Tellus (1978), *30*, 313–322

On nonlinear cascades of atmospheric energy and enstrophy in a two-dimensional spectral index

By TSING-CHANG CHEN,[1] *Joint Institute for Acoustic and Flight Science, George Washington University—Langley Research Center, NASA, Hampton, Virginia 23665, U.S.A.* and AKSEL WIIN-NIELSEN, *European Center for Medium Range Weather Forecasts, Bracknell, Berkshire, England*

(Manuscript received July 13; in final form December 2, 1977)

ABSTRACT

Diagnostic analyses of the spectra and nonlinear exchange of the kinetic energy (K), enstrophy (E), and available potential energy (A) are made using the degree of the Legendre polynomial expansion, n, as a two-dimensional spectral index. Results are presented for a data set covering the period from December 19, 1971 to February 27, 1972. The spectra of K, E, and A have slopes close to -3, -1, and -3, respectively for large n. K and E cascade nonlinearly from the scales of intermediate n to the scales of large and small n. A cascades from $n = 1$ and 2, particularly the latter, to scales with larger n. The fluxes of nonlinear exchanges of K, E, and A are also evaluated to investigate if an inertial subrange exists. No firm conclusion for the existence of this subrange can be made.

1. Introduction

Diagnostic study of energy spectra and nonlinear exchange of energy in the large-scale atmospheric flow is an interesting meteorological problem. A -3 power law over a certain range of kinetic and available potential energy spectra has been observed, e.g. Wiin-Nielsen (1967). The kinetic energy has been found to be transferred from the intermediate scales to the planetary and small scales (Saltzman, 1970; Steinberg et al., 1971), while the enstrophy and available potential energy are transferred from large to small scales (Steinberg et al., 1971).

The nonlinear exchanges of kinetic energy and enstrophy among different wave components have been investigated by Fjørtoft (1953) in a theoretical analysis of nonlinear interaction of a barotropic nondivergent flow in two-dimensional motion. From theoretical studies of two-dimensional turbulence, Kraichnan (1967) and Leith (1968) suggested that an inertial subrange corresponding to a -3 power law may exist. It is characterized by a constant flux of enstrophy and a vanishing flux of kinetic energy through the spectrum. These theoretical proposals were shown to exist by Lilly (1969 and 1972) in a numerical simulation of two-dimensional turbulence. Furthermore, Charney (1971) showed analytically that the available potential energy spectrum should possess a -3 slope over a spectral range where the kinetic energy spectrum displays the same sort of slope. For models with low vertical resolution, Merilees and Warn (1972) show that the available potential energy should obey a -5 power law, while the kinetic energy should follow a -3 power law.

Efforts have been made observationally (Steinberg et al., 1971) and numerically (Steinberg, 1973; Barros and Wiin-Nielsen, 1974) to investigate the possible existence of an inertial subrange in the large-scale atmospheric flow. Difficulties were encountered most likely because the one-dimensional spectral index of conventional longitudinal wavenumber is used in those studies, while the theory of Fjørtoft, Kraichnan, and Leith are developed in a two-dimensional flow.

In order to take the anisotropy of atmospheric data into account and to extend the conventional empirical study of atmospheric energy spectra and nonlinear exchanges of atmospheric energy and

[1] Present affiliation: Department of Meteorology, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, U.S.A.

enstrophy to a two-dimensional domain, the degree of Legendre polynomial expansion, n, has been proposed by Baer (1972 and 1974) and Wiin-Nielsen (1972) to replace the one-dimensional longitudinal wavenumber. Baer (1972) indicated that the inertial subrange possessing a -3 slope in the kinetic energy spectra might exist between $14 \leqslant n \leqslant 25$, while Wiin-Nielsen (1972) suggested the evaluation of the nonlinear cascades of the kinetic energy and enstrophy using the two-dimensional spectral index.

A data set for December 19, 1971–February 27, 1972, consisting of 10 levels (1000, 850, 700, 500, 400, 300, 250, 200, 150, and 100 mb), has been used to analyze the spectra of kinetic energy, enstrophy, and available potential energy, and their nonlinear exchanges as a function of the two-dimensional spectral index. Fluxes of nonlinear exchanges are also evaluated to verify the possible existence of an inertial subrange in the large-scale atmospheric flow following Leith and Kraichnan's criteria. However, the present investigation can be regarded as an extension of Wiin-Nielsen's (1972) and Steinberg et al.'s studies.

2. Formulation

The kinetic energy (K), enstrophy (E), and available potential energy (A) per unit area of non-divergent flow can be expressed in terms of stream-function (ψ) and temperature (T) as follows:

$$K = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\frac{1}{2}\nabla\psi\cdot\nabla\psi\, d\lambda\, d\mu = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}-\frac{1}{2}\psi\zeta\, d\lambda\, d\mu \tag{1}$$

$$E = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\frac{1}{2}\zeta^2\, d\lambda\, d\mu \tag{2}$$

$$A = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\frac{1}{2\bar{\sigma}}\left(\frac{R}{\bar{P}}\right) T^2\, d\lambda\, d\mu \tag{3}$$

where $\mu = \sin\phi$, ϕ = latitude, and other notation for physical variables is conventional. The vorticity and thermodynamic equations of nondivergent, adiabatic, frictionless atmospheric motion are,

$$\frac{\partial\zeta}{\partial t} = -J(\psi,\zeta) - \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} \tag{4}$$

and

$$\frac{\partial T}{\partial t} = -J(\psi, T) \tag{5}$$

where J's are Jacobians.

Streamfunction and temperature fields can be expressed in a series of the form,

$$\psi(\lambda,\mu,t) = \sum_{n=0}^{\infty}\sum_{l=-n}^{n}\psi_n^l(t)P_n^l e^{il\lambda} \tag{6}$$

$$T(\lambda,\mu,t) = \sum_{n=0}^{\infty}\sum_{l=-n}^{n}T_n^l(t)P_n^l(\mu)e^{il\lambda} \tag{7}$$

Since $\zeta = \nabla^2\psi$, we also have,

$$\zeta(\lambda,\mu,t) = \sum_{n=0}^{\infty}\sum_{l=-n}^{n}\frac{-n(n+1)}{a^2}\psi_n^l(t)P_n^l(\mu)e^{il\lambda} \tag{8}$$

where $P_n^l(\mu)$ is the associated Legendre function. Baer (1972) and Wiin-Nielsen (1972) proposed to use the degree of the Legendre polynomial, n, as a two-dimensional spectral index for the atmospheric energy and enstrophy spectra. Substituting (6)–(8) into (1)–(3), and using n as spectral index, we can express K, E, and A, in the forms

$$K = \sum_{n=0}^{\infty}K_n = \sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}K_n^l\right\} = \sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\frac{n(n+1)}{2a^2}(2-\delta_{0l})\,|\psi_n^l|^2\right\} \tag{9}$$

$$E = \sum_{n=0}^{\infty}E_n = \sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}E_n^l\right\} = \sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\frac{n^2(n+1)^2}{2a^4}(2-\delta_{0l})\,|\psi_n^l|^2\right\} \tag{10}$$

$$A = \sum_{n=0}^{\infty}A_n = \sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}A_n^l\right\} = \sum_{n=0}^{\infty}\left\{\frac{1}{2\sigma}\left(\frac{R}{C_p}\right)(2-\delta_{0l})\,|T_n^l|^2\right\} \tag{11}$$

where δ is a delta function, that is, $\delta_{0l} = 1$ (or 0) as $l = 0$ (or $l \neq 0$). Comparing (8) and (9), we find $E_n^l = [n(n+1)/a^2]K_n^l$. Notice that the summations over the two scale indices l and n may be interchanged. By doing so, one can express K, E, and A in terms of the longitudinal wavenumber.

Substituting (6)–(8) into the time derivative of (1)–(3) and using the vorticity and thermodynamic equations, (4) and (5), we obtain the n spectral equations of K, E, and A as follows:

$$\frac{dK_n}{dt} = I_n = \sum_{l=0}^{n} I_n^l = \sum_{l=0}^{n} \left[\frac{n(n+1)}{2a^2}\right] \frac{1}{2} (2 - \delta_{0l}) (\psi_n^l B_n^{l*} + \psi_n^{l*} B_n^l) \tag{12}$$

$$\frac{dE_n}{dt} = J_n = \sum_{l=0}^{n} J_n^l = \sum_{l=0}^{n} \left(\frac{n^2(n+1)^2}{2a^4}\right) \frac{n(n+1)}{2a^2} (2 - \delta_{0l}) \times (\psi_n^l B_n^{l*} + \psi_n^{l*} B_n^l) \tag{13}$$

$$\frac{dA_n}{dt} = L_n = \sum_{l=0}^{n} L_n^l = \sum_{l=0}^{n} \frac{1}{2\bar{\sigma}} \left(\frac{R}{C_p}\right) (2 - \delta_{0l}) (T_n^l C_n^{l*} + T_n^{l*} C_n^l) \tag{14}$$

where asterisks signify a complex conjugate. B_n^l and C_n^l are the spherical harmonics coefficients of the Jacobian terms of the vorticity and thermodynamical equations,

$$B_n^l = \frac{1}{4\pi} \int_{-1}^{1} \int_0^{2\pi} J(\psi, \zeta) P_n^{-l} e^{-il\lambda} \, d\lambda \, d\mu \tag{15}$$

and

$$C_n^l = \frac{1}{4\pi} \int_{-1}^{1} \int_0^{2\pi} J(\psi, T) P_n^{-l} e^{-il\lambda} \, d\lambda \, d\mu \tag{16}$$

In addition, we also can find $J_n^l = [n(n+1)/a^2] I_n^l$ as Wiin-Nielsen (1972) remarked.

The major purpose of the present study is to evaluate the n spectra of K, E, and A, and the nonlinear exchanges I_n, J_n, and L_n of K, E, and A, respectively.

3. Data

To calculate the spectra of energy and enstrophy, and the nonlinear exchanges using the two-dimensional spectral index, n, we need the spectral coefficients of streamfunction and temperature. Since the spherical harmonic functions are involved in the computation, and the data cover the Northern Hemisphere only, particular parity conditions must be assigned to meteorological variables. The streamfunction is assumed to be odd, while the temperature is even.

The temperature and wind fields used were obtained from the National Center for Atmospheric Research (NCAR), originated at the National Meteorological Center (NMC) and analyzed by the method of Flattery (1970) or Cooley (1974). The spectral coefficients of the streamfunction are converted from those of vorticity, which were themselves created from the wind fields. The data consist of 10 levels (1000, 850, 700, 500, 400, 300, 250, 200, 150, and 100 mb) available at 0000 GMT and 1200 GMT for the period of 0000 GMT December 19, 1971–0000 GMT February 27, 1972 with 12 missing periods. The spectral coefficients of temperature and streamfunction are created following Ellsaesser (1966). An outline of the transformation procedures from the grid-point values to the spectral coefficients can also be found in Baer (1972).

A triangular truncation with $n = 31$ and $l = 30$ is used for the expansion of the data in spectral coefficients. A truncation higher than $n = 25$ is used to insure recovery of the highest analysis modes. It must be kept in mind that contributions beyond $n = 25$ are not representative of observed data.

The values of static stability in each level are computed by linear interpolation from those calculated by Gates (1960) for January case.

4. Energy and enstrophy spectra

Several investigations have been dedicated to the energy spectra as a function of the two-dimensional spectral index, n. Wiin-Nielsen (1972) used the spectral coefficients of height and kinetic energy created by Eliasen and Machenhauer (1965 and 1969) and by Merilees (1966) at the 500 mb level, and obtained slopes of -3 and -5 respectively, in the kinetic and available potential energy spectra for large values of n. Burrows (1976) used a multi-level wind field for the first 2 weeks of August, 1970, and found that a -3 slope of the kinetic energy spectra occurred in the upper levels. Based upon a 5-level wind field of January and February, 1969, and a 5-level temperature field of January, February, and March, 1970, Baer (1972 and 1974) found that a -3 slope exists in the kinetic and available potential energy spectra between $14 \leqslant n \leqslant 25$.

The data set used in the present study have better vertical resolution (10 levels). The kinetic energy and enstrophy may be evaluated directly from the spectral coefficients of the streamfunction and are connected by a simple relation shown in Section 2. It is desirable to re-examine the energy and enstrophy spectra in the two-dimensional spectral index, n, by using the current data sample. The primary assumption for the inertial subrange of the two-dimensional turbulence is that this range of the spectrum does not contain the energy-generating eddies. Based on the stability analysis of a quasi-geostrophic model and data accuracy, Baer (1972) concluded that the most proper scale range for testing two-dimensional turbulent energy exchange from current atmospheric data is $14 \leqslant n \leqslant 25$ where no baroclinically forced eddies exist.

Empirical studies of energy spectra of large-scale atmospheric flow in the two-dimensional spectral index mentioned above have shown that a power relationship exists between the energies and the scale index, n, e.g. the kinetic energy in the form of $K_n = an^b$, where a and b are constant determined by regression methods. In order to determine the power law between $14 \leqslant n \leqslant 25$ in our results, we write the relation between the spectral components of K, E, A, and n in the form,

$$\log \begin{Bmatrix} K_n \\ E_n \\ A_n \end{Bmatrix} = \log \begin{Bmatrix} a_K \\ a_E \\ a_A \end{Bmatrix} + \begin{Bmatrix} b_K \\ b_E \\ b_A \end{Bmatrix} \log n \tag{17}$$

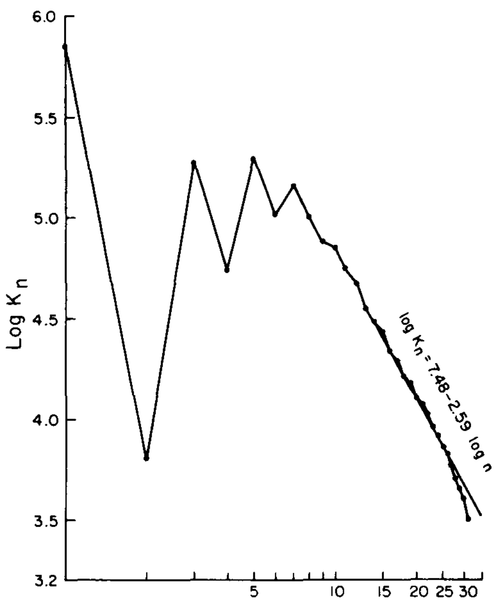

Fig. 1. The kinetic energy K for the total atmosphere as a function of a two-dimensional spectral index (n) plotted on logarithmic scales.

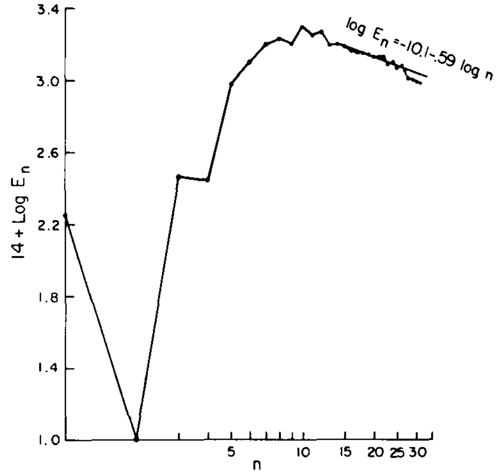

Fig. 2. The vertical mean enstrophy E as a function of a two-dimensional spectral index (n) plotted on logarithmic scales.

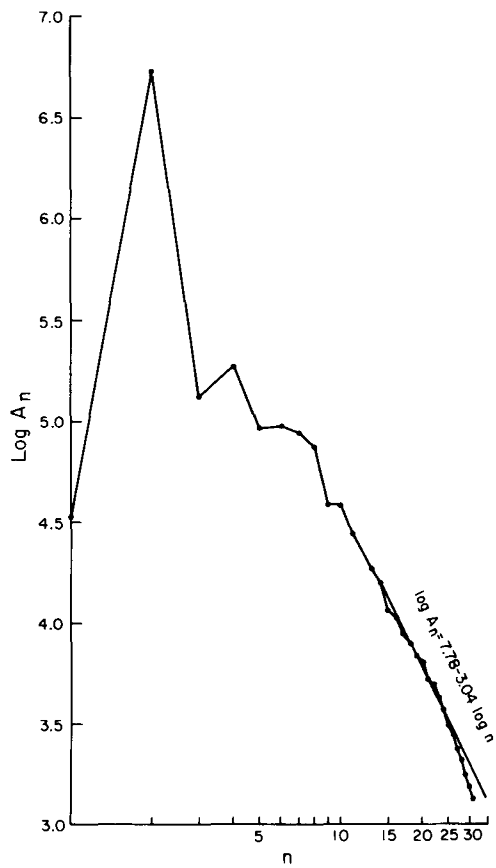

Fig. 3. The available potential energy A for the total atmosphere as a function of a two-dimensional spectral index (n) plotted on logarithmic scales.

where a's and b's can be determined by linear regression. The lower subscripts of a and b in eq. (17) denote the associated physical variable. The time mean and vertical integral spectra of kinetic energy, enstrophy, and available potential energy are respectively shown in Figs. 1–3.

Fig. 1 shows that a vertical mean slope of $b_K = -2.6$ exists for $14 \leqslant n \leqslant 25$. The b_K values at various pressure levels are displayed in Table 1. The vertical mean value of b_K in this study is slightly smaller than Baer's (1972). In addition, Baer also found that b_K has a maximum value at 250 mb, while our b_K is almost constant from 500 mb to 200 mb. Firstly, we have to keep in mind that Baer's analysis was based upon the total wind field, while our results are evaluated by using the rotational component of the wind only. Nevertheless, it has been shown by Chen and Wiin-Nielsen (1976) that the rotational wind component gives a good representation of the total wind. Baer used only the wind field of 5 levels, 700, 500, 300, 250, and 200 mb. Table 1 shows that b_K has a value of approximately -3 at those levels. It is apparent that the vertical mean value of b_K is reduced by the smaller values of b_K in the lower and upper layers. The small values of b_K at the lower levels indicate that the smaller scales have more energy as would be expected from baroclinic stability theory. Furthermore, the dynamics in the stratosphere are different from that in the troposphere. We cannot expect that the values of b_K at 100 mb should be similar to those in the troposphere. The values of b_K at various pressure levels in Table 1 indicate therefore that the two-dimensional turbulence theory may be properly applied between 700 and 200 mb.

It has been pointed out in Section 2 that the n spectral component of enstrophy, E_n, may be obtained simply by multiplying K_n with $n(n+1)/a^2$. In other words, we can predict $|b_K| - |b_E| \simeq 2$. Fig. 2 and Table 1 show that the vertical mean $b_E = -0.6$. Steinberg et al. (1971) found that the transient part of enstrophy in the large wave-number range possesses a slope of about -1, but their result was evaluated for a one-dimensional scale index. The argument for the vertical variation of b_K above applies equally well to b_E. Thus, b_E has a value of about -1 between 700 and 200 mb levels.

The available potential energy spectrum in Fig. 3 shows that the vertical mean value of b_A is about -3. This value is similar to the one obtained by Baer (1974) and is identical to Charney's (1971) prediction. The variation of b_A at various pressure levels is also displayed in Table 1 and is similar to Baer's (loc. cit.) result. b_A has two minimum values at 1000 and 300 mb, and a maximum value at 700 mb. The vertical variation of b_A is very different from those of b_K and b_E. A possible reason for the small b_A at 300 mb was discussed by Baer and is probably due to the pronounced variation of static stability. The small b_A at 1000 mb may be due to the relative high energy contained in the eddies with higher values of n. At any rate, the comparison of the vertical variation between b_K and b_A shows that it would be best to test the existence of an inertial subrange with -3 slope by using the stream-function rather than the temperature field as Baer (1974) suggested.

Finally, we would like to comment on the influence of spectral components $n > 25$. It is shown clearly in Figs. 1–3 that the slopes of K, E, and A are steeper than between $14 \leqslant n \leqslant 25$. This indicates that the smoothing of the data may reduce energy in the shorter scales. Nevertheless, this speculation has to be tested with data accurate in the short scales (Baer, 1972).

5. Nonlinear cascades of energy and enstrophy, and inertial change

The first step to evaluate the nonlinear exchange of kinetic energy, enstrophy, and available potential energy among different n scales, I_n, J_n, and L_n, is to

Table 1. *Values of the slope b_K, b_E, and b_A as function of pressure*

Pressure (mb)	1000	850	700	500	400	300	250	200	150	100	Vertical average
b_K	−1.3	−2.4	−2.7	−3.0	−2.9	−2.9	−2.9	−2.9	−2.6	−2.0	−2.6
b_E	0.7	−0.4	−0.7	−1.0	−0.9	−0.9	−0.9	−0.9	−0.6	0.0	−0.6
b_A	−1.8	−2.8	−3.8	−3.5	−3.5	−2.1	−2.4	−2.9	−2.9	−3.1	−3.0

calculate the spectral coefficients of Jacobian terms of the vorticity and thermodynamic equations, i.e. B_n^l and C_n^l in eqs. (15) and (16). There are two methods of computing B_n^l and C_n^l. One is the interaction coefficient method (Platzman, 1960), and the other is the transform method (Machenhauer and Rasmussen, 1972). To save computation, the latter method is applied in the present study. I_n, J_n, and L_n can be evaluated by (12), (13), and (14). The time mean and vertical integral of I_n, J_n, and L_n are shown in Table 2.

Based upon the one-dimensional spectral index, l, i.e. the latitudinal wavenumber, it has been observed from the diagnostic studies of atmospheric data that the kinetic energy is transferred from intermediate scales to large and small scales (e.g. Saltzman, 1970; Steinberg et al., 1971). The energy source of intermediate scales comes from the conversion of available potential energy due to the occurrence of baroclinic instability. The results of observational studies are consistent with Fjørtoft's theory. In fact, Fjørtoft's theory was developed in a two-dimensional spectral domain. In order to take the anisotropy of atmospheric data into account, it is more proper to examine how the energy and enstrophy is nonlinearly transferred in a two-dimensional spectral domain. Table 2 shows clearly that the kinetic energy, in general, is transferred from intermediate n scales to large and small n scales. This result confirms the theoretical argument of Fjørtoft's blocking theorem in n index by Wiin-Nielsen (1974) and Merilees and Warn (1974). If we separate n scales into three groups:

$$1 \leqslant n \leqslant 7, \quad 8 \leqslant n \leqslant 14,^1 \quad \text{and} \quad 15^1 \leqslant n \leqslant 31,$$

we find that a large portion (830 ergs cm^{-2} s^{-1}) and a very small portion (47 ergs cm^{-2} s^{-1}) of kinetic energy are exported from intermediate-n scales ($8 \leqslant n \leqslant 14$) respectively to small-$n$ scales ($1 \leqslant n \leqslant 7$) and large-$n$ scales ($15 \leqslant n \leqslant 31$). The nonlinear cascades of kinetic energy in the two-dimensional spectral index are similar to those in the one-dimensional spectral index.

Table 2. *Time mean and vertical integral and/or mean of nonlinear exchange of kinetic energy, I_n, of enstrophy, J_n, and of available potential energy, L_n. Unit: I_n, erg cm^{-2} s^{-1}; J_n, 10^{-18} s^{-3}; L_n, erg cm^{-2} s^{-1}*

n	I_n		J_n		L_n
1	−69		−1		−12
2	964		28		−5108
3	114		5		83
4	−52	840	−4	39	375
5	−9		1		217
6	−598		−81		184
7	480		87		475
8	−268		−59		598
9	−60		−16		380
10	−141		−46		473
11	−101	−877	−40	−328	347
12	−35		−15		321
13	−139		−73		283
14	−133		−79		202
15	−56		−38		182
16	−90		−68		143
17	−30		−24		128
18	−42		−38		104
19	10		10		78
20	−19		−22		89
21	18		22		72
22	26		35		68
23	23	47	34	289	64
24	30		57		64
25	41		69		46
26	27		49		29
27	37		72		26
28	16		33		25
29	30		89		22
30	13		30		25
31	13		32		17

It is interesting to compare our results of I_n with Burrows (1976). The general features of both studies are similar, but there are two distinct differences: (1) The magnitude of I_n is larger in the current study. This is because the data used in this study are a winter case, while Burrows used summer wind fields; (2) No contribution comes from $n = 0$ to I_n in this study. It is because B_0^0 when integrated over the whole globe is zero when the streamfunction alone is used to evaluate the Jacobian term in the vorticity equation.

It was pointed out in Section 2 that J_n is related to I_n by a factor of $n(n+1)/a^2$. In other words, J_n can be obtained simply by multiplying I_n by $n(n+1)/a^2$. J_n displayed in Table 2 shows similar features to I_n. That is, enstrophy is exported from intermediate-n scales to large- and small-n scales. There are very marked differences between

[1] Baer (1972) suggested the division between intermediate- and large-n scales is at $n = 14$. In order to reconcile the results of this study with conventional studies of one-dimensional spectral indices, we chose $n = 15$.

I_n and J_n. The enstrophy cascade is very small (39 $\times$ 10^{-11} s^{-3}) to small-n scales and large (393) to large-n scales from intermediate-n scales. Steinberg et al. (1971) found that enstrophy is transferred from the large- to small-scale motion with a very small accumulation in the intermediate scales in one-dimensional domain. It is obvious that the nonlinear exchange of enstrophy in the two-dimensional spectral domain is very different from that in the one-dimensional spectral domain.

The nonlinear cascade of available potential energy, L_n, is shown in the last column of Table 2. Available potential energy is exported from $n = 1$ and 2, with an extremely large amount associated with the latter, to scales of other n values. Intermediate-n scales obviously receive a greater amount of available potential energy than large-n scales. Notice that L_0 makes no contribution to the nonlinear exchange of available potential energy in the spectral index. It is because T_0^0 has been excluded when A is evaluated. It implies, in turn, $C_0^0 = 0$. The spectral distribution of L_n is similar to the non-linear exchange of A in a one-dimensional spectral domain as obtained by Steinberg et al.

From their theoretical study of two-dimensional turbulence, Leith (1968) and Kraichnan (1967) suggested the existence of an inertial subrange, corresponding to a -3 power in the kinetic energy spectrum, and characterized by a constant flux of enstrophy and a zero flux of kinetic energy. In order to examine whether an inertial subrange satisfying Leith's and Kraichnan's criteria exists, let us introduce flux functions. $F_K = F_K(n)$ for kinetic energy, $F_E = F_E(n)$ for enstrophy, and $F_A = F_A(n)$ for available potential energy. Following Leith, we define:

$$\frac{\partial F_K}{\partial n} = -I_n$$
$$\frac{\partial F_E}{\partial n} = -J_n \tag{18}$$
$$\frac{\partial F_A}{\partial n} = -L_n$$

Using the observational value of I_n, J_n, and L_n shown in Table 2, we can evaluate flux function, F_K, F_E, and F_A by integrating (18). The integral of horizontal advection terms of kinetic energy and enstrophy equations over a closed domain vanishes. This indicates that the integrations of I_n, J_n, and L_n through all n scales are zero. It also

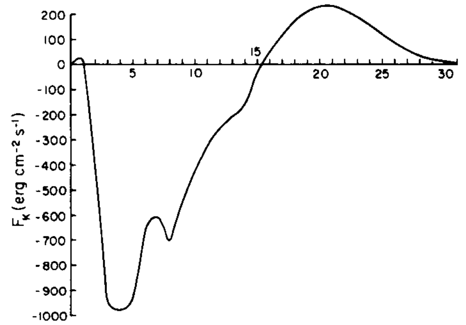

Fig. 4. The flux function for kinetic energy as a function of a two-dimensional spectral index (n).

implies that $F(0) = F(n_{max})$. We shall select $F_K(0) = F_E(0) = F_A(0) = 0$.

Fig. 4 displays the flux of kinetic energy as a function of n. The n spectral distribution of F_K is similar to that obtained by Steinberg et al. (1971), and Barros and Wiin-Nielsen (1974) in the one-dimensional index domain. The most interesting difference between observational studies of Steinberg et al.'s one-dimensional F_K and our two-dimensional F_K is that the ratio of the largest positive F_K to the largest negative F_K is much smaller in the current study.

The flux of enstrophy is depicted in Fig. 5. The spectral distribution of F_E using the two-dimensional index is somewhat different from that in the one-dimensional index obtained by Steinberg et al. It is found that $F_E < 0$ for small-n scales in the current study, while Steinberg et al.'s flux of entrophy is positive over all wavenumbers. The reason for negative values of F_E in the small-n scales is that the enstrophy is transferred from intermediate-n scales to small- and large-n scales in two-dimensional spectral domain. On the contrary,

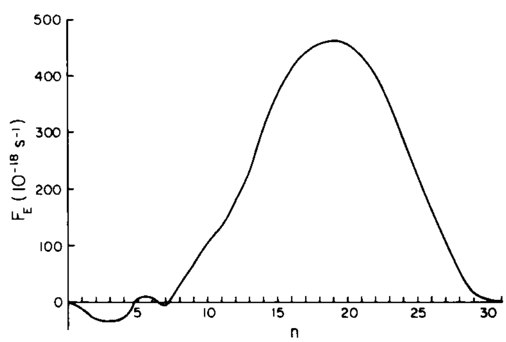

Fig. 5. The flux function for enstrophy as a function of a two-dimensional spectral index (n).

the enstrophy cascades from large scales to small scales with little accumulation in intermediate scales in the one-dimensional spectral domain.

It is interesting to note that the studies of Steinberg et al., Barros and Wiin-Nielsen, and the present study find that the flux of enstrophy decays to insignificant values for large n. Barros and Wiin-Nielsen discussed that the decrease of F_E for large n implies a viscous range in which a balance exists between the flux of enstrophy from small to large n values and the dissipation of enstrophy by various processes. Thus, they concluded that the cascade of enstrophy is better described by an exponential decrease by taking the viscous effect into account.

It has been shown in the last section that the spectra of kinetic energy and enstrophy obey certain power laws in the range $14 \leqslant n \leqslant 25$. Furthermore, neither F_K is zero nor F_E is constant over this spectral range. But F_K in Fig. 4 is not significant, while F_E in Fig. 5 is dominant in the range $14 \leqslant n \leqslant 25$. In view of the n spectral distribution of F_K and F_E in the present study or compared with those in Steinberg et al.'s, it seems more resonable to postulate that an inertial subrange might exist in the spectral range between $14 \leqslant n \leqslant 25$.

The flux of available potential energy as a function of n shown in Fig. 6 is very similar to the results of Barros and Wiin-Nielsen's numerical experiment. F_A for $14 \leqslant n \leqslant 25$ is not so small as Barros and Wiin-Nielsen's result for large wavenumbers.

Following Leith's (1968) formula for the spectral distribution of kinetic energy, Wiin-Nielsen (1972) suggested that kinetic energy over the -3 inertial subrange obeys the expression,

$$K_S = C\eta^{2/3} S^{-3} \tag{19}$$

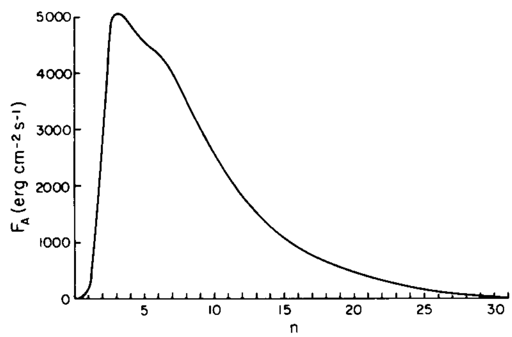

Fig. 6. The flux function for available potential energy as a function of a two-dimensional spectral index (n).

where $S \simeq n/r$, r is the radius of the earth, and η is the enstrophy flux. The exact expression of S which Wiin-Nielsen (1972) used is $S = \sqrt{n(n+1)}/r \sim n/r$ as n is large. In order to determine the nondimensional constant C, let us follow Steinberg et al. (1971). The kinetic energy spectrum can be expressed as

$$K_n(n) = an^{-3} \tag{20}$$

in the n index. If we adjust the slope of b_K in Fig. 1 to be -3, then we may find $a \simeq 1.0155 \times 10^8$ joule m^{-2}. Combining (19) and (20), and noting that (20) has to be integrated over the whole depth of the atmosphere, we obtain

$$K_n = C\frac{P_0}{g}\eta^{2/3} r^2 n^{-3} \tag{21}$$

Taking the average of enstrophy flux over $14 \leqslant n \leqslant 25$ as shown in Fig. 5, we have $\eta \simeq 385 \times 10^{-18}$ s^{-3}. The nondimensional constant in (19) can be evaluated by (21), $C \simeq 4.6$. This value is very close to Lilly's (1969) value.

6. Concluding remarks

10-level (1000, 850, 700, 500, 400, 300, 250, 200, 150, and 100 mb) temperatures and streamfunctions for the period of December 19, 1971 to February 27, 1972 are used to compute the spectra and nonlinear exchange of kinetic energy (K), enstrophy (E), and available potential energy (A) using a two-dimensional spectral index, n, which is the degree of the Legendre polynomial.

The spectra of K and E are generated from the spectral coefficients of the streamfunction, while the spectrum of A is obtained from the spectral coefficients of temperature. K, E, and A have, respectively, vertical–mean slopes of -2.6, -0.6, and -3.0 in the spectral range $14 \leqslant n \leqslant 25$, the spectral range over which Baer (1972) suggested that there might be an inertial subrange of large-scale atmospheric flow.

K is nonlinearly transferred with a large fraction from intermediate-n scales to small-n scales, and with a small fraction to large-n scales. The nonlinear exchange of enstrophy is related to that of kinetic energy by a factor of $n(n+1)/a^2$. It is found that the enstrophy is exported nonlinearly with a small portion to small-n scales and with a large portion to large-n scales. In fact, the nonlinear exchange of K in the two-dimensional spectral scales is similar to that in the one-dimensional ones, while the nonlinear exchange of E is very different. A cascades from $n = 1$ and 2, particularly the latter, to scales of other n-values. It is observed that a large amount of A is transferred to intermediate-n scales and a small amount to small-n scales.

F_K, F_E and F_A, the fluxes of nonlinear interactions of K, E, and A among different n scales, are obtained, respectively, by integrating the nonlinear interactions of K, E, and A through their spectra. It is found that the spectral distributions of F_K, F_E, and F_A are generally in agreement with the one-dimensional spectral results obtained by Steinberg et al. (1971), and Barros and Wiin-Nielsen (1974). Although K and E spectra between $14 \leqslant n \leqslant 25$ obey certain power laws, they are not characterized by a constant flux of enstrophy and a zero flux of kinetic energy as Kraichnan (1967) and Leith (1968) suggested. The nondimensional constant in the spectral distribution of kinetic energy presented by Leith (1968) for $14 \leqslant n \leqslant 25$ is also evaluated. The value of this constant in the present study is 4.6 which agrees well with Lilly's (1969) estimate.

7. Acknowledgements

We would like to express our appreciation to Professor Ferdinand Baer of the University of Michigan for his generosity in providing us with the spectral coefficients of streamfunction and temperature fields which made this study possible. Mr Jordan Alpert prepared the data tape. The manuscript was prepared while one of us (T.C.C.) was affiliated with the Department of Meteorology, Massachusetts Institute of Technology. The author T.C.C. is grateful to the stratospheric modelling group which is supported by NASA under Grant No. NSG-2010 to M.I.T.

REFERENCES

Baer, F. 1972. An alternate scale representation of atmospheric energy spectra. *J. Atmos. Sci. 29*, 649–664.

Baer, F. 1974. Hemispheric spectral statistics of available potential energy. *J. Atmos. Sci. 31*, 932–941.

Barros, V. R. and Wiin-Nielsen, A. 1974. On quasi-geostrophic turbulence: a numerical experiment. *J. Atmos. Sci. 31*, 609–621.

Burrows, W. R. 1976. A diagnostic study of atmospheric spectral kinetic energetics. *J. Atmos. Sci. 33*, 2308–2321.

Charney, J. G. 1971. Geostrophic turbulence. *J. Atmos. Sci. 28*, 1087–1095.

Chen, T. C. and Wiin-Nielsen, A. 1976. On the kinetic energy of the divergent and nondivergent flow in the atmosphere. *Tellus 28*, 486–497.

Cooley, D. S. 1974. A description of the Flattery global analysis method #1. *Tech. Procedures Bulletin #105* (W 111X1), Department of Commerce, NOAA, NWS, Silver Springs, Maryland.

Eliasen, E. and Machenhauer, B. 1965. A study of the fluctuations of the atmospheric planetary flow patterns represented by spherical harmonics. *Tellus 17*, 220–238.

Eliasen, E. and Machenhauer, B. 1969. On the observed large-scale atmospheric waves. *Tellus 21*, 149–165.

Ellsaesser, H. W. 1966. Expansion of hemispheric meteorological data in antisymmetric surface spherical harmonic (Laplace) series. *J. Appl. Meteor. 5*, 263–276.

Fjørtoft, R. 1953. On the changes in the spectral distribution of kinetic energy for two-dimensional, nondivergent flow. *Tellus 5*, 225–230.

Flattery, T. W. 1970. Spectral models for global analysis and forecasting. *Proceedings of the Sixth A.M.S. Technical Exchange Conference*, U.S. Naval Academy, 21–24 September, 1970. Air Weather Service Tech. Rep. 242.

Gates, W. L. 1961. Static stability measures in the atmosphere. *J. Meteor. 18*, 526–533.

Kraichnan, R. H. 1967. Inertial subranges in two-dimensional turbulence. *Phys. Fluids 10*, 1417–1423.

Leith, C. E. 1968. Diffusion approximation for two-dimensional turbulence. *Phys. Fluids 11*, 671–673.

Lilly, D. K. 1969. Numerical simulation of two-dimensional turbulence. *Phys. Fluids 12*, Suppl. II-240 to II-249.

Lilly, D. K. 1972. Numerical simulation studies of two-dimensional turbulence. *Geophys. Fluid. Dyn. 3*, 289–319.

Machenhauer, B. and Rasmussen, E. 1972. On the integration of the spectral hydrodynamic equations by a transformed method. Report No. 3, Copenhagen University, Institute for Theoretical Meteorology, 44 pp.

Merilees, P. E. 1966. Harmonic representation applied to large-scale atmospheric wave. Publication No. 83, *Meteorology*, 174 pp.

Merilees, P. E. and Warn, T. 1972. The resolution implications of geostrophic turbulence. *J. Atmos. Sci. 29*, 990–991.

Merilees, P. E. and Warn, T. 1975. On energy and enstrophy exchanges in two-dimensional non-divergent flow. *J. Fluid Dyn. 69*, part 4, 625–630.

Platzman, G. W. 1960. The spectral form of the vorticity equation. *J. Meteor. 17*, 635–644.

Saltzman, B. 1970. Large-scale atmospheric energetics in the wavenumber domain. *Rev. Geophys. Space Phys. 8*, 289–302.

Steinberg, H. L., Wiin-Nielsen, A. and Yang, C. H. 1971. On nonlinear cascades in large-scale atmospheric flow. *J. Geophys. Res. 76*, 8829–8840.

Steinberg, H. L. 1973. Numerical simulation of quasi-geostrophic turbulence. *Tellus 25*, 233–246.

Wiin-Nielsen, A. 1967. On the annual variation and spectral distribution of atmospheric energy. *Tellus 19*, 540–559.

Wiin-Nielsen, A. 1972. A study of power laws in the atmospheric kinetic energy spectrum using spherical harmonic functions. *Meteor. Ann. 6*, 107–124.

Wiin-Nielsen, A. 1974. A note on Fjørtoft's blocking theorem. *Report of the International Symposium of Spectral Methods in Numerical Weather Prediction*, Copenhagen, 12–16 August 1974. The GARP Programme on Numerical Experimentation, Report No. 7, December 1974, 105–117.

О НЕЛИНЕЙНЫХ КАСКАДАХ АТМОСФЕРНОЙ ЭНЕРГИИ И ЭНСТРОФИИ ДЛЯ ДВУМЕРНОГО СПЕКТРАЛЬНОГО ИНДЕКСА

Выполнен диагностический анализ спектров и нелинейных обменов для кинетической энергии (K), энстрофии (E) и доступной потенциальной энергии (A) с использованием в качестве двумерного спектрального индекса степень n в разложении полей по полиномам Лежандра. Результаты представлены для данных за период с 19 декабря 1971 г. по 27 февраля 1972 г. Спектры K, E и A имеют наклоны, близкие к -3, -1 и -3, соответственно, для больших n. Для величин K и E нелинейность распространяется от промежуточных значений n к большим и малым n. Для A каскад идет от $n = 1$ и 2, особенно от последнего значения, к масштабам с бо́льшими значениями n. Вычислены также потоки нелинейных обменов для K, E и A с целью исследования возможности существования инерционных подинтервалов. Однако твердого заключения об их существовании сделать не удается.