31° ahead of the 500 mb trough so that the upward motion and the warm sector are nearly in phase.

Table 2. *Amplitude (A_τ) and phase (ϕ_τ) of the thermal field, and phase of the 750 mb stream function (ϕ_3) and vertical motion field (ϕ_w) relative to the 500 mb stream function for $W = 20$ m s^{-1} and $\alpha' = 0$. The phase is in degrees*

Wavelength (km)	500	6000	7000	8000
A_τ/A_ψ	0.53	0.66	0.74	0.80
ϕ_τ	−72	−66	−58	−36
ϕ_3	31	39	46	52
ϕ_w	98	104	116	121

The 750 mb trough leads the 500 mb trough by 31° to 52° so that the 500 mb upward motion is about 22° upstream from that which would be present from the Helmholtz term.

In a similar manner the structure of unstable waves for $\alpha' = 10$ is given in Table 3. Greatest change is in the phase of the thermal wave toward a more barotropic state. This has the effect also of reducing the amplitude of the 750 mb wave, relative to that at 500 mb, which tends to offset somewhat the effects of the Helmholtz term. The change is greatest for the shorter waves, and the warm sector and region of upward motion are now from 46° to 52° out of phase.

Table 3. *As for Table 2, but for $\alpha' = 10$*

Wavelength (km)	5000	6000	7000	8000
A_τ/A_ψ	0.43	0.62	0.74	0.84
ϕ_τ	−35	−37	−32	−26
ϕ_3	21	36	47	56
ϕ_w	93	97	101	108

The reason for the change in structure is related to the reduction in low level convergence ahead of the trough, thereby slowing the eastward movement of the 750 mb wave and tending to make the system more vertical.

4. Energetics of the model

The details of the energetics of the two-layer model are well known (see Holton 1972, p. 196), but the inclusion of the Helmholtz term introduces an additional form of energy. We define

$$K_i = \frac{\Delta p}{2g} \int (\nabla \psi_i)^2 dS; \quad i = 1, 3 \tag{17}$$

as the area integrated kinetic energy of a layer Δp thick.

$$A = \frac{f_0^2}{2g\sigma\Delta p} \int (\psi_3 - \psi_1)^2 dS \tag{18}$$

is the area integrated available potential energy of layer Δp thick and we put

$$P = \frac{\alpha'^2 \Delta p}{g} \int \psi_3^2 dS \tag{19}$$

is a potential energy term associated with the distribution of total mass as given by the surface pressure. Using (5)

$$\frac{\partial P}{\partial t} = \frac{f_0}{g} \int \omega_4 \psi_3 dS \tag{20}$$

The energetics equations may then be written

$$\frac{\partial}{\partial t}(K_1 + K_3) = \frac{f_0}{g} \int \omega_2(\psi_3 - \psi_1) dS$$

$$- \frac{f_0}{g} \int \omega_4 \psi_3 dS \tag{21}$$

$$\frac{\partial (A + P)}{\partial t} = -\frac{f_0}{g} \int \omega_2(\psi_3 - \psi_1) dS$$

$$+ \frac{f_0}{g} \int \omega_4 \psi_3 dS \tag{22}$$

so that total energy is conserved. We put

$$\mathrm{CE} = \frac{f_0}{g} \int \omega_2(\psi_3 - \psi_1)\, dS \tag{23}$$

which describes the process of warm air rising and cold air sinking converting available energy into kinetic energy. However, we can show that $P = (2\alpha^2/k^2)K_3$. Therefore, for a given amount of energy converted (= CE)

$$\frac{\partial}{\partial t}\,[K_1 + K_3(1 + 2\alpha^2/k^2)] = \mathrm{CE}$$

Clearly this limits the growth of the kinetic energy of the perturbation, especially at the lower level for longer waves, and some of the energy is used to redistribute the mass of the atmosphere.

However, as we have already seen, a much greater effect is the reduction of CE arising from the change in phase of the thermal wave relative to the vertical motion. This is more important for the shorter unstable waves, and is the main reason growth rates are reduced with the introduction of the Helmholtz term.

5. Discussion and conclusions

We have shown that as well as reducing the phase speeds of all waves, the Helmholtz term also reduces the growth rates of unstable waves.

There appears to be an inherent tendency for quasi-geostrophic models without an enhanced Helmholtz term to overdevelop systems. The reason for this may lie in the omission of certain ageostrophic effects. Krishnamurti (1968) showed that deformation and divergence terms both act to counteract vertical motions from quasi-geostrophic terms and are clearly important near a major storm undergoing cyclogenesis. Also, as noted in the introduction, the excessive retrograde phase speeds of very long waves arises from the inadequacy of the scaling assumptions in the quasi-geostrophic model when applied to ultra-long forced waves. Both problems can apparently be controlled by the Helmholtz term, although improvements in diabatic heating should also help. The Helmholtz term therefore emerges as a piece of engineering, albeit very effective, to control problems brought about by shortcomings of the quasi-geostrophic set of equations.

The Helmholtz term is shown to be important, even if not given an enhanced value, and should be included in all quasi-geostrophic models.

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ВЛИЯНИЕ ЧЛЕНА ГЕЛЬМГОЛЬЦА НА БАРОКЛИННУЮ МОДЕЛЬ АТМОСФЕРЫ

Исследуется эффект включения члена Гельмгольца за счет нижнего краевого условия в двухуровенную квазигеострофическую модель, Член Гельмгольца приводит к двум главным эффектам, причем оба желательны для квазигеострофической модели. Уменьшая фазовые скорости всех волн и тем самым контролируя сильное обратное движение ультрадлинных волн Россби, он приводит к уменьшению скоростей роста неустойчивых волн. Член Гельмгольца уменьшает наклон неустойчивых волн с высотой к западу, причем эффект больше проявляется для коротких волн.

Показано, что член Гельмгольца важен для всех квазигеострофических моделей. Однако его использование с увеличением в 10 раз хотя и желательно для оптимизации прогнозов, имеет малое физическое обоснование и служит просто преодолению недостатков модели.