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A note on the finite amplitude dynamics of spatially growing baroclinic waves on an f-plane

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ABSTRACT

Finite amplitude behavior of inviscid spatially growing baroclinic waves in a two-layer model on an f-plane is investigated. In the framework of the weakly nonlinear theory it is found that nonlinearity can stabilize or destabilize the flow. In the former case, the nonlinear wave experiences spatial vacillations. In the latter case, the possibility of high frequency finite amplitude subcritical destabilization of the flow field exists. The study is relevant to the problem of mountain-induced cyclogenesis.

1. Introduction

Recently there has been an increasing interest in the dynamical response of the large-scale atmospheric and oceanic systems to topographic forcing. Meteorologists are interested in understanding the mechanisms responsible for lee-cyclogenesis, a phenomenon describing the development of storms in the lee of major mountain complexes (Petterssen, 1956; Manabe and Terpstra, 1965). The same interest is also shared by oceanographers, since bottom relief plays an important role in the dynamics of oceanic flows.

While the dynamics of such flows are far from being steady, the earlier studies were mainly concerned with the steady-state response of such systems (see, for example, Hogg, 1973; Huppert, 1975; Merkine and Rivas, 1976). None the less, these studies, while not always motivated by the problem of lee-cyclogenesis, were valuable since they have demonstrated that baroclinicity is increased in the vicinity of mountain complexes. This led immediately to the speculation that lee-cyclogenesis might be attributed to baroclinic instability (Merkine, 1975).

Analytical studies of the stability properties of such systems is not feasible. However, great simplification can be achieved if the mountainous region is assumed to contain an increased noise level resulting from the highly distorted flow pattern in its neighborhood. It is then necessary to study only the development of the disturbances as they are carried downstream into a less distorted flow. This is justified since the mountains are too narrow to affect the evolution of a disturbance. In that respect mountains might be considered as localized sources of disturbances. Since such regions are only a few hundred kilometers in width we expect the energy content of the disturbances generated to be concentrated in the high frequency components of the spectrum of the large-scale atmospheric flows. Furthermore, since the mountains are quite high, the excitation of finite amplitude disturbances is also possible.

It is therefore suggested that a study of the response of a baroclinic system to a time-dependent source is warranted (this is also emphasized by Hart, 1972). Oceanographers have recently considered such an approach for the purpose of finding an explanation for the Gulf Stream meanders (Robinson and Gadgil, 1970; Gadgil, 1976).

The long time response of an unstable baroclinic system to a localized excitation is crucially dependent on the nature of the transient. It can be classified according to the system's response to a pulse disturbance. This has been recently studied by Thacker (1976) and Merkine (1977). In an

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infinite system which is unstable, a linearized pulse disturbance may propagate along the system so that its amplitude eventually decreases with time at any fixed point in space (convective instability), or it may grow in time without limit at every spatial point reached by the pulse (absolute instability). For the simple inviscid nonconducting case of the two-layer model on an f-plane, criterion for the onset of absolute instability has been determined (Merkine, 1977). The f-plane geometry is adequate for our purposes, since high frequencies are only marginally influenced by the β-effect.

If the system operates in the right parameter space such that absolute instability is prohibited, the long time response of the system depends on the time-dependent characteristics of the source. In the most simple case the source may consist of a small amplitude simple wave maker oscillating with a fixed frequency. In this case, spatially amplifying baroclinic waves oscillating with the frequency of the source will be present downstream of the source (in the limit of zero frequency spatially growing waves have been studied recently by Hogg (1976). However, it is not clear how his analysis is related to the long time response of the appropriate initial value problem.

At sufficiently downstream distances the amplitude of the linearized amplifying baroclinic waves becomes finite such that the linear analysis ceases to apply. In this work we wish to study the finite behavior of such waves. A tractable analysis is possible only in the framework of the weakly nonlinear theory developed by Stuart (1960) for timewise growing waves and extended by Watson (1962) to spatially growing waves.

Pedlosky (1970) and Drazin (1970) applied the weakly nonlinear theory to timewise growing baroclinic waves, and subsequent works dealt mainly with the limit cycle behavior of such waves. Pedlosky (1972, 1976) discusses some aspects of the dynamics of spatially growing waves. His analysis is carried out for marginally unstable waves on a β-plane. These waves are found to be modulated in space in much the same way as the timewise growing waves of his 1970 work. In this early work Pedlosky determines also the finite amplitude dynamics of marginally unstable temporally growing waves on an f-plane. The limit cycle behavior is dynamically identical to the β-plane case.

No such similarity exists when we consider the finite amplitude dynamics of marginally unstable spatially growing baroclinic waves on an f-plane. A somewhat different equation is found to govern the amplitude behavior of the waves. In particular, the amplitude of such waves is not always bounded as a result of the weakly nonlinear dynamics introduced. If the frequency of the source is sufficiently high the possibility of finite amplitude subcritical destabilization of the flow field exists. This effect may be related to the phenomenon of lee-cyclogenesis discussed earlier. It should be pointed out that Huppert and Bryan (1976) suggested that lee-cyclogenesis could be explained in terms of vorticity redistribution resulting from the interaction of a time-dependent flow with bottom relief. Their analysis was established, however, only for a system which is baroclinically stable.

2. Formulation

We consider the two-layer model for a quasi-geostrophic system describing an inviscid nonconducting fluid on an f-plane. The system is bounded above and below by rigid horizontal planes separated by a distance D. It is also bounded laterally by vertical walls which are separated by a distance L. In the framework of the Boussinesq approximation, the fractional change in density, $\Delta\rho/\rho$, across the interface separating the two fluids is assumed to be small. In the absence of motion the two layers are of equal depth. The relevant nondimensional equations are (Pedlosky, 1970)

$$\left(\frac{\partial}{\partial t}+\frac{\partial \psi_1}{\partial x}\frac{\partial}{\partial y}-\frac{\partial \psi_2}{\partial y}\frac{\partial}{\partial x}\right)(\nabla^2 \psi_1 + F(\psi_2-\psi_1)) = 0 \tag{2.1a}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial \psi_2}{\partial x}\frac{\partial}{\partial y}-\frac{\partial \psi_2}{\partial y}\frac{\partial}{\partial y}\right)(\nabla^2 \psi_2 + F(\psi_1-\psi_2)) = 0. \tag{2.1b}$$

Subscripts 1 and 2 denote the upper and lower layers respectively. Distances are measured in units of L and time in units of L/U_m where U_m is the average speed of the unperturbed baroclinic current. The stream function ψ is measured in units of

$U_m L$. The positive x direction points downstream. $F = L^2/L_R^2$ is the rotational Froude number. $L_R = ((\Delta\rho/\rho) g\frac{1}{2}D)^{1/2}/f$ is the radius of deformation. g is the gravitational acceleration and f is the Coriolis parameter. $\frac{1}{2}f$ is the rotational rate of the system.

The inviscid boundary conditions at the lateral walls are

$$\frac{\partial \psi_n}{\partial x} = 0 \quad \text{for } y = 0, 1; \quad n = 1, 2 \tag{2.2}$$

We assume that a wavemaker oscillating with a fixed frequency is placed at $x = 0$. The linear theory indicates that such a system is unstable to small perturbations provided F is large enough. In the framework of the nonlinear theory we seek to determine the finite amplitude behavior of the system.

Let F_c be the critical rotational Froude number such that a wave oscillating with a real frequency, ν, is marginally unstable. If $F = F_c + \Delta$, $0 < \Delta \ll 1$, such a wave is slightly spatially unstable according to the linear theory with a growth rate proportional to $\Delta^{1/2}$. This is suggested by the linear analysis for temporally growing waves (Pedlosky, 1970) and by Merkine's (1977) analysis for the spatial growth of long waves. The functional dependence relating the frequency of the wavemaker, F_c, and the other free parameters of the problem will be determined later.

Following Pedlosky (1970) we assume that

$$\psi_n = -U_n y + \phi_n(x, X, y, t); \quad X = |\Delta|^{1/2} x$$

$$\phi_n = |\Delta|^{1/2} \phi_n^{(1)} + |\Delta| \phi_n^{(2)} + |\Delta|^{3/2} \phi_n^{(3)} + \ldots$$

$$U_1 = 1 + \tfrac{1}{2}\varepsilon, \quad U_2 = 1 - \tfrac{1}{2}\varepsilon. \tag{2.3}$$

ε is the shear of the unperturbed zonal flow, and $U_m = 1$.

When the above expressions are substituted into the equations of motion and the coefficients of like powers of $|\Delta|^{1/2}$ are set equal to zero a hierarchy of equations governing the development of the various orders of the asymptotic expansion of ϕ_n are obtained.

The linear problem is obtained at $O(|\Delta|^{1/2})$

$$\mathscr{L}_1(\phi_1^{(1)}, \phi_2^{(1)}) \equiv \left(\frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \phi_1^{(1)}}{\partial x^2} + \frac{\partial^2 \phi_1^{(1)}}{\partial y^2} + F_c(\phi_2^{(1)} - \phi_1^{(1)})\right) + F_c \varepsilon \frac{\partial \phi_1^{(1)}}{\partial x} = 0$$

$$\mathscr{L}_2(\phi_1^{(1)}, \phi_2^{(1)}) \equiv \left(\frac{\partial}{\partial t} + U_2 \frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \phi_2^{(1)}}{\partial x^2} + \frac{\partial^2 \phi_2^{(1)}}{\partial y^2} + F_c(\phi_1^{(1)} - \phi_2^{(1)})\right) - F_c \varepsilon \frac{\partial \phi_2^{(1)}}{\partial x} = 0. \tag{2.4}$$

The wavelike solution oscillating with the frequency ν is given by

$$\begin{bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{bmatrix} = \tfrac{1}{2}A(X) \begin{bmatrix} 1 \\ \gamma \end{bmatrix} e^{i(\alpha x - \nu t)} \sin m\pi y + \text{c.c.}, \quad m = 1, 2, \ldots \tag{2.5}$$

c.c denotes the complex conjugate part. α is the wave number. The amplitude of the lower layer wave relative to the upper layer wave is given by

$$\gamma = 1 + \frac{\alpha^2 + m^2\pi^2}{F_c} + \frac{\varepsilon\alpha}{\nu - \alpha U_1} = \frac{1}{1 + \dfrac{\alpha^2 + m^2\pi^2}{F_c} + \dfrac{\varepsilon\alpha}{\nu - \alpha U_2}} \tag{2.6}$$

which also yields the dispersion relation

$$(1 - \tfrac{1}{4}\varepsilon^2)\alpha^4 - 2\nu\alpha^3 + [\nu^2 + 2F_c(1 + \tfrac{1}{4}\varepsilon^2) + m^2\pi^2(1 - \tfrac{1}{4}\varepsilon^2)]\alpha^2 - 2(2F_c + m^2\pi^2)\nu\alpha + (2F_c + m^2\pi^2)\nu^2 = 0. \tag{2.7}$$

(For $F > F_c$ (2.7) yields spatially amplifying waves for real frequencies, i.e. α is complex with $\mathrm{Im}(\alpha) < 0$.)

In the temporal case the neutral stability curve and γ are given by

$$2F_c - m^2\pi^2 - \alpha^2 = 0, \quad \gamma = 1, \tag{2.8}$$

independently of ε. In the spatial case this curve is ε dependent (Merkine, 1977). It is determined by solving (2.7) subject to the side condition for the existence of marginally unstable spatially growing waves. This side condition is determined at the next order of the expansion. $O(|\Delta|)$ yields

$$\mathscr{L}_1(\phi_1^{(2)}, \phi_2^{(2)}) = [(-\nu + \alpha U_1) 2\alpha + U_1(\alpha^2 + m^2\pi^2) - F_c(\gamma - 1) - \varepsilon F_c] \frac{dA}{dX} \cdot \tfrac{1}{2} e^{i(\alpha x - \nu t)} \sin m\pi y + \text{c.c.}$$

$$\mathscr{L}_2(\phi_1^{(2)}, \phi_2^{(2)}) = \gamma[(-\nu + \alpha U_2) 2\alpha + U_2(\alpha^2 + m^2\pi^2) - F_c(\gamma^{-1} - 1) + \varepsilon F_c] \frac{dA}{dX} \cdot \tfrac{1}{2} e^{i(\alpha x - \nu t)} \sin m\pi y + \text{c.c.} \tag{2.9}$$

The solution of (2.9) can be written as

$$\begin{bmatrix}\phi_1^{(2)}\\ \phi_2^{(2)}\end{bmatrix} = \tfrac{1}{2}\begin{bmatrix}A_1^{(2)}(X)\\ A_2^{(2)}(X)\end{bmatrix} e^{i(\alpha x - \nu t)} \sin m\pi y + \text{c.c.}$$

$$+ \begin{bmatrix}\Phi_1(X,y)\\ \Phi_2(X,y)\end{bmatrix} \tag{2.10}$$

which when substituted into (2.9) yields

$$-\gamma A_1^{(2)} + A_2^{(2)} = \frac{1}{iF_c}\left[2\alpha + \frac{\varepsilon\nu F_c}{(-\nu+\alpha U_1)^2}\right]\frac{dA}{dX}$$

$$-\gamma A_1^{(2)} + A_2^{(2)} = -\frac{1}{iF_c}\gamma^2$$

$$\left[2\alpha - \frac{\varepsilon\nu F_c}{(-\nu+\alpha U_2)^2}\right]\frac{dA}{dX}. \tag{2.11}$$

Since $dA/dX \neq 0$ we require that

$$2\alpha(\gamma^2+1) + \varepsilon\nu F_c\left[\frac{1}{(-\nu+\alpha U_1)^2} - \frac{\gamma^2}{(-\nu+\alpha U_2)^2}\right] = 0 \tag{2.12}$$

which is the necessary condition for the existence of marginally unstable spatially growing waves. The neutral stability curve has been determined by solving (2.7) subject to (2.12). Fig. 1 shows various cases of ε. The amplitude ratio, γ, is depicted in Fig. 2. If $\varepsilon > 2^{1/2}$ absolute instability exists in a laterally open domain and spatially growing waves cannot be realized. In a channel flow the critical value of ε is larger (Merkine, 1977). However, since the lateral walls may also be thought of as an artifice for fixing the lateral wavelength in an open domain, no calculations have been performed for $\varepsilon > 2^{1/2}$. We present the results only for $m = 1$ since it yields the largest growth rates. (Long mountain ranges like the Rockies have a meridional scale which is much larger than the radius of deformation and the meridional structure of the disturbance may possess many wiggles. The naturally occurring meridional scale of the disturbance depends on the radius of deformation and on the lateral variation of the basic flow (Simonns, 1974; Gent, 1974, 1975). Atmospheric and oceanic flows always possess a lateral shear. No such effect has been incorporated into the present nonlinear analysis. Needless to say, however, such a study is highly desirable.)

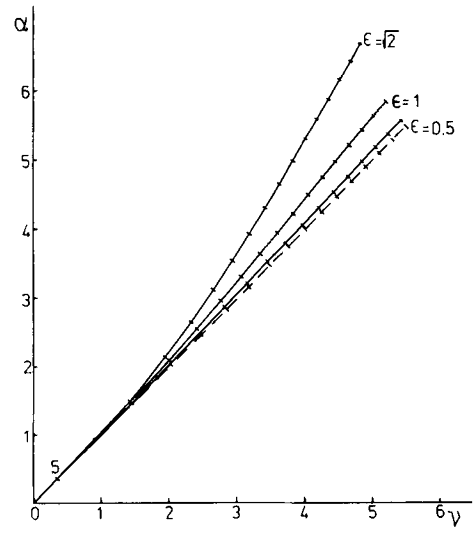

Fig. 1. Neutral curves for the spatial problem—full lines. Neutral curve for the temporal problem— dashed line. $m = 1$. The small lines denote the critical Froude numbers in the range $5 \leq F_c \leq 20$.

Without any loss of generality we can write

$$A_1^{(2)} = 0 \tag{2.13}$$

$$A_2^{(2)} = \frac{1}{iF_c}\left[2\alpha + \frac{\varepsilon\nu F_c}{(-\nu+\alpha U_1)^2}\right]\frac{dA}{dX},$$

since any homogeneous solution of (2.11) can be absorbed in the solution of the $O(|\Delta|^{1/2})$ problem.

The evolution of $A(X)$ and $\phi_n(X,y)$ is determined at the next order of the expansion, which symbolically can be written as

$$\begin{aligned}\mathcal{L}_1(\phi_1^{(3)}, \phi_2^{(3)}) &= \mathcal{M}_1(\phi_1^{(1)}, \phi_2^{(1)}, \phi_1^{(2)}, \phi_2^{(2)})\\ \mathcal{L}_2(\phi_1^{(3)}, \phi_2^{(3)}) &= \mathcal{M}_2(\phi_1^{(1)}, \phi_2^{(1)}, \phi_1^{(2)}, \phi_2^{(2)}).\end{aligned} \tag{2.14}$$

For the sake of brevity the functional form of $\mathcal{M}_n$ will not be presented here. It was pointed out by Pedlosky (1970) that the $\mathcal{M}_n$ must satisfy a certain orthogonality condition if secularity is to be prevented. In our case this leads to two equations

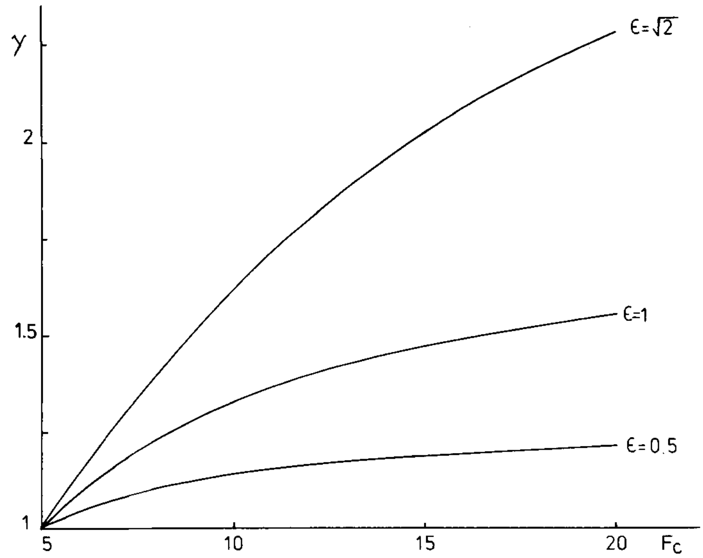

Fig. 2. The amplitude ratio of the lower layer wave to the upper layer wave for marginally unstable spatially growing waves. $m = 1$.

governing Φ_n and to an equation determining $A(X)$. We obtain that

$$\frac{\partial}{\partial X}\left\{U_1\left[\frac{\partial^2\Phi_1^{(2)}}{\partial y^2} + F_c(\phi_2^{(2)} - \phi_1^{(2)})\right] + \varepsilon F_c\phi_1^{(2)}\right\} = b_1\frac{m\pi}{8}\sin 2m\pi y\frac{d|A|^2}{dX}.$$

$$\frac{\partial}{\partial X}\left\{U_2\left[\frac{\partial^2\Phi_2^{(2)}}{\partial y^2} + F_c(\Phi_1^{(2)} - \Phi_2^{(2)})\right] - \varepsilon F_c\Phi_2^{(2)}\right\} = b_2\frac{m\pi}{8}\sin 2m\pi y\frac{d|A|^2}{dX} \tag{2.15}$$

$$b_1 = 2\alpha\frac{\varepsilon\nu F_c}{(-\nu + \alpha U_1)^2};$$

$$b_2 = -2\alpha\frac{\varepsilon\nu F_c}{(-\nu + \alpha U_2)^2}\gamma^2.$$

(2.15) is integrated with respect to X and the constants of integration are determined by requiring that the mean flow is distorted wherever the amplitude of the wave is different from zero including, of course, the location of the wavemaker. Such a requirement is necessary if the distortion of the mean flow is attributed solely to the wavemaker. The boundary condition employed by Pedlosky (1972, 1976) requires that no mean flow distortion exists at the wavemaker location. Such a requirement is inconsistent with the model considered here since it implies that the mean flow is affected by causes other than the wavemaker excitation.

The solution of (2.15) which satisfies (2.2) is

$$\begin{bmatrix}\Phi_1^{(2)}\\ \Phi_2^{(2)}\end{bmatrix} = \begin{bmatrix}c_1\\ c_2\end{bmatrix}|A|^2\frac{m\pi}{8}\sin 2m\pi y$$

$$\begin{bmatrix}c_1\\ c_2\end{bmatrix} = -\frac{\begin{bmatrix}b_1U_2(2m\pi)^2 - 4\alpha^2(1+\gamma^2)U_1F_c\\ b_2U_1(2m\pi)^2 - 4\alpha^2(1+\gamma^2)U_2F_c\end{bmatrix}}{(2m\pi)^2[U_1U_2(2m\pi)^2 + F_c(U_1^2 + U_2^2)]}. \tag{2.16}$$

The equation governing A is found to be

$$\frac{d^2A}{dX^2} = \frac{\Delta}{|\Delta|}a_0A - a_1A|A|^2 \tag{2.17}$$

$$a_0 = \{-(-\nu + \alpha U_1)(1-\gamma)^2 + \alpha\varepsilon[1 - \gamma^2(-\nu + \alpha U_1)/(-\nu + \alpha U_2)]\}/b_0$$

$$a_1 = -\tfrac{1}{8}(m\pi)^2\alpha\{(3m^2\pi^2 - \alpha^2)(c_1 + \delta\gamma c_2) + F_c(1-\delta)(\gamma c_1 - c_2)\}/b_0$$

$$b_0 = -(-\nu + \alpha U_1)[1 + \gamma^2 + U_2(a_3 + 2\alpha\gamma^2)/(-\nu + \alpha U_2) - a_3^2/(\gamma F_c)] + U_1a_3 - 2\alpha U_1$$

$$\delta = \gamma(-\nu + \alpha U_1)/(-\nu + \alpha U_2)$$

$$a_3 = 2\alpha + \nu\varepsilon F_c/(-\nu + \alpha U_1)^2.$$

Equation (2.14) is different than the corresponding equation in Pedlosky's (1972, 1976) analysis. This is a consequence of the different boundary condition as discussed above.

3. The finite amplitude behavior

Following Pedlosky (1970) A is writen as

$$A = Re^{i\theta}, \tag{3.1}$$

where R is the modulus and θ is the phase. When (3.1) is substituted into (2.17) we obtain

$$\frac{1}{2}\left(\frac{dR}{dX}\right)^2 + V(R) = E \tag{3.2}$$

$$V(R) = \frac{1}{2}\left(-\frac{\Delta}{|\Delta|} a_0 R^2 + \tfrac{1}{2}a_1 R^4 + \frac{L^2}{R^2}\right)$$

$$L = R^2 \frac{d\theta}{dX}$$

$$E = \frac{1}{2}\left(\left(\frac{dR(0)}{dX}\right)^2 - \frac{\Delta}{|\Delta|} a_0 R^2(0) + \tfrac{1}{2}a_1 R^4(0) + \frac{L^2}{R^2(0)}\right).$$

E and L are constants of integration which may be thought of as the energy and angular momentum integrals respectively. The linear stability problem corresponds to the class of solutions of (3.2) for which $L = 0$. We shall discuss only this case, although the analysis of the more general case is possible. We further restrict ourselves to the initial conditions

$$\left(\frac{dR(0)}{dX}\right)^2 - \frac{\Delta}{|\Delta|} a_0 R^2(0) = 0, \tag{3.3}$$

again corresponding to the linear theory in the limit $R(0) \to 0$, for which

$$E = \tfrac{1}{4}a_1 R^4(0). \tag{3.4}$$

$R(0)$ is the amplitude of the wavemaker.

The development of the amplitude is crucially dependent on the signs of a_0, a_1 and Δ. a_0 is always positive since the perturbation analysis is about the neutral curve and for supercritical conditions, $\Delta > 0$, spatially growing waves exist. Fig. 3 depicts $a_0^{1/2}$ as a function of F_c and ε. Note that for supercritical conditions $a_0^{1/2}$ corresponds to the spatial growth rate of the linear theory. The numerical values of a_1 as depicted in Fig. 4 indicate

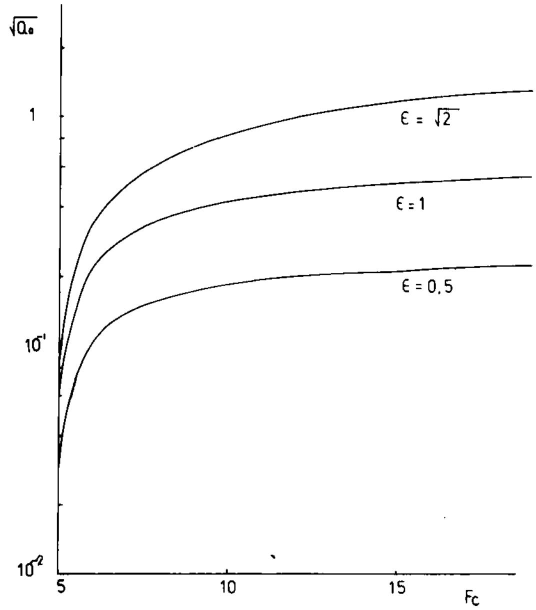

Fig. 3. $a_0^{1/2}$, the linear growth rate, for marginally unstable spatially growing waves.

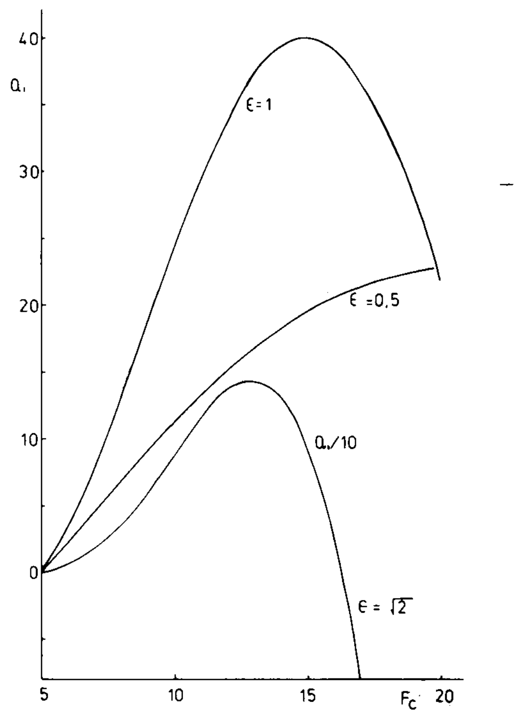

Fig. 4. a_1 as a function of the critical Froude number for various values of shear. $m = 1$.

that a_1 can change its sign. Consequently the $a_1 > 0$ and $a_1 < 0$ cases should be considered separately.

3.1. $a_1 > 0$, *supercritical conditions* ($\Delta > 0$)

Under the constraints discussed earlier, the maximum amplitude that can be attained by the wave is given by

$$R_{\max} = \left\{\frac{a_0}{a_1}\left\{1 + \left[1 + \left(\frac{a_1}{a_0}\right)^2 R^4(0)\right]^{1/2}\right\}\right\}^{1/2}. \quad (3.5)$$

The minimum amplitude is

$$R_{\min} = 0 \quad (3.6)$$

which implies that R undergoes a change in phase when $R = 0$.

The spatial development of the amplitude is consequently given by

$$A = R_{\max}\, cn\left[\left(\frac{a_1 R^2_{\max}}{2k^2}\right)^{1/2}(X - X_0); k\right] \quad (3.7)$$

where

$$k^2 = \frac{\dfrac{a_1}{a_0} R^2_{\max}}{2\left(\dfrac{a_1}{a_0} R^2_{\max} - 1\right)}$$

$$X_0 = \left(\frac{2k^2}{a_1 R^2_{\max}}\right)^{1/2} cn^{-1}\left(\frac{R(0)}{R_{\max}}; k\right) \quad (3.8)$$

k is the modulus of the Jacobian elliptic function cn.

At X_0 the wave attains its maximum amplitude. For very small amplitudes of the wavemaker such that $(a_1/a_0)^2 R^4(0) \ll 1$

$$R_{\max} \sim (2a_0/a_1)^{1/2}$$

$$R \sim R_{\max}\,\mathrm{Sech}[a_0^{1/2}(X - X_0)] \quad (3.9)$$

$$X_0 \sim \mathrm{Sech}^{-1}(R(0)/R_{\max})/a_0^{1/2}.$$

3.2. $a_1 < 0$, *subcritical conditions* ($\Delta < 0$)

Once more we consider the case of $L = 0$. Equations (3.2) can be written as

$$\frac{1}{2}\left(\frac{dR}{dX}\right)^2 + V(R) = E.$$

$$V(R) = \tfrac{1}{2}(a_0 R^2 - \tfrac{1}{2}|a_1| R^4)$$

$$E = \frac{1}{2}\left(\left(\frac{dR(0)}{dX}\right)^2 + a_0 R^2(0) - \tfrac{1}{2}|a_1| R^4(0)\right) \quad (3.10)$$

The potential $V(R)$ possesses a maximum at $R_m = (a_0/|a_1|)^{1/2}$ contrary to case 3.1 for which $V(R)$ has a minimum at R_m. This implies an unbounded amplitude development for $R(0) > R_m$. (If $dR(0)/dX < 0$ the amplitude will decrease first up to the value of R corresponding to $V(R) = E$ and then it will start increasing indefinitely.) This behavior indicates finite amplitude subcritical destabilization of the flow. In other words, the flow field which is stable according to the linear theory for infinitesimal disturbances and which is also stable for finite amplitude disturbances for $R(0) < R_m$, may be destabilized by finite amplitude disturbances provided they are sufficiently strong. The phenomenon of finite amplitude subcritical destabilization of a fluid motion is well known in fluid mechanics, and it has also been found for plane Poiseuille flow (Stewartson and Stuart, 1971).

The nature of the finite amplitude destabilization process can be studied in the following way. Let

$$A = (a_0/|a_1|)^{1/2} + r; \quad r > 0 \quad (3.11)$$

and substitute this expression into (2.17). We obtain

$$\frac{d^2 r}{dX^2} = a_0 r[2 + 3(|a_1|/a_0)^{1/2} r + (|a_1|/a_0) r^2] \quad (3.12)$$

For $r \ll 1$ the growth rate is exponential but $2^{1/2}$ times faster than the corresponding growth rate of the linear theory. Equation (3.12) indicates also that the growth rate increases with the amplitude with the consequence of an unbounded amplitude development. If the flow field is supercritical and $a_1 < 0$ (2.17) implies qualitatively the same super exponential growth, but for arbitrarily small initial amplitudes. The super exponential growth can be attributed to the fact that the wave distorts the mean flow to such an extent that more energy can be extracted from it. Obviously an unbounded amplitude development is unacceptable and consequently higher order amplitude effects must be considered. However, such considerations are of little practical interest since the higher order effects will be felt over distances which are much larger than $O(1/\Delta^{1/2})$. $O(1/\Delta^{1/2})$ is the spatial range of validity of the perturbation expansion.

It should be emphasized that for temporally growing inviscid waves on an f-plane (Pedlosky, 1970), $a_1 > 0$. This is fundamentally different from the dynamics of the spatially growing waves

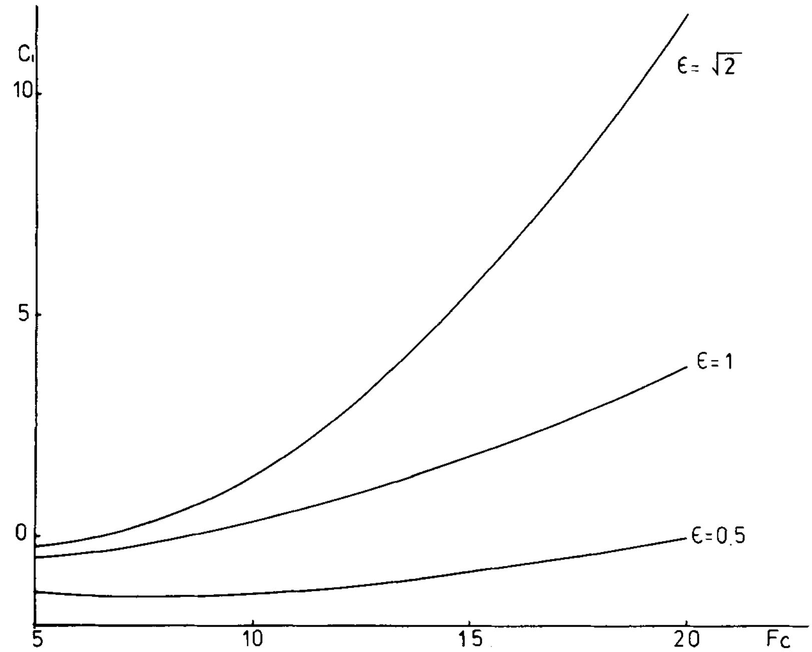

Fig. 5. The mean flow distortion coefficient for the upper layer, c_1, as a function of F_c for various ε. Equation (2.16). m 2 1.

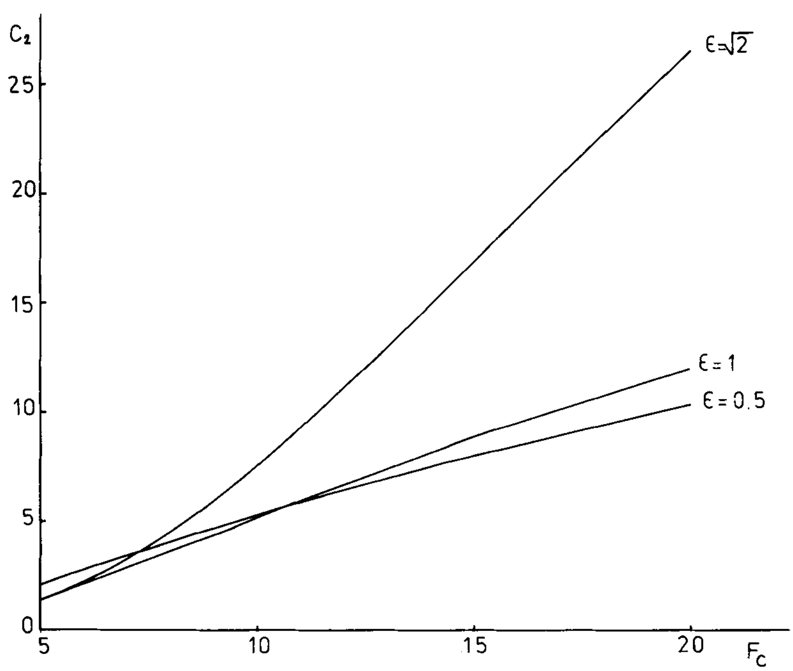

Fig. 6. The mean flow distortion coefficient for the lower layer, c_2, as a function of F_c for various ε. Equation (2.16). $m = 1$.

considered here (temporally growing waves are important for the dynamics of the atmosphere as a whole, while spatially growing waves are more important for local phenomena (Merkine, 1977)). Note that negative values of a_1 can be obtained for temporally growing waves, if the β-effect is included, however such negative values are restricted to a parameter range which is not relevant for the temporal problem (Pedlosky, 1970).

Fig. 4 indicates that negative values of a_1 are obtained for large values of F_c or equivalently for high frequencies (see Fig. 1). This only underscores the importance of the high frequency dynamics which we believe to be relevant to the problem of lee-cyclogenesis.

Figs. 5 and 6 depict the coefficients c_1 and c_2, given by (2.16), which characterize the nature of the distortion of the basic flow caused by the wave. We see that the basic flow is strongly modified by the distortion. This behavior becomes more pronounced as ε and F_c (frequency) increase. The same is true for the amplitude ratio γ (see Fig. 2). This suggests that phenomena described by spatially growing waves are rather shallow. It should be pointed out that in the temporal problem the two layers are characterized by the same wave activity (Pedlosky, 1970).

In this work Merkine's (1977) study of linear spatially growing waves is extended to include nonlinear dynamics in the framework of the weakly nonlinear theory. It should be emphasized that such a theory, while providing us with further insight into the flow dynamics, depends on the smallness of a certain parameter and we have no way of judging its applicability to highly amplifying disturbances. However, even in the framework of the weakly nonlinear theory a very challenging goal is to determine the finite amplitude dynamics of absolute instability.

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БАРОКЛИННЫЕ ВОЛНЫ КОНЕЧНОЙ АМПЛИТУДЫ, РАСТУЩИЕ В ПРОСТРАНСТВЕ

В двуслойной невязкой модели на *f*-плоскости изучается поведение бароклинных волн конечной амплитуды, растущих в пространстве. В рамках слабо нелинейной теории найдено, что нелинейность может стабилизировать или дестабилизировать поток. В первом случае нелинейная волна испытывает пространственные василяции. Во втором случае существует возможность докритической дестабилизации потока высокочастотным возмущением конечной амплитуды. Эта задача имеет отношение к проблеме циклогенеза, индуцируемого горами.