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A small perturbation approximation for stochastic dynamic weather prediction

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ABSTRACT

The stochastic dynamic method for optimum weather forecasting and forecast variance estimation suffers from computational complexity sufficient to obviate its utility as an operational tool. The present paper suggests the use of a linear approximation for a perturbed set of stochastic dynamic equations aimed at reducing computation time. The linearization is based upon an assumption of small departures of the stochastic dynamic solution from the usual deterministic forecast. The resultant linear scheme was tested by evaluating its predictions for a simple mathematical model of the atmosphere, that described by Lorentz's "minimum hydrodynamic equations".

The linearization was found to be as effective as was the standard non-linear stochastic dynamics method, which approximates the exact stochastic dynamic solution by ignoring third-order moments. The problem of developing efficient numerical algorithms that take advantage of the simplification attendant upon linearization is not taken up in this paper, but will be reported upon as the work is pursued.

1. Introduction

Following the pioneering work of Gleeson (1968), Epstein (1969) demonstrated the potential utility of statistical solutions of the equations used in numerical weather forecasting. The technique developed by Epstein, which he labelled the "stochastic dynamic method", enabled error estimates to be assigned to forecast predictions made at every grid point of the numerical simulation, as well as improving on the prediction itself.

The fundamental assumption of stochastic dynamics is that the error in numerical forecasts is attributable to errors in initial and boundary values and forcing terms whose statistical influence can be computed via the forecasting equations by assigning particular values to these inputs. The effects, it has usually been assumed, are dominated by the error associated with insufficiency or inaccuracy of information needed for initialization of the numerical prediction algorithm. Prediction error arising from lack of knowledge of boundary or forcing functions is more difficult to treat, due primarily to the less-well-developed mathematical tools that are available for this problem, as compared with the rather complete formulation that exists for stochastic forecasting in which the uncertainty arises just from initial value error (see Fleming, 1972). For completeness we shall summarize the basic approach for dealing with the latter case, although Epstein gave a comprehensive and readable description in his original paper (1969).

Following his formulation, we represent the state of the atmosphere by a state vector x_i, corresponding to all the prognostic variables in the governing differential equations of the atmosphere, the latter being written in generalized canonical form as

$$\frac{dx_i}{dt} = \sum_{j,k} a_{ijk} x_j x_k - \sum_j b_{ij} x_j + c_i \tag{1}$$

in which a_{ijk}, b_{ij} and c_i are constants and the summation extends over $j, k = 1, 2, \ldots N$, where N is the total number of dependent variables defining the state of the atmosphere. The latter is theoretically infinite in a continuum description, but is of course finite, but large, in any real description or analysis of atmospheric motions.

Equation (1), it will be noted, contains no spatial derivatives, and is thus of a reduced form corresponding, for example, to a spectral decomposition, or to a finite difference description of the system. This form, however, does not admit of inclusion of the integro-differential structure required for description of radiatively coupled processes in the atmosphere. In weather forecasting the effect of the latter is incorporated as a prescribed set of heating and cooling sources distributed in the atmosphere. Apart from this restriction, eq. (1) would seem to encompass all varieties of differential systems that might be used for describing atmospheric circulation.

It is useful to conceptualize the stochastic description in terms of an N-dimensional phase space of the state variables. In these terms the instantaneous state of the atmosphere at a given moment is represented by a single point in phase space, and its evolution in time corresponds to a particular (presumably non-intersecting) trajectory in the space. True atmospheric conditions are thus representable by such a unique curve, but the *known* state of the atmosphere (past, present, or future) has an uncertainty that corresponds to a spread around the true trajectory, the volume of which is indicative of this degree of uncertainty. More specifically, for a given numerical forecasting scheme, as given by the deterministic formulation of equations of the form (1), a prognostic solution for specific initial values of the state variables will lead to a particular phase space trajectory, which should correspond to the true atmospheric conditions if the initial values are chosen correctly (and if the equations are solved exactly). Actually, we need to represent the initial values as covering a certain volume in phase space, representing the range of possible initial states that correspond to our degree of uncertainty in their values. Each point within this volume, when used for initialization of eq. (1), will result in a different trajectory, and the ensemble of all points in the volume will generate a cylinder (enclosing the true trajectory), whose cross section would be expected to increase with time, corresponding to growth in uncertainty of the prediction in time.

This conceptual picture can be readily put in mathematical terms. Thus, following conventional statistical mechanics terminology, we introduce the notion of density of points in phase space as an ensemble mean of a set of solutions of eq. (1), and define the moments of the density distribution as follows:

$$\mu_i = \langle x_i \rangle, \tag{2}$$

$$\sigma_{ij} = \langle (x_i - \mu_i)(x_j - \mu_j) \rangle, \tag{3}$$

$$\tau_{ijk} = \langle (x_i - \mu_i)(x_j - \mu_j)(x_k - \mu_k) \rangle, \tag{4}$$

$$\lambda_{ijkl} = \langle (x_i - \mu_i)(x_j - \mu_j)(x_k - \mu_k)(x_l - \mu_l) \rangle, \tag{5}$$

etc. in which $\langle\ \rangle$ represents the ensemble mean over all possible values (and given, in our case, by the set of trajectories corresponding to a prescribed initial distribution of initial values of the state variables).

Application of the averaging operation $\langle\ \rangle$ to eq. (1) and to its products with $(x_i - \mu_i)$, $(x_i - \mu_i)(x_j - \mu_j)$, etc., yields a set of equations for the mean μ_i, the variances and convariances σ_{ij}, and all the higher order moments defined above. Their derivation is straightforward, if lengthy, and will not be reproduced here (see Fleming, 1970 for details). Up to the third order, these equations are

$$\frac{d\mu_i}{dt} = \sum_{j,k} a_{ijk}(\mu_j\mu_k + \sigma_{jk}) - \sum_j b_{ij}\mu_j + c_i \tag{6}$$

$$\begin{aligned}\frac{d\sigma_{ij}}{dt} = \sum_{k,l} [a_{ikl}(\mu_k\sigma_{jl} + \mu_l\sigma_{jk} + \tau_{jkl})\\ + a_{jkl}(\mu_k\sigma_{il} + \mu_l\sigma_{ik} + \tau_{ikl})]\\ - \sum_k (b_{ik}\sigma_{jk} + b_{jk}\sigma_{ik})\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{d\tau_{ijk}}{dt} = \sum_{l,m} [a_{ilm}(\mu_l\tau_{jkm} + \mu_m\tau_{jkl} - \sigma_{lm}\sigma_{jk} + \lambda_{jklm})\\ + a_{jlm}(\mu_l\tau_{ikm} + \mu_m\tau_{ikl} - \sigma_{lm}\sigma_{ik} + \lambda_{iklm})\\ + a_{klm}(\mu_l\tau_{ijm} + \mu_m\tau_{ijl} - \sigma_{lm}\sigma_{ij} + \lambda_{ijlm})]\\ - \sum_l (b_{il}\tau_{jkl} + b_{jl}\tau_{ikl} + b_{kl}\tau_{ijl})\end{aligned} \tag{8}$$

Each of the above moment equations is coupled to its next higher order neighbor via linear terms on the right-hand side, and thus, in general, only approximate solutions are possible. The usual approach is to terminate the formally infinite set of equations at as low an order as possible, with the expectation that the higher-order neglected moments play relatively minor roles. Such ad hoc procedures have in fact been found to work quite well for the weather-forecasting problem, as studied at length, not only in the paper by Epstein already referred to, but also in later work by him (1971), (Epstein and Fleming, 1971; Epstein and Pitcher, 1972) as well as by Fleming (1970, 1971a,b, 1972) and by Pitcher (1977). Indeed Epstein found that truncation of the moment equations at the 3rd order, i.e. putting $\tau_{ijk} \equiv 0$, seemed adequate for forecast periods of up to five days when applied to a simple forecasting model, and his results have been confirmed by later studies (Pitcher, 1974).

In spite of the seeming success of these results it has not yet been possible to consider implementing the stochastic dynamic method operationally. This is a direct consequence of the inordinate amount of computation required to solve the moment equations; for example, if just the first two moment equations are retained, the number of dependent variables increases from N for the usual deterministic forecast, to $N(N + 3)/2$, so that the number of operations in the numerical solution increases by a factor of the order of N. N may be of the order of 100, for a simple hemi-spheric model, but is more likely of the order of 10^5 for the type of advanced model currently used in numerical weather forecasting. In general, therefore, the use of stochastic dynamic forecasting is prohibitively expensive in computer time.

The only way of avoiding this difficulty that had been suggested prior to the work undertaken in the present study, was to achieve essentially the same results as does stochastic dynamics by using the Monte Carlo approach (Leith, 1974). In the Monte Carlo method the statistical ensemble is constructed from a set of deterministic forecasts made with differing, randomly selected, initial states. Leith (ibid.) found that perhaps as few as eight deterministic runs were needed for satisfactory accuracy, although he refined the method by introducing a regression step and employing an optimum filtering technique. A further advantage of this Monte Carlo method is that it automatically avoids the theoretical (if not practical) difficulty of closure that exists for the stochastic dynamic equations. However, it should also be remarked that a Monte Carlo simulation has a problem that is not shared with the stochastic dynamic method, namely that of sampling approximations.

In this paper we propose another technique for avoiding the excessive computation time problem. We suggest that dramatic simplification of the stochastic dynamic equations is possible and appropriate for sufficiently small forecast errors (or, equivalently, sufficiently short forecasting times), by linearizing for small perturbations about the deterministic solution (the latter being given by normal numerical forecasting techniques).

The present paper develops this theme and describes its verification using a simple model equation for the atmospheric motions. Thus, in Section 2 we derive the linearized perturbation equations in a general form, and Section 3 describes their application to the so-called "minimum hydrodynamic equation" of Lorenz (1960), Section 4 details the results of some numerical tests of the linearization, and finally Section 5 discusses the import of the results and outlines the steps that are needed to develop the technique towards operational utility.

The work performed to date and reported upon here, covers verification of the validity of the concept of linearization. The second major requirement, namely the development of numerical algorithms for forecasting that take advantage of the simplification attendant on linearization, has yet to be undertaken, or indeed, proven to be possible.

2. Linearization of the stochastic dynamic equations

The linearization we introduce is simple in principle, although algebraically complicated in detail. The following outlines the main steps, omitting intermediate algebraic manipulations.

The canonical form we use is that already given in Section 1, eq. (1), i.e.

$$\frac{dx_i}{dt} = \sum_{j,k} a_{ijk}x_jx_k - \sum_j b_{ij}x_j + c_i \tag{9}$$

Let us assume that (9) is already expressed in terms of non-dimensional variables and coefficients. To typify the perturbation of x from a particular solution of eq. (9) we can define a single small parameter

$$s^2 = \frac{1}{N^2}\sum_{i,j} \sigma_{ij}|_{t=0} \tag{10}$$

which is thus representative of the magnitude of the initial error at time $t = 0$ if its distribution is multivariate normal.

We next introduce the perturbation variables

$$\tilde{\mu}_i = \mu_i$$
$$\tilde{\delta}_i = \frac{\delta_i}{s}$$
$$\tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{s^2}$$
$$\tilde{\tau}_{ijk} = \frac{\tau_{ijk}}{s^3} \tag{11}$$

etc., in which the moments μ_i, σ_{ij}, τ_{ijk} . . . are defined as in eqs. (2)–(5), and

$$\delta_i = \mu_i - \mu_i^{(0)} \tag{12}$$

where $\mu_i^{(0)} = x_i^{(0)}$ corresponds to the deterministic result for which all moments higher than the first are identically zero. The scaling introduced in eqs. (11) ensures that the newly defined variables are of order unity or less in the parameter s.

Substituting (11) into the moment eqs. (6)–(8), we get

$$\frac{d\tilde{\mu}_i}{dt} = \sum_{j,k} a_{ijk}(\tilde{\mu}_j\tilde{\mu}_k + s^2\tilde{\sigma}_{jk}) - \sum_j b_{ij}\tilde{\mu}_j + c_i \tag{13}$$

$$\frac{d\tilde{\delta}_i}{dt} = \sum_{j,k} a_{ijk}(\tilde{\mu}_j^{(0)}\tilde{\delta}_k + \tilde{\mu}_k^{(0)}\tilde{\delta}_j + s\tilde{\delta}_k\tilde{\delta}_j + s\tilde{\sigma}_{jk}) - \sum_j b_{ij}\tilde{\delta}_j \tag{14}$$

$$\begin{aligned}\frac{d\tilde{\sigma}_{ij}}{dt} = \sum_{k,l} [&a_{ikl}(\tilde{\mu}_k\tilde{\sigma}_{jl} + \tilde{\mu}_l\tilde{\sigma}_{jk} + s\tilde{\tau}_{jkl}) \\ &+ a_{jkl}(\tilde{\mu}_k\tilde{\sigma}_{il} + \tilde{\mu}_l\tilde{\sigma}_{ik} + s\tilde{\tau}_{ikl})] \\ &- \sum_k (b_{ik}\tilde{\sigma}_{jk} + b_{jk}\tilde{\sigma}_{ik})\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{d\tilde{\tau}_{ijk}}{dt} = \sum_{l,m} [&a_{ilm}(\tilde{\mu}_l\tilde{\tau}_{jkm} + \tilde{\mu}_m\tilde{\tau}_{jkl} - s\tilde{\sigma}_{lm}\tilde{\sigma}_{jk} + s\tilde{\lambda}_{jklm}) \\ &+ a_{jlm}(\tilde{\mu}_l\tilde{\tau}_{ikm} + \tilde{\mu}_m\tilde{\tau}_{ikl} - s\tilde{\sigma}_{lm}\tilde{\sigma}_{ik} + s\tilde{\lambda}_{iklm}) \\ &+ a_{klm}(\tilde{\mu}_l\tilde{\tau}_{ijm} + \tilde{\mu}_m\tilde{\tau}_{ijl} - s\tilde{\sigma}_{lm}\tilde{\sigma}_{ij} + s\tilde{\lambda}_{ijlm})] \\ &- \sum_l (b_{il}\tilde{\tau}_{jkl} + b_{jl}\tilde{\tau}_{ikl} + b_{kl}\tilde{\tau}_{ijl})\end{aligned} \tag{16}$$

The small perturbation assumption is next introduced by making the following expansions of the moments:

$$\tilde{\mu}_i = \tilde{\mu}_i^{(0)} + \sum_{n=1}^{\infty} s^n\tilde{\mu}_i^{(n)}$$
$$\tilde{\delta}_i = \sum_{n=1}^{\infty} s^{n-1}\tilde{\delta}_i^{(n)}$$
$$\tilde{\sigma}_{ij} = \sum_{n=1}^{\infty} s^{n-1}\tilde{\sigma}_{ij}^{(n)}$$
$$\tilde{\tau}_{ijk} = \sum_{n=1}^{\infty} s^{n-1}\tilde{\tau}_{ijk}^{(n)} \tag{17}$$

etc. The first two of these definitions imply that $\tilde{\mu}_i^{(n)} = \tilde{\delta}_i^{(n)}$, $n > 0$. Substituting into eqs. (14)–(16), there results

$$\begin{aligned}\sum_{n=1}^{\infty} s^n \frac{d\tilde{\mu}_i^{(n)}}{dt} = \sum_{j,k} a_{ijk}\Bigg[&\tilde{\mu}_j^{(0)}\sum_{n=1}^{\infty} s^n\tilde{\mu}_k^{(n)} + \tilde{\mu}_k^{(0)}\sum_{n=1}^{\infty} s^n\tilde{\mu}_j^{(n)} \\ &+ \sum_{n=1}^{\infty} s^n\tilde{\mu}_k^{(n)}\sum_{n=1}^{\infty} s^n\tilde{\mu}_j^{(n)} + s^2\sum_{n=1}^{\infty} s^{n-1}\tilde{\sigma}_{jk}^{(n)}\Bigg] \\ &- \sum_j b_{ij}\sum_{n=1}^{\infty} s^n\tilde{\mu}_j^{(n)}\end{aligned} \tag{18}$$

$$\begin{aligned}\sum_{n=1}^{\infty} s^{n-1}\frac{d\tilde{\sigma}_{ij}^{(n)}}{dt} = \sum_{l,k}\Bigg[&a_{ikl}\Bigg\{\Bigg(\tilde{\mu}_k^{(0)} + \sum_{n=1}^{\infty} s^n\tilde{\delta}_k^{(n)}\Bigg)\sum_{n=1}^{\infty} s^{n-1}\tilde{\sigma}_{jl}^{(n)} \\ &+ \Bigg(\tilde{\mu}_l^{(0)} + \sum_{n=1}^{\infty} s^n\tilde{\delta}_l^{(n)}\Bigg)\sum_{n=1}^{\infty} s^{n-1}\tilde{\sigma}_{jk}^{(n)} \\ &+ s\sum_{n=1}^{\infty} s^{n-1}\tilde{\tau}_{jkl}^{(n)}\Bigg\} \\ &+ a_{jkl}\Bigg\{\Bigg(\tilde{\mu}_k^{(0)} + \sum_{n=1}^{\infty} s^n\tilde{\delta}_k^{(n)}\Bigg)\sum_{n=1}^{\infty} s^{n-1}\tilde{\sigma}_{il}^{(n)} \\ &+ \Bigg(\tilde{\mu}_l^{(0)} + \sum_{n=1}^{\infty} s^n\tilde{\delta}_l^{(n)}\Bigg)\sum_{n=1}^{\infty} s^{n-1}\tilde{\sigma}_{ik}^{(n)} \\ &+ s\sum_{n=1}^{\infty} s^{n-1}\tilde{\tau}_{ikl}^{(n)}\Bigg\}\Bigg] \\ &- \sum_k\Bigg(b_{ik}\sum_{n=1}^{\infty} s^{n-1}\tilde{\sigma}_{jk}^{(n)} + b_{jk}\sum_{n=1}^{\infty} s^{n-1}\tilde{\sigma}_{ik}^{(n)}\Bigg)\end{aligned} \tag{19}$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} s^{n-1} \frac{d\tilde{\tau}_{ijk}^{(n)}}{dt} = \sum_{l,m} \Bigg[a_{ilm} \Bigg\{ \Bigg(\Bigg(\tilde{\mu}_l^{(0)} + \sum_{n=1}^{\infty} s^n \tilde{\delta}_l^{(n)} \Bigg)
&+ \sum_{n=1}^{\infty} s^{n-1} \tilde{\tau}_{jkm}^{(n)} + \Bigg(\tilde{\mu}_m^{(0)} \\
&+ \sum_{n=1}^{\infty} s^n \tilde{\delta}_m^{(n)} \Bigg) \sum_{n=1}^{\infty} s^{n-1} \tilde{\tau}_{jkl}^{(n)} \\
&- s \sum_{n=1}^{\infty} s^{n-1} \tilde{\sigma}_{lm}^{(n)} \sum_{n=1}^{\infty} s^{n-1} \tilde{\sigma}_{jk}^{(n)} \\
&+ s \sum_{n=1}^{\infty} s^{n-1} \tilde{\lambda}_{jklm}^{(n)} \Bigg\} + a_{jlm} \Bigg\{ \Bigg(\tilde{\mu}_l^{(0)} \\
&+ \sum_{n=1}^{\infty} s^n \delta_l^{(n)} \Bigg) \sum_{n=1}^{\infty} s^{n-1} \tau_{ikm}^{(n)} \\
&+ \Bigg(\tilde{\mu}_m^{(0)} \\
&+ \sum_{n=1}^{\infty} s^n \tilde{\delta}_m^{(n)} \Bigg) \sum_{n=1}^{\infty} s^{n-1} \tilde{\tau}_{ikl}^{(n)} \\
&- s \sum_{n=1}^{\infty} s^{n-1} \tilde{\sigma}_{lm}^{(n)} \sum_{n=1}^{\infty} s^{n-1} \tilde{\sigma}_{ik}^{(n)} \\
&+ s \sum_{n=1}^{\infty} s^{n-1} \tilde{\lambda}_{iklm}^{(n)} \Bigg\} \\
&+ a_{klm} \Bigg\{ \Bigg(\tilde{\mu}_l^{(0)} + \sum_{n=1}^{\infty} s^n \tilde{\delta}_l^{(n)} \Bigg) \sum_{n=1}^{\infty} s^{n-1} \tilde{\tau}_{ijm}^{(n)} \\
&+ \Bigg(\tilde{\mu}_m^{(0)} + \sum_{n=1}^{\infty} s^n \tilde{\delta}_m^{(n)} \Bigg) \sum_{n=1}^{\infty} s^{n-1} \tilde{\tau}_{ijl}^{(n)} \\
&- s \sum_{n=1}^{\infty} s^{n-1} \tilde{\sigma}_{lm}^{(n)} \sum_{n=1}^{\infty} s^{n-1} \tilde{\sigma}_{ij}^{(n)} \\
&+ s \sum_{n=1}^{\infty} s^{n-1} \tilde{\lambda}_{ijlm}^{(n)} \Bigg\} \Bigg] \\
&- \sum_l \Bigg(b_{il} \sum_{n=1}^{\infty} s^{n-1} \tilde{\tau}_{jkl}^{(n)} + b_{jl} \sum_{n=1}^{\infty} s^{n-1} \tilde{\tau}_{ikl}^{(n)} \\
&+ b_{kl} \sum_{n=1}^{\infty} s^{n-1} \tilde{\tau}_{ijl}^{(n)} \Bigg) \qquad (20)
\end{aligned}
$$

The governing equations for the various orders of approximation in s are then given by identifying terms of like order in s. In the general case, inspection of eqs. (18)–(20) shows that the lowest order approximations, $\tilde{\mu}_i^{(1)}$, $\tilde{\sigma}_{ij}^{(1)}$, $\tilde{\tau}_{ijk}^{(1)}$, depend only on the first order moment $\tilde{\mu}_i^{(0)}$. Successively higher orders of approximation in s introduce higher and higher order moments with every order of approximation being governed by a closed finite set of equations. We thus see that the small perturbation approach provides a rationale for closure of the set of moment equations by neglecting moments above a chosen order, the latter being determined by the magnitude of the perturbation.

Rather than document the equations for this general case we shall go directly to a particular approximation that has been used for closure of the moment equations, namely the so-called quasi-normal eddy-damped closure for the third-order moment. This approximation was first suggested by Orzag (1970) and applied to stochastic dynamic forecasting by Fleming (1970). In it the fourth-order moments are assumed to be related to the second-order moment as if the distribution were multivariate normal, that is by

$$
\lambda_{ijkl} = \sigma_{ij}\sigma_{kl} + \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk} \qquad (21)
$$

and this relationship is introduced into the equation for the third-order moments (20), which is further modified by adding a damping term $k\tau_{ijk}$ on the right-hand side. The coefficient k is an empirical constant determined by Fleming as 0.025 for best fitting in stochastic dynamic predictions.

In addition to using the quasi-normal approximation we shall particularize the equations to the usual type of initial conditions used in stochastic dynamic forecasting, that is, we choose the initial state for the moment equations to be given by

$$
\delta_i = \tau_{ijk} = 0; \quad \sigma_{ij} \equiv 0, \quad i \neq j; \quad t = 0. \qquad (22)
$$

which is in accord with an assumption of multivariate normality of the initial distribution of error about the mean $\mu_i^{(0)}$. Under these conditions, equations for the first- and second-order approximations, as derived from eqs. (18)–(20) by equating the multipliers of s, s^2, s^3 and s^4 to zero, are

$$
\begin{aligned}
&\tilde{\mu}_i^{(1)} \equiv 0 \\
&\frac{d\tilde{\mu}_i^{(2)}}{dt} = \sum_{j,k} a_{ijk} (\tilde{\mu}_j^{(0)} \tilde{\mu}_k^{(2)} + \tilde{\mu}_k^{(0)} \tilde{\mu}_j^{(2)} + \tilde{\sigma}_{jk}^{(1)}) - \sum_j b_{ij} \tilde{\mu}_j^{(2)} \qquad (23) \\
&\tilde{\mu}_i^{(3)} \equiv 0 \\
&\frac{d\tilde{\mu}_i^{(4)}}{dt} = \sum_{j,k} a_{ijk} (\tilde{\mu}_j^{(0)} \tilde{\mu}_k^{(4)} + \tilde{\mu}_k^{(0)} \tilde{\mu}_j^{(4)} + \tilde{\mu}_k^{(2)} \tilde{\mu}_j^{(2)} \\
&\qquad + \tilde{\sigma}_{jk}^{(3)}) - \sum_j b_{ij} \mu_j^{(4)} \qquad (24)
\end{aligned}
$$

$$\frac{d\tilde{\sigma}_{ij}^{(1)}}{dt} = \sum_{l,k} [a_{ikl}(\tilde{\mu}_k^{(0)}\tilde{\sigma}_{jl}^{(1)} + \tilde{\mu}_l^{(0)}\tilde{\sigma}_{jk}^{(1)}) + a_{jkl}(\tilde{\mu}_k^{(0)}\tilde{\sigma}_{il}^{(1)} + \tilde{\mu}_l^{(0)}\tilde{\sigma}_{ik}^{(1)})] - \sum_k (b_{ik}\tilde{\sigma}_{jk}^{(1)} + b_{jk}\tilde{\sigma}_{ik}^{(1)}) \tag{25}$$

$$\tilde{\sigma}_{ij}^{(2)} \equiv 0$$

$$\frac{d\tilde{\sigma}_{ij}^{(3)}}{dt} = \sum_{l,k} [a_{ikl}(\tilde{\mu}_k^{(0)}\tilde{\sigma}_{jl}^{(3)} + \tilde{\mu}_l^{(0)}\tilde{\sigma}_{jk}^{(3)} + \tilde{\mu}_k^{(2)}\tilde{\sigma}_{jl}^{(1)} + \tilde{\mu}_l^{(2)}\tilde{\sigma}_{jk}^{(1)} + \tilde{\tau}_{jkl}^{(2)}) + a_{jkl}(\tilde{\mu}_k^{(0)}\tilde{\sigma}_{il}^{(3)} + \tilde{\mu}_l^{(0)}\tilde{\sigma}_{ik}^{(3)} + \tilde{\mu}_k^{(2)}\tilde{\sigma}_{il}^{(1)} + \tilde{\mu}_l^{(2)}\tilde{\sigma}_{ik}^{(1)} + \tilde{\tau}_{ikl}^{(2)})] - \sum_k (b_{ik}\tilde{\sigma}_{jk}^{(3)} + b_{jk}\tilde{\sigma}_{ik}^{(3)}) \tag{26}$$

$$\tilde{\tau}_{ijk}^{(1)} \equiv 0$$

$$\frac{d\tilde{\tau}_{ijk}^{(2)}}{dt} = \sum_{l,m} [a_{ilm}(\tilde{\mu}_l^{(0)}\tilde{\tau}_{jkm}^{(2)} + \tilde{\mu}_m^{(0)}\tilde{\tau}_{jkl}^{(2)} + \tilde{\sigma}_{jl}^{(1)}\tilde{\sigma}_{km}^{(1)} + \tilde{\sigma}_{jm}^{(1)}\tilde{\sigma}_{kl}^{(1)}) + a_{jlm}(\tilde{\mu}_l^{(0)}\tilde{\tau}_{ikm}^{(2)} + \tilde{\mu}_m^{(0)}\tilde{\tau}_{ikl}^{(2)} + \tilde{\sigma}_{il}^{(1)}\tilde{\sigma}_{km}^{(1)} + \tilde{\sigma}_{im}^{(1)}\tilde{\sigma}_{kl}^{(1)}) + a_{klm}(\tilde{\mu}_l^{(0)}\tilde{\tau}_{ijm}^{(2)} + \tilde{\mu}_m^{(0)}\tilde{\tau}_{ijl}^{(2)} + \tilde{\sigma}_{il}^{(1)}\tilde{\sigma}_{jm}^{(1)} + \tilde{\sigma}_{im}^{(1)}\tilde{\sigma}_{jl}^{(1)})] - \sum_k (b_{ik}\tilde{\tau}_{jkl}^{(2)} + b_{jl}\tilde{\tau}_{ikl}^{(2)} + b_{kl}\tilde{\tau}_{ijl}^{(2)}) - 0.025\ \tilde{\tau}_{ijk}^{(2)} \tag{27}$$

In the above $\tilde{\mu}_i^{(1)}$ and $\tilde{\tau}_{ijk}^{(1)}$ are identically zero because of homogeneity.

At this stage we can see that the parameter s can be eliminated from explicit consideration by defining the new perturbation variables

$$\mu_i^{(2)} = s^2\tilde{\mu}_i^{(2)}, \quad \mu_i^{(4)} = s^4\tilde{\mu}_i^{(4)}$$
$$\sigma_{ij}^{(1)} = s^2\tilde{\sigma}_{ij}^{(1)}, \quad \sigma_{ij}^{(3)} = s^4\tilde{\sigma}_{ij}^{(3)}$$
$$\tau_{ijk}^{(2)} = s^4\tau_{ijk}^{(2)} \tag{28}$$

so that, from (11) and (17),

$$\mu_i = \mu_i^{(0)} + \mu_i^{(2)} + \mu_i^{(4)} + \cdots$$
$$\sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(3)} + \cdots$$
$$\tau_{ijk} = \tau_{ijk}^{(2)} + \cdots \tag{29}$$

and the equations satisfied by these newly defined quantities are obtained from (23)–(27) simply by omitting the tilde. It is the latter form of the equations that we shall in fact be employing in the next section where application is made to a particular atmospheric model.

3. Application to the minimum hydrodynamic equations

In this section we apply the linearized forms (23)–(27) to a particular spectral model developed by Lorenz (1960). The so-called "minimum hydrodynamic equations" which define this model have been used a number of times to evaluate the stochastic dynamic method (Epstein, 1969; Pitcher, 1977), and are particularly convenient because of their relative simplicity (deterministic solutions are available in closed form); yet they retain the essential non-linear wave interaction features that characterize the transfer of energy and momentum between zonal and eddy motions, and thus the growth of uncertainty in numerical prediction.

We shall be using the same basic flow configurations for testing the linearized approximation as used in earlier stochastic dynamic studies, and hence we will be brief in discussions of their physical significance, referring the reader to Lorenz's original paper (1960) and to the useful summary of the mathematical features of the equations written by Pitcher (1974).

Lorenz's "minimum hydrodynamic equations" in fact correspond to two-dimensional barotropic flow in which the vorticity ζ is expanded in a double Fourier series over a horizontal rectangular region. If the series is truncated at wave number 1, a solution in terms of only three spectral components is available:

$$\zeta = A_1 \cos ly + A_2 \cos kx + 2A_3 \sin ly \sin kx \tag{30}$$

where $k = 2\pi/L_x$ and $l = 2\pi/L_y$, L_x and L_y being the disturbance wavelengths in the x and y directions respectively. The first term in eq. (30) corresponds to the zonal flow, so that L_x is the distance between successive zonal wind maxima, whilst the last two correspond to the superimposed "eddy" motions. The spectral coefficients A_1, A_2 and A_3 are governed by the following simple non-linear equations:

$$\frac{dA_1}{dt} = C_1 A_2 A_3$$

$$\frac{dA_2}{dt} = C_2 A_1 A_3$$

$$\frac{dA_3}{dt} = C_3 A_2 A_1 \qquad (31)$$

where

$$c_1 = -[\alpha(\alpha^2 + 1)]^{-1},$$
$$c_2 = \alpha^3(\alpha^2 + 1)^{-1},$$
$$c_3 = -(\alpha^2 - 1)/2\alpha \qquad (32)$$

and $\alpha = k/l$.

Equations (31) constitute a particularly simple form of the canonical equations used in Section 3 in deriving the linearized stochastic representation when the following special values for a_{ijk} are selected:

$$a_{ijk} = 0 \quad \text{when} \quad i = j \quad \text{or} \quad i = k \quad \text{or} \quad j = k;$$
$$a_{ijk} = \tfrac{1}{2}c_i, \quad i \neq j \neq k; \quad b_{ij} = 0.$$

We can thus immediately write down the linearized moment equations corresponding to (31) using eqs. (23)–(27). They are as follows:

$$\frac{d\mu_i^{(2)}}{dt} = \sum_{j,k,\neq i} \tfrac{1}{2}c_i(\mu_j^{(0)}\mu_k^{(2)} + \mu_k^{(0)}\mu_j^{(2)} + \sigma_{jk}^{(1)})$$

$$\frac{d\sigma_{ij}^{(1)}}{dt} = \sum_{l,k\neq i} \tfrac{1}{2}c_i(\mu_k^{(0)}\sigma_{jl}^{(1)} + \mu_l^{(0)}\sigma_{jk}^{(1)}) + \sum_{l,k\neq j} \tfrac{1}{2}c_j(\mu_k^{(0)}\sigma_{il}^{(1)} + \mu_l^{(0)}\sigma_{ik}^{(1)}) \qquad (33)$$

$$\frac{d\mu_i^{(4)}}{dt} = \sum_{j,k,\neq i} \tfrac{1}{2}c_i(\mu_j^{(0)}\mu_k^{(4)} + \mu_k^{(0)}\mu_j^{(4)} + \mu_j^{(2)}\mu_k^{(2)} + \sigma_{jk}^{(4)})$$

$$\frac{d\sigma_{ij}^{(3)}}{dt} = \sum_{l,k\neq i} \tfrac{1}{2}c_i(\mu_k^{(0)}\sigma_{jl}^{(3)} + \mu_l^{(0)}\sigma_{jk}^{(3)} + \mu_l^{(2)}\sigma_{kj}^{(1)} + \mu_k^{(2)}\sigma_{lj}^{(1)} + \tau_{jlk}^{(2)}) + \sum_{l,k\neq j} \tfrac{1}{2}c_j(\mu_k^{(0)}\sigma_{il}^{(3)} + \mu_l^{(0)}\sigma_{ik}^{(3)} + \mu_l^{(2)}\sigma_{ik}^{(1)} + \mu_k^{(2)}\sigma_{il}^{(1)} + \tau_{ikl}^{(2)})$$

$$\frac{d\tau_{ijk}^{(2)}}{dt} = \sum_{l,m,\neq i} \tfrac{1}{2}c_i(\mu_l^{(0)}\tau_{jkm}^{(2)} + \mu_m^{(0)}\tau_{jkl}^{(2)} + \sigma_{jl}^{(1)}\sigma_{km}^{(1)} + \sigma_{jm}^{(1)}\sigma_{kl}^{(1)}) + \sum_{l,m,\neq j} \tfrac{1}{2}c_j(\mu_l^{(0)}\tau_{ikm}^{(2)} + \mu_m^{(0)}\tau_{ikl}^{(2)} + \sigma_{il}^{(1)}\sigma_{km}^{(1)} + \sigma_{im}^{(1)}\sigma_{kl}^{(1)}) + \sum_{l,m,\neq k} \tfrac{1}{2}c_k(\mu_l^{(0)}\tau_{ijm}^{(2)} + \mu_m^{(0)}\tau_{ijl}^{(2)} + \sigma_{il}^{(1)}\sigma_{jm}^{(1)} + \sigma_{im}^{(1)}\sigma_{jl}^{(1)}) - 0.025\,\tau_{ijk}^{(2)} \qquad (34)$$

where μ_i, σ_{ij} and τ_{ijk} are now moments of the spectral coefficients, A_i.

The first-order linearized result involving only first- and second-order moments, requires solution of the two coupled eqs. (33). The second-order solution requires evaluation of the third-order moments, $\tau_{ijk}^{(2)}$, so that three coupled eqs. (34) are then involved.

Equation (30) and its linearized approximations are capable of describing qualitatively a number of atmospheric phenomena, including index cycle fluctuations (see Lorenz, 1960). The nature of the atmospheric motions they depict depends on both the choice of the numerical coefficients k and l in eq. (30) and on the initial state. For the former we select $k/l = \alpha = 2$, so that, from eq. (32)

$$c_1 = -0.1, \quad c_2 = 1.6, \quad c_3 = -0.75 \qquad (35)$$

which correspond to the stable flow situations treated in earlier tests of the stochastic dynamic method by Epstein (1969), and repeated here for the linearized approximation. Scaling to global atmospheric motions is obtained by setting $2\pi/l$ = 10,000 km, and then, with $\alpha = 2$, $2\pi/k$ = 5000 km. In the next section we shall describe the results of computing numerical solutions of eqs. (33) and (34) with the c values given by eq. (35) and two choices of initial conditions.

4. Numerical tests of the linearized approximation

Extensive comparisons of the effectiveness of the truncated stochastic dynamic equations have been made by both Epstein (1969) and Pitcher (1974) with $\alpha = 2$ [eq. (35)] and the following initial conditions at $t = 0$:

$$\mu_1 = 0.12; \quad \mu_2 = 0.24; \quad \mu_3 = 0$$

$$\sigma_{ij} = 0, \quad i \neq j; \quad \sigma_{ii} = 10^{-4} \tag{36}$$

$$\mu_1 = 0.12; \quad \mu_2 = 0; \quad \mu_3 = 0.333$$

$$\sigma_{ij} = 0, \quad i \neq j; \quad \sigma_{11} = \sigma_{22} = 4 \times 10^{-4};$$

$$\sigma_{33} = 1 \times 10^{-4} \tag{37}$$

In both of these instances "exact" solutions (i.e. solutions that do not involve closure assumptions) are available, in the form of Monte Carlo solutions (Epstein, ibid.) and Lagrangian quadrature analysis (Pitcher, 1974); these provide the reference for evaluation of the approximate stochastic results.

The first case, as given by eq. (36), corresponds to an initial maximum zonal wind of 64 km/hr [the first term in eq. (30)] with a standard deviation of 5 km/hr. Initially the kinetic energy is distributed equally between the zonal flow and the "eddy flow" [the second and third terms in eq. (30)]. The deterministic solution, corresponding to conditions (36) but with $\sigma_{ij} \equiv 0$ is

$$\mu_1^{(0)} = A_1^* \, dn[h(t - t^*), k_0]$$

$$\mu_2^{(0)} = A_2^* \, sn[h(t - t^*), k_0]$$

$$\mu_3^{(0)} = A_3^* \, cn[h(t - t^*), k_0] \tag{38}$$

where sn, dn, cn are the Jacobian Elliptic functions and the numerical coefficients in (38) have the values

$$A_1^* = (0.018)^{1/2}$$
$$A_2^* = 0.24$$
$$A_3^* = (0.027)^{1/2}$$
$$h = (0.0216)^{1/2}$$
$$k_0 = (0.2)^{1/2}$$

$-ht^* = K(0.2)$, the complete elliptic integral of the first kind.

The zonal flow, given by the first spectral component, is in this case always unidirectional (since the dn function is never negative), and we are dealing here with a relatively mild vacillation of energy between the zonal and eddy flows. The initial uncertainties, as given in eq. (36), are also modest (comparable with observational synoptic errors) and result in a deterministic forecast of good quality, as will be illustrated shortly.

We shall first discuss the results of the stochastic calculations when the equations are truncated after the second-order moments, yielding, for the linearized approximation, eqs. (33). The latter have been solved using a fourth-order Runge–Kutta forward-marching integration scheme with a $1\frac{1}{2}$ h time step. The non-linear results depicted in the illustrations were taken from Pitcher's work (1977).

Figs. 1 and 2 for the second and third spectral coefficients demonstrate the efficacy of the deterministic solutions [as given by eqs. (38)]; they show little departure from the (exact) stochastic result that allows for the initial data error, for a week or more of forecast time. The truncated stochastic solutions are even better, and these remain good for a period of two weeks (much longer, of course, than the period of validity of any current numerical weather forecasting scheme). Significant differences between corresponding linear and non-linear solutions also only appear after some two weeks.

The situation is less perfect for the variance predictions, shown in Figs. 3 and 4. Here the standard deviations for the second- and third-order spectral components, $\sigma_{22}^{1/2}$ and $\sigma_{33}^{1/2}$, computed with truncated stochastic equations, agree with the exact solutions for only some three days, but again the linearization itself can be seen to be good[1] for much longer, perhaps eight to nine days. Thereafter the linear approximation gets rapidly worse, and shows no sign of levelling off to a constant value as does the exact solution.

We therefore find from this test case that the linear approximation is undoubtedly as good as the non-linear stochastic solution that neglects 3rd-order moments, for the range of validity of the latter. Since no one suggests that the stochastic dynamic method can be sensibly expanded to include higher-order moments (although Fleming, 1971a,b asserts that it would be desirable to do so if one could), we conclude that linearization is a perfectly satisfactory approach for simplifying the method. But additionally, since we anticipate the linearization to be an effective means for reducing computation time, we can also expect it to provide a practical means for calculating higher-order closure solutions of the stochastic dynamic moment equations. Figs. 5 and 6 illustrate the

[1] As compared with the non-linear approximate stochastic prediction.

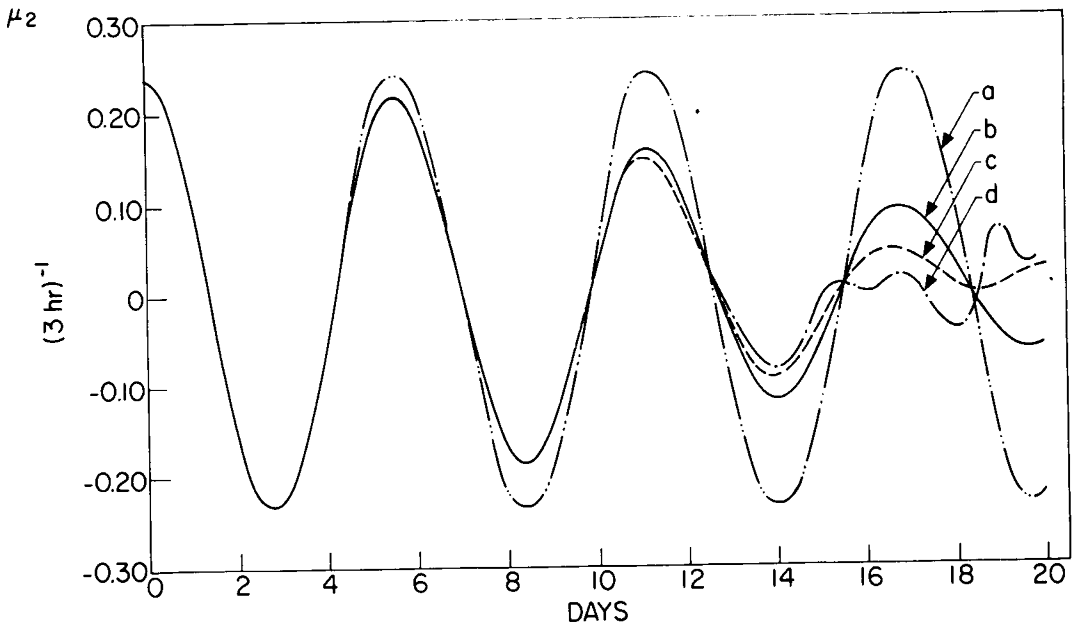

Fig. 1. First case. Mean value predictions versus prediction time. Truncation after the second order. a: deterministic (zero initial error) solution; b: exact stochastic solution; c: approximate stochastic non-linear solution; d: approximate stochastic linear solution.

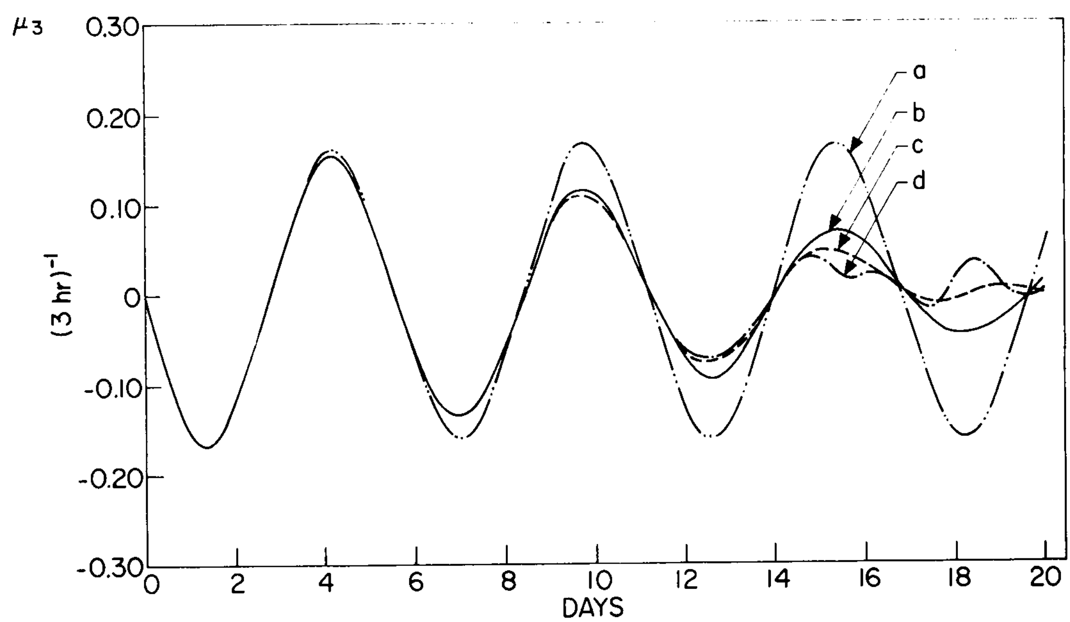

Fig. 2. First case. Mean value predictions versus prediction time. Truncation after the second order (a, b, c, d as in Fig. 1).

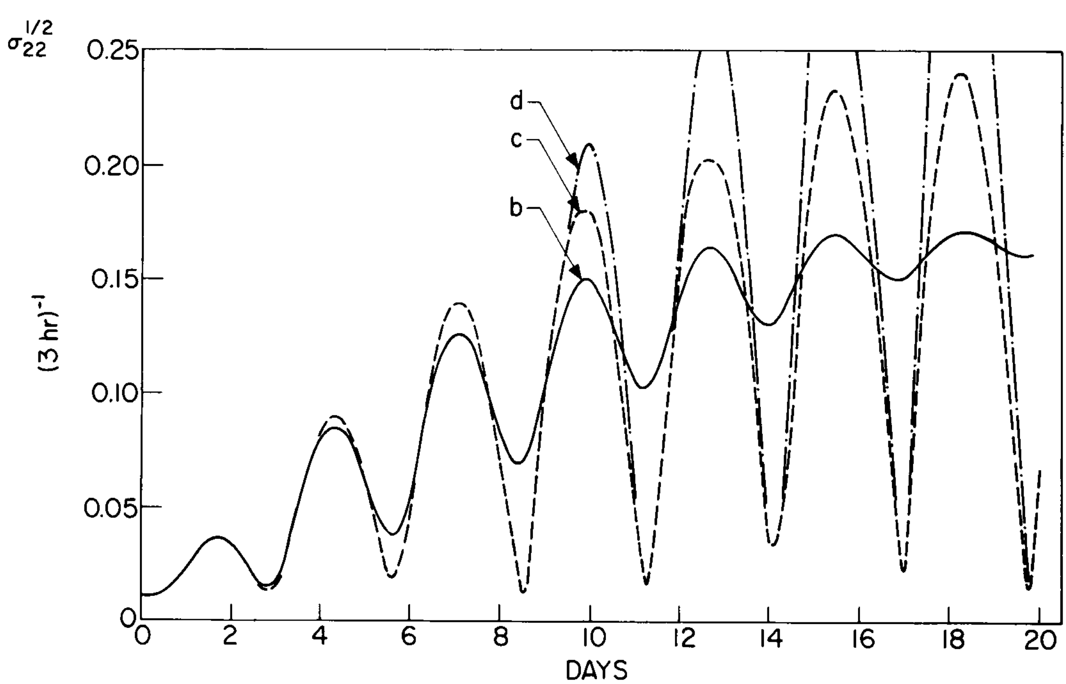

Fig. 3. First case. Standard deviation predictions versus prediction time. Truncation after the second order (b, c, d as in Fig. 1).

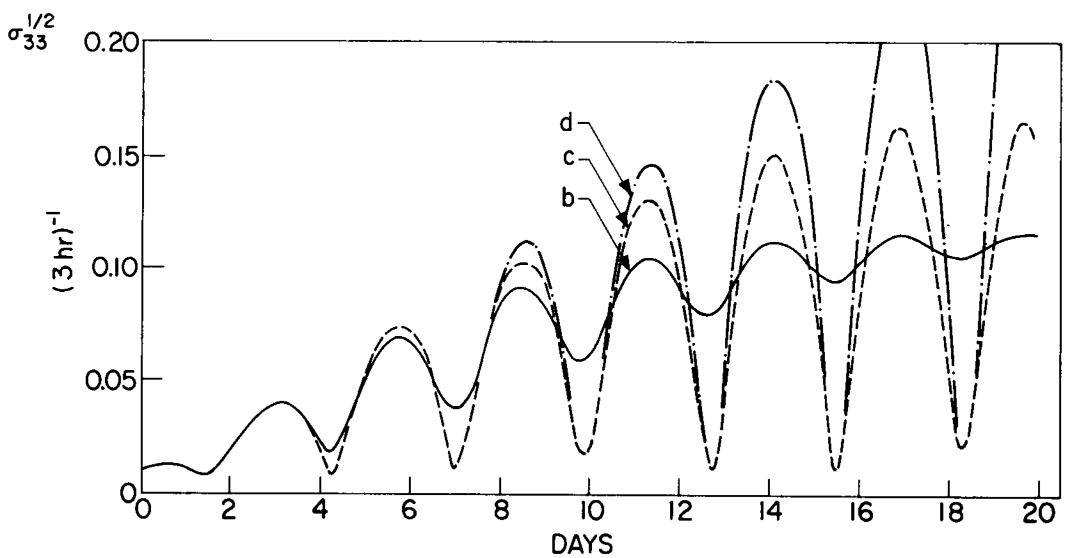

Fig. 4. First case. Standard deviation predictions versus prediction timê. Truncation after the second order (b, c, d as in Fig. 1).

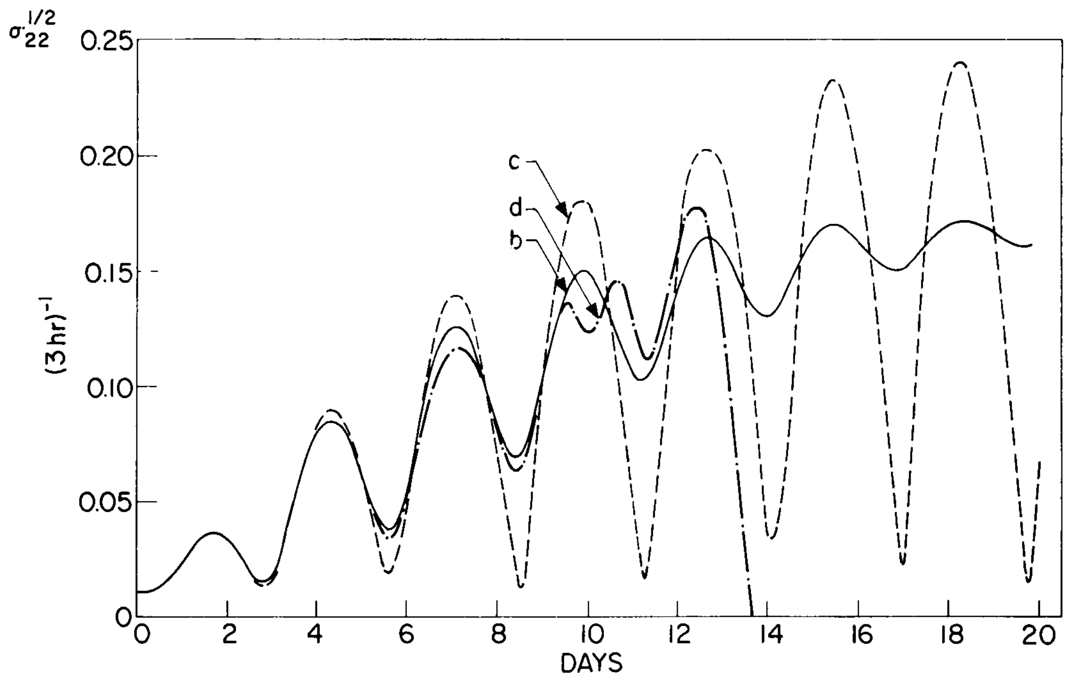

Fig. 5. First case. Standard deviation predictions versus prediction time. Truncation after the third order (quasi-normal eddy damped approximation) (b, c, d as in Fig. 1).

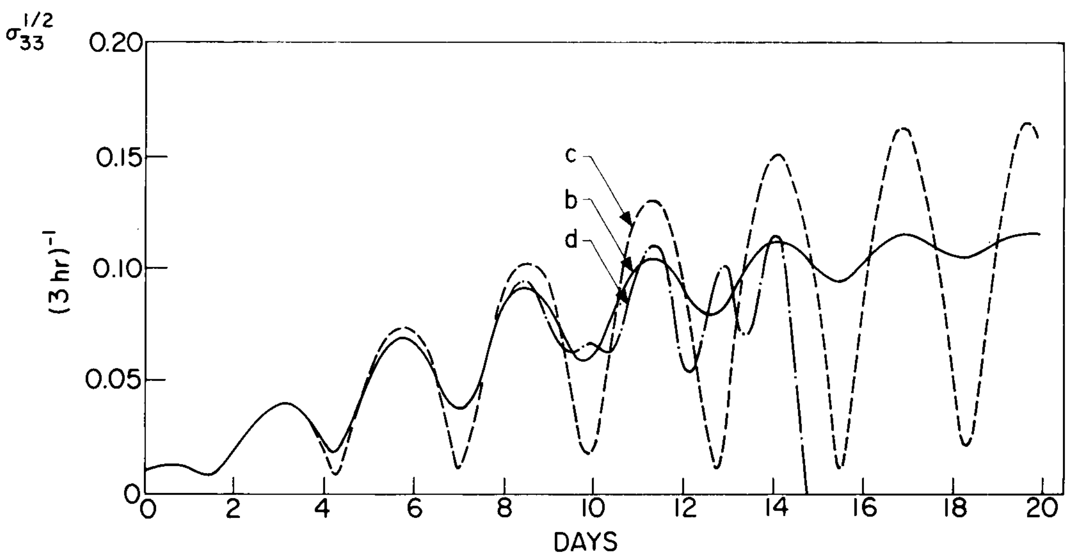

Fig. 6. First case. Standard deviation predictions versus prediction time. Truncation after the third order (quasi-normal eddy damped approximation) (b, c, d as in Fig. 1).

improvement attendant upon using the linearized formulation with truncation after third order, i.e. the quasi-normal eddy-damped equations given by eq. (34). We see from these that higher-order closure, even with linearized equations, gives much better results than the non-linear lower-order approximations for periods of over 10 days, and in rough agreement with the exact solutions for up to two weeks. Runaway divergence of the linear predictions occurs at about 14 days, but we do not know as yet whether this is a fault of the linearization or a result of too large a step-size in the numerical integration.

We now go on to consider the second test case, with initial conditions as given by eq. (37). This is a much less stable condition than in the first case, as can be seen from the deterministic solutions, which are as follows:

$$\mu_{10} = 0.12\ cn(ht, k_0)$$
$$\mu_{20} = 0.48\ sn(ht, k_0)$$
$$\mu_{30} = 0.333\ dn(ht, k_0)$$

with

$$h = 0.1332$$
$$k_0 = 0.986887491 \qquad (39)$$

In this instance the zonal flow, as given by the first equation in (39), and illustrated in Fig. 7, oscillates regularly from east to west. This is the case considered by Epstein (1969) as quite atypical of real conditions and the most severe test for the approximate stochastic dynamic solutions.

Figs. 7 and 8 compare the results for the first-order moments. Agreement between the linear and non-linear predictions, both employing second-order truncation, is satisfactory for only two days, by which time the stochastic solutions differ radically from the deterministic forecast, implying that even in this short time the effect of initial error is important and should be taken into account. Note how markedly this differs from the first test case we have described. After three days the linear solution rapidly diverges, a behavior that may be the result of a break-down of the small perturbation requirement for linearization.

Fig. 9 shows the corresponding results for the standard deviations of the spectral component $\sigma_{11}^{1/2}$. Again, the linearization is satisfactory only for forecast times between two and three days. The approximate non-linear stochastic solution is also valid only for three days or less, and indeed at some times it appears that the linearization is in

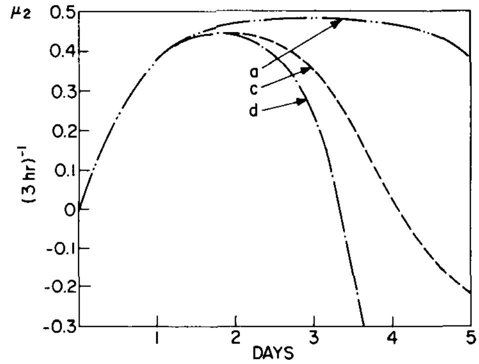

Fig. 8. Second case. Mean value predictions versus prediction time. Truncation after the second order (a, c, d as in Fig. 1).

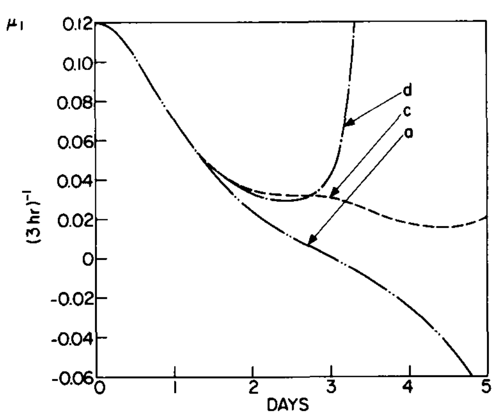

Fig. 7. Second case. Mean value predictions versus prediction time. Truncation after the second order (a, c, d as in Fig. 1).

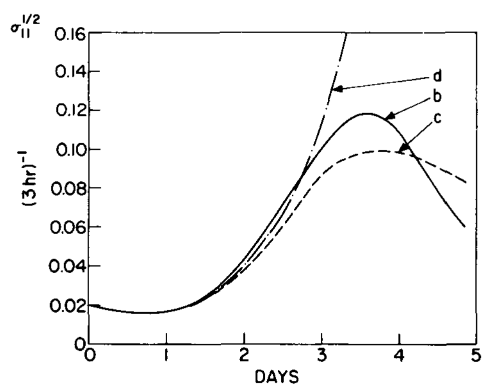

Fig. 9. Second case. Standard deviation predictions versus prediction time. Truncation after the second order (b, c, d as in Fig. 1).

better agreement with the exact result than the non-linear calculation. The "exact" results shown in Fig. 9 was, in fact, computed with Monte Carlo runs by Epstein (1969), who was also responsible for the non-linear stochastic dynamic calculations reproduced here.

In general it appears that the linearization is quite robust. Study of both cases suggests that it tolerates departures of some 50% between the stochastic and deterministic values before gross errors occur. Thus, at this 50% level, the error from linearization is quite small for the first-order moments, of the order of 10%, and is under 50% for the second-order moments when the moment equations are truncated after the third order. These results enable an evaluation of the range of accuracy of any particular linearized stochastic dynamic forecast to be made without appeal to knowledge of the exact stochastic dynamic solution.

5. Conclusions

The two test cases evaluated for assessment of the efficacy of the linearized approximation revealed no conditions under which the linearized predictions were worse than those obtained with the standard non-linear stochastic dynamic method. These comparisons were made with solutions that ignored the effect of third-order moments, and when the latter were included in the linearized approach its results were in considerably better agreement with the exact solutions than were the predictions of the standard approach.

The small perturbation expansion upon which the linearization is founded, provides a logical basis for closure of the stochastic dynamic moment equations by discarding moments above a chosen order, and thus suggests that a truncated non-linear formulation will not in general be an improvement over the corresponding linear expansion. This general conclusion is confirmed by the limited test cases reported upon here, and indicates the potential for computing higher-order moments with a linearized approximation.

We conclude that there is no reason for not using the linearized approximation suggested in this paper in stochastic dynamic prediction. The next requirement is to translate the obvious analytic simplicity attendant upon linearization into effective numerical algorithms that are much faster than those needed for computation of solutions of the non-linear system. The feasibility of such an objective has yet to be demonstrated; its successful attainment should pave the way to operational implementation of stochastic dynamic forecasting, since serious consideration has hitherto been hampered by excessive computation time.

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ПРИБЛИЖЕНИЕ МАЛЫХ ВОЗМУЩЕНИЙ ДЛЯ СТОХАСТИЧЕСКОГО ДИНАМИЧЕСКОГО ПРОГНОЗА ПОГОДЫ

Стохастический динамический метод наилучшего прогнозирования погоды и метод оценивания дисперсии прогноза требуют таких сложных вычислений, что это является достаточным основанием для того, чтобы сомневаться в его пользе для оперативной практики. В настоящей работе предлагается для сокращения вычислительного времени использовать линейную аппроксимацию для возмущенной системы стохастических динамических уравнений. Линеаризация основана на предположении о малости отклонений стохастических динамических решений от обычного детерминированного прогноза. Полученная в результате линейная схема была испытана путем оценивания качества прогноза для простой математической модели атмосферы, описываемой "максимально упрощенными гидродинамическими уравнениями" Лоренца.

Обнаружено, что линеаризованные уравнения дают такой же эффект как и обычный нелинейный стохастический динамический метод, в котором при аппроксимации точного стохастического динамического решения пренебрегают третьими моментами. В настоящей работе не обсуждается проблема разработки эффективных вычислительных алгоритмов, использующих преимущества упрощений, обусловленных линеаризацией; об этом будет сообщено по мере продвижения работы.