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SHORT CONTRIBUTION

A note on the closure in Lilly-type inversion models

By H. KRAUS and E. SCHALLER, *Lehrstuhl für Bioklimatologie der Universität Göttingen, Büsgenweg 1, D-3400 Göttingen, F.R.G.*

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ABSTRACT

Closure conditions in models of lifted inversions are usually based on the buoyancy term in the equation of turbulent energy. The paper discusses this type of closure and compares the condition r = constant (r is the ratio of integral dissipation over integral production of turbulent energy, both due to buoyancy forces; the integral is taken over the whole mixed layer) with the frequently used condition of k = constant (k represents the negative ratio of minimum over maximum flux of virtual static energy). It is shown that in the dry case $r = k^2$. In the wet case including condensation within a St- or Sc-deck, however, it is not possible to set up such a simple general relationship. The problem is illustrated by some model case studies. Finally it is shown that essential characteristics of the inversions are not very sensitive to changes in r or k. This leads to the conclusion that a relatively rough closure assumption suffices for most purposes.

Lilly-type inversion models describing the thermodynamic structure and the flux profiles in and under lifted inversions essentially consist of six thermodynamic equations (two prognostic budget equations for the mixed layer, two budget equations for the inversion layer, which serve as prognostic equations for the height of the inversion, and two diagnostic equations for the surface fluxes) for seven unknown quantities. For more details see for example Lilly (1968), Schubert (1976) or Kraus and Schaller (1978). In this note we will refer to the last paper at several places using the abbreviation KS. The set of equations is usually closed by an incomplete form of the turbulent kinetic energy equation. Since the model does not contain any dynamic relationships, this relatively poor closure consists of the buoyancy term only.

This closure was first proposed by Ball (1960). It is based on the assumption that a certain rate ($0 \leqslant r \leqslant 1$) of the thermally produced turbulent kinetic energy (production under conditions of a positive flux of virtual static energy F_{sv}) is not dissipated by viscosity but again by buoyancy forces in that part of the mixed layer, where $F_{sv} < 0$. For $r = 1$ (for example) one simply can write

$$\int_{p_b}^{p_s} F_{sv}\, dp = 0 \tag{1}$$

(p_s = pressure at the surface, p_b = pressure at the upper boundary of the mixed layer).

Lilly (1968) has consequently defined two extremum conditions, the "maximum entrainment":

$\int_{p_b}^{p_s} F_{sv}\, dp = 0$, but $F_{sv} \neq 0$ somewhere and the "minimum entrainment":

$(F_{sv})_{\min} = 0$, but $\int_{p_b}^{p_s} F_{sv}\, dp > 0$

In the first case $r = 1$, in the second one $r = 0$.

Discussing these conditions for the *dry case* first (in this case the air does not contain any water vapour and F_{sv} is represented by the flux of dry static energy) one may start from the linear profile of F_{sv}, which results from the model assumptions (see e.g. eq. (12a) and Fig. 2 in KS) and which is shown in Fig. 1a. Lilly's maximum entrainment condition is identical with $(F_{sv})_{b^+} = -(F_{sv})_s$ (the subscript b^+ stands for the higher pressure side of the upper boundary of the mixed layer, s for the surface level), the minimum entrainment condition

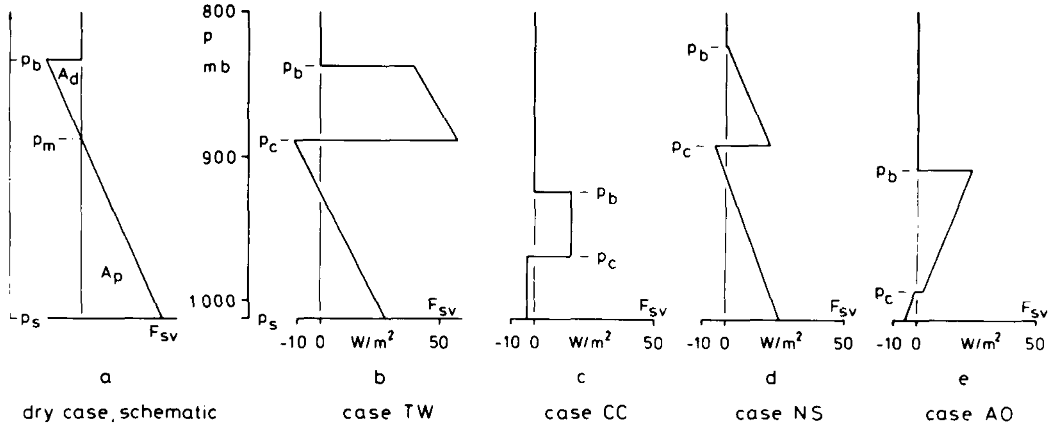

Fig. 1. Examples of profiles of the flux of virtual static energy F_{sv} (stability flux) in concordance with the assumptions of the Lilly-type inversion models. (a) Dry case, schematic; (b to e) wet cases from model computations representing typical conditions of four climatic regions (TW = Trade Wind, CC = West Coast of California, NS = Norwegian Sea, AO = Arctic Ocean). These cases are described in more detail in Kraus and Schaller (1978). p_b pressure at the upper boundary of the mixed layer, p_c pressure at the cloud base, p_s surface pressure.

with $(F_{sv})_{b^+} = 0$. The real dry case schematically depicted in Fig. 1a can be described by

$$(F_{sv})_{b^+} = -c(F_{sv})_s \qquad (2)$$

or by

$$(F_{sv})_{min} = -k(F_{sv})_{max} \qquad (3)$$

The constants c and k are identical for the dry case. Since they play an important role in describing the inversion characteristics, several authors have considered them intensively partly using the results of laboratory (e.g. Deardorff, Willis and Lilly, 1969) and atmospheric (e.g. Cattle and Weston, 1975) experiments. A summary of the literature on k may be found in Stull (1976). The value $k = 0.2$ is used most frequently.

We now want to establish a relationship between r and k. This is done first for the profile of Fig. 1a. Since r is generally defined by

$$r = -\int_{p_b}^{p_s} (F_{sv} < 0)\,dp \Big/ \int_{p_b}^{p_s} (F_{sv} > 0)\,dp \qquad (4)$$

(as a ratio of the integral buoyant energy dissipation over the integral buoyant production instead of a ratio of fluxes) we find:

production area $A_p = \frac{1}{2}(p_s - p_m)(F_{sv})_{max}$
dissipation area $A_d = -\frac{1}{2}(p_m - p_b)(F_{sv})_{min}$
(p_m = pressure level, where $F_{sv} = 0$)

Linearity of the profile results into

$$\frac{p_m - p_b}{p_s - p_b} = \frac{k}{1+k} \quad \text{and} \quad \frac{p_s - p_m}{p_s - p_b} = \frac{1}{1+k}$$

With (3) the integral buoyancy term can be evaluated as

$$\begin{aligned}\int_{p_b}^{p_s} F_{sv}\,dp &= A_p - A_d \\ &= \tfrac{1}{2}[(p_s - p_m)(F_{sv})_{max} + (p_m - p_b)(F_{sv})_{min}] \\ &= -\tfrac{1}{2}(F_{sv})_{min}(p_s - p_b) \times \left[\frac{1}{k}\frac{1}{1+k} - \frac{k}{1+k}\right]\end{aligned}$$

$$\int_{p_b}^{p_s} F_{sv}\,dp = -\tfrac{1}{2}(F_{sv})_{min}(p_s - p_b)\frac{1-k}{k} \qquad (5)$$

This closure is more realistic than the minimum or maximum entrainment condition. The ratio of the dissipation rate over the production rate is

$$r = \frac{A_d}{A_p} = -\frac{(p_m - p_b)}{(p_s - p_m)} \cdot \frac{(F_{sv})_{min}}{(F_{sv})_{max}} = k \cdot k = k^2 \qquad (6)$$

For $k = 0.2$ (for example) $r = 4\%$. The closure (5) and the relation

$$r = k^2 = c^2$$

hold exactly for the linear (F_{sv})-profile depicted in Fig. 1a.

The wet case (including condensation in a St- or Sc-deck) is more complicated than the dry one. Examples of F_{sv}-profiles are drawn as Fig. 1b to 1e. These graphs present the results of model computations which are depicted in full detail in Fig. 8 of KS. They are taken as examples in order to enlighten the discussion of the closure condition. The model used for the computations is of the same type as that of Lilly (1968) and Schubert (1976); it contains a net radiation difference across the inversion $\Delta F_R = -70\ \mathrm{W/m^2}$, variable subsidence and advection of moist static energy and total water content in the mixed layer. The examples of Fig. 1 are for steady-state conditions. At the cloud base a jump of F_{sv} occurs which is partly caused by the destabilizing condensation effect. Another destabilizing effect on the cloud layer is the radiative cooling of the inversion described by ΔF_R. Both effects are responsible for the fact, that in the examples b, c, d and e of Fig. 1 the flux of virtual static energy F_{sv} in the cloud layer is positive. This may be written explicitly by citing the equation, how F_{sv} is connected to the fluxes of moist static energy F_h and total water content F_{q+1}. Within the cloud layer

$$F_{sv} = \beta F_h - \varepsilon L F_{q+l}$$

with L = specific heat of evaporation, $\beta \approx 0.5$ (this is a typical value) and $\varepsilon \approx 0.12$ (for details see KS). Since in the cases b, c, d and e of Fig. 1 F_h is forced to be positive by ΔF_R, according to the moist static energy budget of the inversion layer, and LF_{q+l}, which is also positive and of the same order of magnitude as F_h, enters the equation with a weight of 12% only, the flux F_{sv} is positive in the cloud layer. The slope of the profiles gives some information about the advection term in the mixed layer budget. For steady-state conditions and no radiational heating or cooling in the mixed layer, cases a and b (F_{sv} increases with pressure) represent cases of advection of cold air, d and e of warm air, whereas in case c the advection is zero.

It is evident from these figures that for the wet case the coefficients c and k defined by eqs. (2) and (3) are no longer identical. Also a straightforward deduction of r in dependence on k is not possible, and furthermore the closure (5) cannot be deduced from $k = -(F_{sv})_{\min}/(F_{sv})_{\max}$ as it was done above. The reason is that the shapes of the profiles are no longer linear; they are characterized by the above-mentioned jump and by secondary maxima and minima. However, one can simply use an analogon of the closure (5) writing

$$\int_{p_b}^{p_s} F_{sv}\,dp = -\tfrac{1}{2}(F_{sv})_{\min}(p_s - p_b)\frac{1-k'}{k'} \tag{7}$$

well aware that

$$k' \neq k = -\frac{(F_{sv})_{\min}}{(F_{sv})_{\max}} \tag{8}$$

Thus we have now four different quantities c, k, k' and r, whose interrelationship cannot generally be described (as in the dry case by $c = k = k'$ and $r = k^2$) without knowing further parameters of the F_{sv}-profiles. Equation (7) represents a closure which was first used by Schubert (1976), later in a similar way by Deardorff (1976) and also in KS. Deardorff's closure in his "hypothesis for forced entrainment" however postulated the negative $(F_{sv})_{\min}$ to be located close below the cloudtop, which implies a different feature compared to the use of (7) by Schubert and KS.

In order to give some examples for values of c, k, k' and r we again refer to the profiles of Figs. 1b to 1e. They represent typical conditions for four climatic regions and were computed using typical external parameters (for example surface temperature) of these regions. The computations are based on the closure condition (7) with $k' = 0.20$. Table 1 gives the values of c, k, k' and r. In these four examples the k-values do not differ considerably from k', the r-values, however, cover a range from 3 to 20% for constant k'.

These considerations offer another possibility for a thermal closure, which is much more consistent with the original idea of Ball (1960), i.e.

$$r = -\frac{\int_{p_b}^{p_s}(F_{sv} < 0)\,dp}{\int_{p_b}^{p_s}(F_{sv} > 0)\,dp} = \text{constant} \tag{9}$$

This closure is not based on a ratio of fluxes but on the ratio of the integral buoyant energy dissipation over the integral buoyant energy production. In terms of the integrated turbulent energy equation for steady state and zero flux of turbulent energy at the boundaries of integration this means

$$\int_{p_b}^{p_s}(F_{sv} > 0)\,dp + \int_{p_b}^{p_s}(F_{sv} < 0)\,dp - \int_{p_b}^{p_s}\varepsilon_T\,dp = 0 \tag{10}$$

with ε_T = viscous dissipation of turbulent energy, which was generated by buoyancy. Equation (9) leads to

$$r = -\frac{\int_{p_b}^{p_s}\varepsilon_T\,dp}{\int_{p_b}^{p_s}(F_{sv} > 0)\,dp} \tag{11}$$

Table 1. *Evaluation of different coefficients which are determined by the profile of the virtual static energy flux F_{sv} and thus are essential for the closure condition: c according to eq. (2), k according to eq. (3), k' according to eq. (7) and r according to eq. (4). These terms are presented for four cases, which were described by Kraus and Schaller (1978) and which were plotted in Fig. 1. See text for further explanation. The subscripts of (F_{sv}) mean surface (s), higher and lower pressure side of cloud base (c^+, c^-) and higher pressure side of the boundary of the mixed layer (b^+). D is the divergence of the large-scale horizontal wind field; the large-scale vertical p-velocity ω_b can be obtained from $\omega_b = D(p_s - p_b)$*

Case	p_s mb	p_c mb	p_b mb	D $10^{-6}\,s^{-1}$	$(F_{sv})_s$ W/m²	$(F_{sv})_{c^+}$ W/m²	$(F_{sv})_{c^-}$ W/m²	$(F_{sv})_{b^+}$ W/m²	c	k	k'	r %
TW	1013.2	888.4	837.2	5.0	27.4	−10.1	57.4	39.3	−1.43	0.18	0.20	4.6
CC	1013.2	969.1	923.6	4.0	−3.1	− 3.1	14.8	15.5	+5.00	0.20	0.20	19.9
NS	1013.2	891.7	825.7	4.0	22.3	− 4.6	18.8	0.3	−0.01	0.21	0.20	2.7
AO	1013.2	992.7	908.2	4.0	−4.9	− 0.6	2.9	23.0	+4.69	0.21	0.20	5.1

which together with (9) means that also the ratio of viscous dissipation of thermally produced turbulent energy over thermal production is constant.

Table 2 displays a similar information as Table 1, but only for the TW-case computed with the new closure (9) for different r-values. Both the coefficients k and k' increase with increasing r. Their difference is small for $r \leqslant 10\%$ and increases also with r.

From Tables 1 and 2 it is evident that different closure assumptions lead to different results concerning the inversion characteristics. But there still exists the difficulty to decide which closure is the right one and which values of k' or r should be taken in a certain case. It becomes quite clear from this study (Table 1) that the condition of a constant k' is not identical with the condition of a constant r. What can be done (and what has been done) is (was) to find some guidance from experimental results or from a comparison of model computations with measured inversion characteristics.

However, there is another result from Table 2: a relatively small sensitivity of p_c, p_b and F_{sv} to changes of r or k'. This fact was already clearly illustrated by the diagrams of Schubert's (1976) numerical experiments. The small changes of the inversion characteristics with r or k' should be compared with the errors arising from the rough model assumptions. This leads to the conclusion that the model results do not depend critically on r or k' if these values are chosen within a reasonable range (e.g. $10\% > r > 2\%$ and $0.30 > k' > 0.10$) and that both closures (7) and (9) may be used. Dissatisfaction remains because of the scanty knowledge of the coefficients r or k' and mainly because of the imperfect use of the turbulent energy equation.

The only completely satisfactory way to close

Table 2. *Evaluation of the profile of the flux of virtual static energy and of k' and k with the new closure of eq. (9) for different values of r = ratio of integral buoyant dissipation over integral buoyant production of turbulent energy. This is done for case TW in KS. For more explanation see legend of Table 1*

r %	p_s mb	p_c mb	p_b mb	D $10^{-6}\,s^{-1}$	$(F_{sv})_s$ W/m²	$(F_{sv})_{c^+}$ W/m²	$(F_{sv})_{c^-}$ W/m²	$(F_{sv})_{b^+}$ W/m²	k'	k
2.0	1013.2	893.9	842.7	5.0	28.9	− 6.9	57.9	38.9	0.13	0.12
5.0	1013.2	887.7	836.6	5.0	27.2	−10.5	57.4	39.2	0.21	0.18
10.0	1013.2	880.9	829.9	5.0	25.3	−14.5	56.7	38.4	0.30	0.26
20.0	1013.2	872.1	820.9	5.0	22.7	−19.7	55.8	37.1	0.42	0.35
30.0	1013.2	865.8	814.5	5.0	20.8	−23.5	55.0	36.2	0.52	0.43
50.0	1013.2	856.5	804.8	5.0	18.0	−29.1	53.9	34.6	0.68	0.54

the system of equations in the inversion model by the turbulent energy equation is to use it in the complete form as Stull (1976) and Zeman and Tennekes (1977) have tried to do. However, this demands knowledge of the dynamic quantities (e.g. wind speed, momentum flux) and will also require additional assumptions. The simple thermodynamic inversion models do not contain these dynamic quantities, so that in their framework one cannot go beyond the relative poor thermal closure assumptions as they are given by eqs. (7) and (9).

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