Tellus (1978), *30*, 219–228

A finite element model of a moist atmospheric boundary layer: Part I—Model equations

by M. J. MANTON, *CSIRO, Division of Cloud Physics, P.O. Box 134, Epping, N.S.W. 2121, Australia*

(Manuscript received November 15, 1976; in final form May 2, 1977)

ABSTRACT

A system of equations is derived in order to predict the first moments of the dependent variables in moist, deep convection. Closure of the non-linear system is achieved by making the covariances of the dependent variables proportional to an eddy diffusivity. The eddy diffusivity is proportional to the turbulent kinetic energy and to a characteristic time scale of the turbulence. Although the time scale is simply an algebraic function of the mean strain rate and the Richardson number, an additional partial differential equation is required to determine the turbulent energy. Cloud water is assumed to form instantaneously when the air becomes saturated. Consequently the average amount of cloud water depends upon the probability distribution of the total water (vapour plus liquid). To estimate the average cloud water, the total water is approximated by a binary switching process. Part II of the work describes a finite element method which is used to solve numerically the system of equations.

1. Introduction

The atmospheric boundary layer is invariably turbulent and so the prediction of its properties involves the classical problem of turbulence. Until recently the most successful approach to this problem has been through the use of the mixing-length hypothesis, which assumes that the turbulence acts in a viscous manner. Then the turbulent stresses in the momentum equations are taken to be proportional to the local mean rate of strain, where the constant of proportionality is an eddy diffusivity. The eddy diffusivity may depend upon the local gradients of velocity and temperature. More sophisticated techniques are being used currently and there is particular emphasis on second-order closure models where the explicit equations for the second moments of the variables are considered (e.g. Meller and Yamada, 1974; Wyngaard and Coté, 1974). Perhaps the major disadvantage of second-order models is the number of equations required to close the system: a three-dimensional model with rain and cloud water involves 28 partial differential equations (Manton and Cotton, 1977).

Launder and Spalding (1972) describe the evolution of turbulence modelling from the simple eddy viscosity up to the full second-order closure models. The first improvement on the original mixing-length model is to retain an eddy viscosity, but to make it proportional to the local turbulent energy and a local time scale (Kolmogorov, 1942). This is the strategy adopted in the present work. Thus, in addition to equations for the first moments of the dependent variables, we introduce a differential equation for the rate of change of turbulent energy. The time scale is an algebraic function of the local mean rate of strain and the local Richardson number.

The treatment of the thermodynamic variables in the present model is simplified by assuming that turbulent fluctuations in the temperature, density and pressure are small. In particular, the fractional deviation about the local ensemble average is taken to be small enough for second-order terms to be neglected. Moreover, the turbulent fluctuations are assumed to be incompressible so that the fractional deviation in pressure is negligible in comparison with that in density (Dutton and Fichtl, 1969). The basic equations are equivalent to those suggested by Manton and Cotton (1977) for deep convection, except that the thermodynamic variables are expanded about the local ensemble average in the present work and about a given reference state in Manton and Cotton.

Any liquid water in the present model is assumed to move with the air; that is, the terminal velocity of any condensate is neglected. On the other hand, a rain parameterization could be included readily. The degree of supersaturation in a cloud is observed to be extremely small (Warner, 1968) and so it is taken to be zero: the liquid water specific humidity is set equal to the difference between the total water and saturation specific humidities. While this is valid instantaneously, the presence of turbulent fluctuations implies that the average liquid water specific humidity is not equal to the difference between the average total water and saturation specific humidities. The determination of the average liquid water specific humidity requires knowledge of the probability distributions of the relevant specific humidities. An approach to this problem is given by Manton and Cotton (1977), who assume that the fluctuation in the difference between the total water and saturation specific humidities is normally distributed. However, observations in cumulus cloud (Warner, 1955; Squires, 1958; Warner and Squires, 1958) suggest that the distribution of water is far from normal. There is a sharp boundary between cloudy and clear regions, and the peak value of the liquid water content appears to be conserved in a horizontal flight. In the present work we assume therefore that the probability density function for the total water specific humidity consists of two delta functions. Thus the total water in a region containing some cloud can take only two values: it is a binary switching process. This model requires the introduction of a further differential equation for the variance of the total water specific humidity. The information gained from this is an estimate of the probability of the existence of cloud at a given point.

Systems of partial differential equations such as those proposed here are solved almost invariably by finite difference methods. On the other hand, a finite element method incorporating linear interpolation in both space and time (Manton, 1978) is suitable for solving these problems. One advantage of the method over finite difference methods is that no special technique has to be introduced to determine the pressure. The method is absolutely stable for all time steps and a variable mesh can be used without introducing an explicit transformation in the independent variables. The disadvantages of the finite element technique are that for a given number of time steps and spatial mesh points it requires considerably more computation and storage than finite difference methods.

To demonstrate the accuracy and consistency of the finite element method we use it in Part II of this work to solve the model equations for a one-dimensional moist boundary layer. Similar models of a boundary layer in the absence of condensation have been developed recently (e.g. Wyngaard and Coté, 1974; Yamada and Mellor, 1975). The model equations of Part I are simplified by the neglect of Coriolis forces and subsidence so that only one mean velocity component is considered. The finite element method of solution is shown to produce results which are remarkably insensitive to the time step and spatial mesh size.

2. Basic equations of motion

Although the solution of only a simple one-dimensional model is described in Part II, it is of interest to develop the equations of motion for a general boundary layer. The equations are derived under similar assumptions to those adopted by Manton and Cotton (1977). The thermodynamic variables can be decomposed into their local ensemble average values and turbulent fluctuations about the averages; thus the density, temperature and pressure of the air are

$$\rho = \bar{\rho} + \rho', \quad T = \bar{T} + T' \quad \text{and} \quad p = \bar{p} + p',$$

where an overbar denotes an ensemble average and a prime denotes a turbulent fluctuation. The magnitude of the fractional deviations $\rho'/\bar{\rho}$, $T'/\bar{T}$ and $p'/\bar{p}$ is assumed to be invariably small compared with unity. Therefore, when a thermodynamic variable occurs as a coefficient of a differential term in an equation, fractional deviations are neglected in comparison with unity. Moreover, when a thermodynamic variable occurs in an algebraic equation, the squares of fractional deviations are neglected in comparison with unity: equations are correct to the first order in fractional deviations. The specific humidity is assumed to have a magnitude comparable with $\rho'/\bar{\rho}$ and so it is treated as a fractional deviation quantity. A further assumption is that the turbulent fluctuations have a scale small enough for them to be incompressible; this implies, *inter alia*, that the fractional deviation in the pressure can be neglected in comparison with that of the density (Dutton and Fichtl, 1969).

From the above assumptions it follows readily that the equations for the conservation of momentum, mass and total water (vapour plus cloud water) are

$$\bar{\rho}\, du_i/dt - \bar{\rho}\varepsilon_{ijk} f_k u_j + \partial p/\partial x_i + \rho g_i = \bar{\rho}\nu\nabla^2 u_i, \tag{2.1}$$

$$d\rho/dt + \bar{\rho}\partial\bar{u}_j/\partial x_j = 0, \quad \partial u_j'/\partial x_j = 0, \tag{2.2}$$

$$dq_t/dt = \nu_t\nabla^2 q_t, \tag{2.3}$$

where (x_1, x_2, x_3) form a Cartesian coordinate system with x_3 increasing vertically upward; t is time; $\mathbf{u}$ is the velocity vector; $\mathbf{f}$ is the Coriolis frequency vector; $\mathbf{g} = (0, 0, g)$ is the gravitational acceleration; q_t is the total water specific humidity; ν is the kinematic viscosity of air and ν_t is the effective diffusivity of water in air.

The equations of state for dry air and water vapour are

$$p_a = \rho_a R_a T \quad \text{and} \quad p_v = \rho_v R_v T, \tag{2.4}$$

where R is a gas constant; the subscripts a and v denote dry air and water vapour, respectively. The dry air and vapour densities are related to the total air (air plus water) and water densities by

$$\rho_a = (1 - q_t)\rho \quad \text{and} \quad \rho_v = (q_t - q_c)\rho, \tag{2.5}$$

where q_c is the cloud water specific humidity. The air pressure is the sum of the partial pressures, i.e.

$$p = p_a + p_v. \tag{2.6}$$

From (2.4)–(2.6) and by making the assumptions discussed above, we find the appropriate equation of state to be

$$\bar{p} = R_a\bar{\rho}\bar{T}\{1 + \rho'/\bar{\rho} + T'/\bar{T} + (R_v/R_a - 1)q_t - (R_v/R_a)q_c\}. \tag{2.7}$$

The application of the first law of thermodynamics to a system containing unit mass of dry air yields

$$\{C_{pa}\, dT - (R_a T/p_a)\, dp_a\} + (\rho_v/\rho_a)\{C_{pv}\, dT - (R_v T/p_v)\, dp_v\} + (\rho/\rho_a)q_c C_w\, dT = T\, dS, \tag{2.8}$$

where C_p is a specific heat at constant pressure; C_w is the specific heat of liquid water; S is the entropy of the fluid. The entropy is increased by the condensation of water vapour and by the molecular diffusion of heat, and so

$$dS/dt = (L/T)\, d((\rho/\rho_a)q_c)/dt + D, \tag{2.9}$$

where L is the latent heat of vaporization for water and D represents the effects of molecular diffusion. Combining (2.8) and (2.9) and neglecting second-order terms in the deviations, we find that the conservation of energy is expressed by

$$d\theta_L/dt = \nu_\theta\nabla^2\theta_L, \tag{2.10}$$

where ν_θ is the diffusivity of heat and the potential temperature θ_L is given by

$$T = \bar{\theta}_L(\bar{p}/p_0)^{R_a/C_{pa}}(1 + \theta_L'/\bar{\theta}_L) + Lq_c/C_{pa}; \tag{2.11}$$

the reference pressure $p_0 = 1000$ mb. It is seen that the natural dependent variable in the energy equation is the potential temperature introduced by Betts (1973).

Any liquid water is assumed to move with the local fluid and the air does not become supersaturated. Hence the cloud water specific humidity is given by

$$q_c = \max(0, q_t - q_s), \tag{2.12}$$

where q_s is the saturation specific humidity. The Clausius–Clapeyron equation is

$$dp_s/dT = Lp_s/R_v T^2, \tag{2.13}$$

where $p_s = q_s\rho R_v T$ is the saturation vapour pressure. By taking L to be constant and neglecting terms that are small compared with unity, it is found from (2.13) that

$$q_s = \{1 + \alpha_{s0}(T_{s0}/\bar{T})(T'/\bar{T})\} \times (R_a/R_v)(p_{s0}/\bar{p})\exp\{\alpha_{s0}(1 - T_{s0}/\bar{T})\} \tag{2.14}$$

where α_{s0}, p_{s0} and T_{s0} are constants. With $L = 2.5154 \times 10^{10}$ erg g^{-1}, $p_{s0} = 12.272$ mb, $\alpha_{s0} = 19.2484$ and $T_{s0} = 283.16$ K, the mean saturation vapour pressure obtained from (2.14) is accurate to 1.7% over the range $-30\,°\mathrm{C} \leqslant \bar{T} \leqslant 20\,°\mathrm{C}$; L varies by less than 2.5% over this range.

Equations (2.1), (2.2), (2.3), (2.7), (2.10), (2.11), (2.12) and (2.14) form a system which is valid for moist, deep convection in the atmosphere. The equations can be adapted readily to include the effects of rain (that is liquid water which does not move with the local fluid). The system of equations involves turbulent fluctuations in addition to the mean variables. However, for most practical situations, sufficient information is obtained from the first (and perhaps the second) moments of the dependent variables. This introduces the problem of closure of the non-linear system. Although the present system can be used for a full second-order closure approximation (e.g. Manton and Cotton, 1977), we consider a simpler case: the turbulent fluxes appearing in the equations for the first moments of the dependent variables are assumed to behave in a viscous manner and they are characterized by an eddy diffusivity. The eddy diffusivity is proportional to the turbulent energy and a turbulent time scale.

3. First moments of dependent variables

The equations for the first moments of the dependent variables are found by taking the ensemble averages of the equations in Section 2. Thus, from (2.1), (2.2), (2.3), (2.7), (2.10), (2.11) and (2.14), we find

$$D\overline{u_i}/Dt - \varepsilon_{ijk}f_k\overline{u_j} + \bar{\rho}^{-1}\partial p/\partial x_i + g_i + \partial(\overline{u_i'u_j'})/\partial x_j = 0, \quad (3.1)$$

$$D\bar{\rho}/Dt + \bar{\rho}\partial\overline{u_j}/\partial x_j + \partial(\overline{\rho' u_j'})/\partial x_j = 0, \quad (3.2)$$

$$D\overline{q_t}/Dt + \partial(\overline{q_t'u_j'})/\partial x_j = 0, \quad (3.3)$$

$$\bar{p} = R_a\bar{\rho}\bar{T}\{1 + (R_v/R_a - 1)\overline{q_t} - (R_v/R_a)\overline{q_c}\}, \quad (3.4)$$

$$D\overline{\theta_L}/Dt + \partial(\overline{\theta_L'u_j'})/\partial x_j = 0, \quad (3.5)$$

$$\bar{T} = \bar{\theta}_L(\bar{p}/p_0)^{R_a/C_{pa}} + L\overline{q_c}/C_{pa}, \quad (3.6)$$

$$\overline{q_s} = (R_a/R_v)(p_{s0}/\bar{p}) \exp\{\alpha_{s0}(1 - T_{s0}/\bar{T})\}, \quad (3.7)$$

where $D/Dt = \partial/\partial t + \bar{u}_j\partial/\partial x_j$.

The effects of molecular diffusion are neglected in these equations; that is, the Reynolds number of the mean motion is taken to be large. The determination of the second moments in (3.1)–(3.5) is considered in Section 4 below.

The average cloud water specific humidity is given by the average of (2.12). Because the right-hand side of this equation involves a Heaviside unit step function, its average value requires some knowledge of the probability distribution of $q_t' - q_s'$. If the fluctuations in q_c about its mean are small then the effects of these fluctuations can be neglected. However, observations of q_c measured by horizontal flights through warm cumulus clouds show that, while a uniform peak value is maintained over much of a flight, the instantaneous value of q_c can become very low (Warner, 1955; Warner and Squires, 1958). Moreover the boundary between cloudy and clear regions is very sharp, and q_c occasionally may even fall to zero within the interior of a cloud (Squires, 1958). It would seem therefore that fluctuations in q_c about its local ensemble average ought not to be neglected. The fluctuations in q_c become more significant when q_c represents a spatial average. For example, in a one-dimensional model where the variables are averaged over a horizontal plane, q_c' must account for the difference between in-cloud and out-of-cloud conditions when the cloud cover is less than unity.

The observed behaviour of q_c suggests that it can be modelled by a binary switching process which alternates between a given maximum value and a minimum value; the latter correspondng to either the low in-cloud value when $\overline{q_c}$ is a local ensemble average or to the (zero) out-of-cloud value when $\overline{q_c}$ is horizontal spatial average. Thus the probability density function of q_c is approximated by

$$f_c(q_c) = (1 - \gamma)\delta(q_c - q_0^*) + \gamma\delta(q_c - q_l^*), \quad (3.8)$$

where $\delta(x)$ is the Dirac delta function; q_l^* and q_0^* are the two possible values of q_c; γ is the probability of the occurrence of the peak value q_l^*. It follows from (2.12) and (3.8) that $q_t - q_s$ also is essentially a binary switching process. (The detailed distribution of $q_t - q_s$ when it is less than zero is irrelevant to the distribution of q_c.)

The fluctuations in q_c must be considered because their magnitude is observed to be comparable with the mean value $\overline{q_c}$. If the fluctuations in the saturation specific humidity are small compared with $\overline{q_c}$ then they can be neglected and (2.12) reduces to

$$q_c = \max(0, q_t - \overline{q_s}); \quad (3.9)$$

that is, any fluctuations in q_c *are associated* primarily with fluctuations in the total water. We have from (2.14) and (3.7) that

$$q_s' = \alpha_{s0}(T_{s0}/\bar{T})(T'/\bar{T})\overline{q_s},$$

where $\alpha_{s0} \sim 20$. Thus temperature fluctuations of about 1 K lead to saturation specific humidity fluctuations of about $0.1\overline{q_s}$. Taking a typical value of 1 g kg^{-1} for $\overline{q_c}$ and hence for q_c', we see that the magnitude of q_s' is somewhat less than that of q_c' when $\overline{q_s}$ is less than about 10 g kg^{-1}; that is when the mean temperature is less than about 10 °C at 900 mb. It would seem that the validity of (3.9) increases with decreasing mean temperature, and it ought to be satisfactory at least in temperate regions.

Equation (3.9) can possibly be applied to trade-wind cumulus-cloud fields when a spatial average is taken. Warner (1967) finds that in horizontal flights below cloud base there are temperature variations of about 1 K between air directly under cloud and air not under cloud; this corresponds to variations in q_s of less than 1 g kg^{-1}. On the other hand, q_t varies by up to 4 g kg^{-1}. Thus fluctuations in q_c are expected to be dominated by the variations in total water.

Some simultaneous measurements of q_t, q_s and q_c in warm cumulus clouds (J. Warner, private communication) indicate that these three variables are strongly positively correlated with each other; that is, peak values of liquid water are associated with peak values of buoyancy. It follows from (2.12) that the magnitude of the fluctuations in q_s must be less than those in q_t; in particular, $q_s' \sim \beta q_t'$ where β is less than unity. Thus the fluctuations in q_c are essentially proportional to the fluctuations in q_t. Although (3.9) is not necessarily expected to be valid in warm clouds (i.e. β is not extremely small), the probability distribution of q_c can be estimated from the probability distribution of q_t. The concepts used in the present analysis could therefore be applied to more general situations.

Assuming that (3.8) and (3.9) are valid in regions of cloud, we find that the probability density function, mean and variance of the total water specific humidity are given by

$$\begin{aligned} f(q_t) &= (1-\gamma)\delta(q_t - q_0) + \gamma\delta(q_t - q_l), \\ \overline{q_t} &= \gamma q_l + (1-\gamma)q_0, \\ \overline{q_t'^2} &= \gamma(1-\gamma)(q_l - q_o)^2, \end{aligned} \tag{3.10}$$

where q_l and q_0 are the two possible values of q_t. From (3.9) and (3.10) we find that if the air is unsaturated when $q_t = q_0$ then the average cloud water specific humidity is

$$\overline{q_c} = \gamma \max(0, q_l - \overline{q_s}). \tag{3.11}$$

(Clearly the case where both q_0 and q_l are greater than $\overline{q_s}$ could be of interest. Then γ is not a measure of the probability of cloud, which is unity; γ is simply the probability of the occurrence of the peak value q_l.) Elimination of q_l from (3.10) and (3.11) yields

$$\overline{q_c} = \max\{0, \overline{q_t} - (1-\gamma)q_0 - \gamma\overline{q_s}\},$$

where (3.12)

$$\gamma = (\overline{q_t} - q_0)^2/\{(\overline{q_t} - q_0)^2 + \overline{q_t'^2}\}.$$

Hence the average amount of liquid water and the probability of cloud are determined by the specification of q_0. The variance $\overline{q_t'^2}$ is evaluated in Section 4 below.

Limiting values of $\overline{q_c}$ and γ are given when $q_0 = 0$ (corresponding to the clear air being completely dry) and when $q_0 = \overline{q_s}$ (corresponding to the clear air being just saturated). It is found from (3.12) that in the former case ($q_0 = 0$)

$$\overline{q_c} = \max(0, \overline{q_t} - \gamma\overline{q_s}), \quad \gamma = \overline{q_t}^2/(\overline{q_t}^2 + \overline{q_t'^2}), \tag{3.13}$$

and in the latter case ($q_0 = \overline{q_s}$)

$$\overline{q_c} = \max(0, \overline{q_t} - \overline{q_s}), \gamma = (\overline{q_t} - \overline{q_s})^2/\{(\overline{q_t} - \overline{q_s})^2 + \overline{q_t'^2}\}. \tag{3.14}$$

The behaviour of $\overline{q_c}$ and γ given by (3.13) and (3.14) is shown in Figs. 1 and 2. For a given mean saturation $\overline{q_t}/\overline{q_s}$ and normalized variance $\overline{q_t'^2}/\overline{q_s}^2$, (3.13) always yields a value of q_c larger than that given by (3.14). Moreover (3.13) gives rise to cloud water even when the mean saturation is less than unity. The difference between the formulations increases with increasing total water variance. The fractional cloud cover γ increases monotonically and approaches unity with increasing saturation. The cloud cover decreases with increasing total water variance.

It is seen from Figs. 1 and 2 that in general the mean cloud water specific humidity depends significantly on the closure assumption whenever the total water variance is large. When the total water variance is small, however, the detailed probability distribution of q_t does not affect the value of $\overline{q_c}$. The

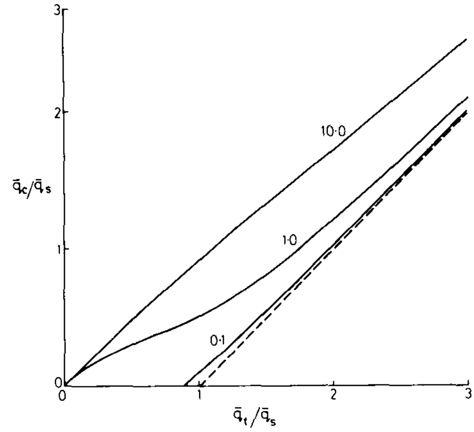

Fig. 1. Variation of normalized mean cloud water specific humidity $\overline{q_c}/\overline{q_s}$ with mean saturation $\overline{q_t}/\overline{q_s}$. Numbers refer to the normalized total water variance $\overline{q_t'^2}/\overline{q_s}^2$. —— (3.13); --- (3.14).

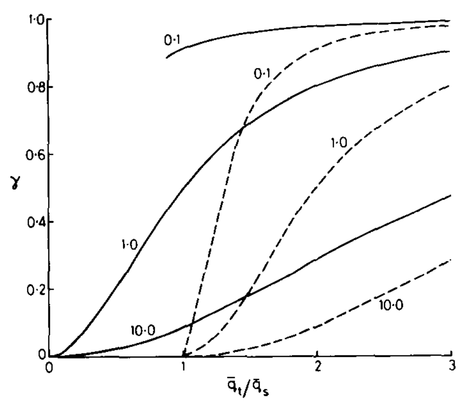

Fig. 2. Variation of fractional cloud cover γ with mean saturation $\overline{q_t}/\overline{q_s}$. Numbers refer to the normalized total water variance $\overline{q_t'^2}/\overline{q_s}^2$. —— (3.13); --- (3.14).

two closures presented here correspond to different physical situations. The first ($q_s = 0$) ought to be a suitable approximation of the initial development of a cloud field when moist elements with $q_t \simeq q_i$ first penetrate into dry air with $q_t \simeq 0$. This approximation is adopted in the present work. The second case ($q_0 = \overline{q_s}$) is an approximation to an almost uniform cloud layer in which any isolated region of air without cloud is on the point of becoming cloudy.

Other estimates for q_0 clearly can be incorporated into the model. However, further field observations of the relationship between q_t, q_c and T are necessary in order to obtain the most appropriate closure for the parameterization of $\overline{q_c}$. Perhaps the primary aim of this discussion has been to point out that the fluctuations in $\overline{q_c}$ are observed to be large and not to be normally distributed. The consequences of these fluctuations must be taken into account.

Equations (3.1)–(3.7) and (3.13) form a system of equations for the first moments of the dependent variables. The system is closed when the second moments appearing in them are specified.

4. Second moments of dependent variables

The second moments of the dependent variables which occur in Section 3 are assumed to have a diffusive nature. Thus we introduce an eddy diffusivity K such that the turbulent Reynolds stress is given by

$$\overline{u_i' u_j'} = \tfrac{2}{3}E\,\delta_{ij} - K\left\{\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} - \frac{2}{3}\frac{\partial \overline{u_k}}{\partial x_k}\,\delta_{ij}\right\} \quad (4.1)$$

where $E = \tfrac{1}{2}\overline{u_i'^2}$ is the turbulent kinetic energy per unit mass. The last term on the right-hand side of (4.1) ensures that the equation is consistent when the mean velocity field is compressible. Similarly the flux of a scalar quantity s, which is conserved (in the absence of molecular diffusion) by following a fluid particle, is represented by

$$\overline{s' u_i'} = -K_s\,\frac{\partial \bar{s}}{\partial x_i}, \quad \text{where } K_s = a_3 K; \quad (4.2)$$

a_3 is a constant to be determined. It follows from (2.3), (2.10) and (4.2) that

$$\overline{q_t' u_i'} = -K_s\,\frac{\partial \overline{q_t}}{\partial x_i} \quad \text{and} \quad \overline{\theta_L' u_i'} = -K_s\,\frac{\partial \overline{\theta_L}}{\partial x_i}. \quad (4.3)$$

Now the air density ρ is not conserved when following a fluid particle. However, it is found from (2.7), (2.11), (3.6) and (3.7) that to the first order in fractional deviations

$$\rho'/\bar{\rho} = -\theta_L'/\bar{\theta}_L - (R_v/Ra - 1)q_t' - (L/C_{pa}\bar{T} - R_v/R_a)q_c', \quad (4.4)$$

and hence

$$\overline{\rho' u_i'}/\bar{\rho} = -\overline{\theta_L' u_i'}/\theta_L - (R_v/Ra - 1)\overline{q_t' u_i'} - (L/C_{pa}\bar{T} - R_v/R_a)\overline{q_c' u_i'}. \tag{4.5}$$

Only the cloud water flux is not known on the right-hand side of (4.5).

An estimate of the cloud water flux can be found from the assumed probability distribution of the total water. Thus, setting $q_0 = 0$ in (3.8) and (3.9), we see that

$$\overline{q_c' u_i'} = \overline{q_t' u_i'}\, H(\overline{q_c}), \tag{4.6}$$

where $H(x)$ is the Heaviside unit step function. Equations (4.3), (4.5) and (4.6) yield

$$\overline{\rho' u_i'}/\bar{\rho} = K_s N_i/g, \tag{4.7}$$

where

$$N_i = g\left\{\frac{1}{\overline{\theta_L}}\frac{\partial \overline{\theta_L}}{\partial x_i} + [R_v/R_a - 1 + (L/C_{pa}\bar{T} - R_v/R_a)H(\overline{q_c})]\frac{\partial \overline{q_t}}{\partial x_i}\right\};$$

$N_3^{1/2}$ is the Brunt-Vaisala frequency when the fluid is stably stratified.

The eddy diffusivity is specified in the form suggested by Kolmogorov (1942); that is

$$K = a_2 E\tau, \tag{4.8}$$

where τ is a characteristic time scale of the turbulence, and a_2 is a constant. The time scale is assumed to be of the form proposed by Manton and Cotton (1977) when deriving a full second-order closure approximation; hence

$$\tau^{-2} = (\partial\overline{u_i}/\partial x_j)^2\, \psi(Ri), \tag{4.9}$$

where the Richardson number $Ri = N_3/(\partial\overline{u_i}/\partial x_j)^2$.

Equation (4.9) is simply a generalization of the usual assumption, used to derive the logarithmic velocity profile in a neutral boundary layer, that the time scale of the mean flow and the turbulence time scale are equivalent.

The stability function ψ is chosen to be given by

$$\psi(Ri) = \begin{cases} 1 - a_5\, Ri, & Ri \leqslant 0; \\ 1 - a_3\, Ri, & 0 < Ri < Ri_{\text{crit}}; \\ a_4\, Ri, & Ri \geqslant Ri_{\text{crit}}; \end{cases} \tag{4.10}$$

where $a_4 = Ri_{\text{crit}}^{-1} - a_3$; a_5 and Ri_{crit} are constants. It is seen that ψ is continuous and that it is unity for neutral conditions (i.e. for $Ri = 0$). The reasons for the detailed form of (4.10), which is somewhat different from that used by Manton and Cotton (1977), is discussed below.

To determine E in (4.1) and (4.2) we use the turbulent energy equation which is obtained by multiplying (2.1) by u_i' and then averaging; thus

$$\frac{DE}{Dt} + \overline{u_i' u_j'}\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial}{\partial x_j}\overline{(u_j' u_i')} + \frac{1}{\bar{\rho}}\frac{\partial}{\partial x_j}\overline{(p' u_i')} + \frac{\overline{\rho' u_3'}}{\bar{\rho}}g + \nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} = 0. \tag{4.11}$$

Since the Reynolds number of the mean motion is high, molecular diffusion is neglected in (4.11). On the other hand, viscous dissipation cannot be neglected and the last term in (4.11) is modelled by

$$\varepsilon \equiv \nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} = \frac{a_1}{\tau}E, \tag{4.12}$$

where a_1 is a constant. Equation (4.12) is consistent with the observation that, while dissipation occurs at small scales, the rate of dissipation of energy is controlled by the large-scale motion (Batchelor, 1953). In accordance with (4.1), the turbulent diffusion terms are approximated by

$$\frac{\partial}{\partial x_j}\overline{(u_j' u_i'^2)} + \frac{1}{\bar{\rho}}\frac{\partial}{\partial x_i}\overline{(p' u_i')} = -\frac{\partial}{\partial x_j}\left(K\frac{\partial E}{\partial x_j}\right). \tag{4.13}$$

Putting (4.12) and (4.13) into (4.11), we obtain the turbulent energy equation

$$\frac{DE}{Dt} + \overline{u_i' u_j'}\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\overline{\rho' u_3'}}{\bar{\rho}}g + \frac{a_1}{\tau}E = \frac{\partial}{\partial x_j}\left(K\frac{\partial E}{\partial x_j}\right). \tag{4.14}$$

The eddy diffusivity can now be calculated from (4.9), (4.10) and (4.14) provided that the constants a_1 to a_5 and Ri_{crit} are known.

The constants in the turbulence parameterization are evaluated by comparing the theory with observations in a steady-state equilibrium surface layer. Then (4.14) and (4.9) reduce to

$$\overline{u_1' u_3'}\frac{\partial \bar{u}_1}{\partial x_3} + \frac{\overline{\rho' u_3'}}{\bar{\rho}} g + \frac{a_1}{\tau} E = \frac{\partial}{\partial x_3}\left(K \frac{\partial E}{\partial x_3}\right), \quad (4.15)$$

with $\tau^{-2} = (\partial \bar{u}_1/\partial x_3)^2 \psi(Ri)$, where $Ri = N_3/(\partial \bar{u}_1/\partial x_3)^2$.

In the equilibrium layer the fluxes u_*^2 and H_* are constants, where

$$u_*^2 = -\overline{u_1' u_3'} = K \partial \bar{u}_1/\partial x_3$$

and (4.16)

$$H_* = \overline{\rho' u_3'}/\bar{\rho} = a_3 K N_3/g.$$

It is convenient to define the dimensionless profile functions (Monin and Yaglom, 1971)

$$\phi_m = \frac{k x_3}{u_*}\frac{\partial \bar{u}_1}{\partial x_3} \quad \text{and} \quad \phi_h = \frac{k x_3 u_* N_3}{g H_*}, \quad (4.17)$$

where k is the von Karman constant.

Measurements in a neutral layer ($H_* = 0$) show that $u_*^2/E = 0.32$ (Mellor, 1973). It follows from (4.8), (4.10), (4.15) and (4.16) that

$$a_1 = a_2 = 0.32. \quad (4.18)$$

Equations (4.2), (4.16) and (4.17) imply that $\phi_h/\phi_m = 1/a_3$. Businger et al. (1971) observe that this ratio equals 0.74 in neutral conditions. We therefore take

$$a_3 = 1.35. \quad (4.19)$$

Businger et al. (1971) find that turbulence cannot be sustained in the surface layer when the Richardson number Ri is greater than 0.21, and so we set

$$Ri_{\text{crit}} = 0.21 \quad (4.20)$$

in (4.10). When Ri is less than this critical value in stable conditions, Wyngaard and Coté (1971) observe an approximate balance between the net production and dissipation of turbulent energy; that is, the right-hand side of (4.15) is negligible. Then eqs. (4.9), (4.15) and (4.16) reduce to (4.10) over the range $0 < Ri < Ri_{\text{crit}}$. Above the critical Richardson number, there is a net dissipation of turbulence. Assuming that the rate of dissipation is proportional to the Brunt-Vaisala frequency and independent of the mean shear, we obtain (4.10) from (4.9) when $Ri \geqslant Ri_{\text{crit}}$.

In free convection conditions ($Ri \to -\infty$), the turbulence is sustained by the unstable stratification. The turbulent energy, the turbulent time scale, the rate of dissipation of turbulent energy and the mean density gradient are expected to be independent of the mean shear. Dimensional arguments consequently imply that as $Ri \to -\infty$ (Monin and Yaglom, 1971)

$$\begin{aligned} \tau^{-2} &\sim -a_5 N_3, \\ \varepsilon &\sim c_1(-gH_*), \\ E &\sim c_2(-gH_* k x_3)^{2/3}, \\ N_3 &\sim -(c_3/a_3)(-gH_*)^{2/3}(k x_3)^{-4/3}, \end{aligned} \quad (4.21)$$

where c_1, c_2 and c_3 are constants. It is apparent from (4.9) that (4.10) for $Ri \leqslant 0$ is the simplest extrapolation of (4.21a) to the origin under the constraint that $\psi = 1$ at $Ri = 0$. Putting (4.21) into (4.12), (4.15) and (4.16), we obtain the following relationships between the constants:

$$\begin{aligned} c_1 &= a_1 c_2 (a_5 c_3/a_3)^{1/2}, \\ 1 - a_5/a_3 &= 2c_2 k^2/3c_3, \\ 1 &= a_2 c_2 (a_3 c_3/a_5)^{1/2}. \end{aligned} \quad (4.22)$$

(It is of interest to note that eqs. (4.15) and (4.21) together imply that in free convection there is locally a net sink of energy which is balanced by turbulent diffusion. This is consistent with the results of Wyngaard and Coté (1971), in moderately unstable conditions at least.)

Now Wyngaard and Coté (1971) find that the rate of dissipation of energy in unstable conditions can be approximated by

$$k x_3 \varepsilon/u_*^3 = \{1 + 0.5\,(-k x_3 g H_*/u_*^3)^{2/3}\}^{3/2}.$$

This implies that $c_1 = 0.35$ and hence from (4.18), (4.19) and (4.22) with $k = 0.35$ (Businger et al., 1971),

$$a_5 = 0.47 \quad (4.23)$$

with $c_2 = 3.00$ and $c_3 = 0.38$. All the constants appearing in the parameterization of the turbulent fluxes are now specified.

It remains to determine the variance of the total water specific humidity which arises in (3.13). The equation for $\overline{q_t'^2}$ is found by multiplying (2.3) by q_t'

and averaging; thus

$$\frac{D\overline{q_t'^2}}{Dt} + 2\overline{q_t' u_j'}\frac{\partial \overline{q_t}}{\partial x_j} + \frac{\partial}{\partial x_j}(\overline{u_j' q_t'^2}) + 2\nu_t \overline{\left(\frac{\partial q_t'}{\partial x_j}\right)^2} = 0, \quad (4.24)$$

where molecular diffusion is neglected. The turbulent transport term and the dissipation term in (4.24) are modelled in the same manner as those terms in (4.11); in particular,

$$\overline{u_j' q_t'^2} = -K_s \partial \overline{q_t'^2}/\partial x_j,$$
$$2\nu_t \overline{(\partial q_t'/\partial x_j)^2} = a_6 \tau^{-1} \overline{q_t'^2}, \quad (4.25)$$

where a_6 is a constant. Equations (4.24) and (4.25) imply that

$$\frac{D\overline{q_t'^2}}{Dt} + 2\overline{q_t' u_j'}\frac{\partial \overline{q_t}}{\partial x_j} + \frac{a_6}{\tau}\overline{q_t'^2} = \frac{\partial}{\partial x_j}\left(K_s \frac{\partial \overline{q_t'^2}}{\partial x_j}\right). \quad (4.26)$$

As before, the constant a_6 is found from consideration of the equilibrium surface layer in which the scalar flux Q_* is constant, where

$$Q_* = -\overline{q_t' u_3'} = K_s \partial \overline{q_t}/\partial x_3. \quad (4.27)$$

Defining the profile function

$$\phi_t = \frac{k x_3 u_*}{Q_*}\frac{\partial \overline{q_t}}{\partial x_3}, \quad (4.28)$$

we see from (4.16), (4.17), (4.27) and (4.28) that

$$\phi_t = \phi_h. \quad (4.29)$$

Equation (4.26) in the equilibrium layer reduces to

$$2\overline{q_t' u_3'}\frac{\partial \overline{q_t}}{\partial x_3} + \frac{a_6}{\tau}\overline{q_t'^2} = \frac{\partial}{\partial x_3}\left(K_s \frac{\partial \overline{q_t'^2}}{\partial x_3}\right). \quad (4.30)$$

In free convection, the variance of a scalar is expected to be independent of u_* and so dimensional arguments imply that as $Ri \to -\infty$ (Monin and Yaglom, 1971)

$$\overline{q_t'^2} \sim c_4^2 Q_*^2 (-gH_* k x_3)^{-2/3}. \quad (4.31)$$

Smedman-Högström (1973) and Leavitt and Paulson (1975) observe that $c_4 = 1.2$. It therefore follows from (4.27)–(4.31) that

$$a_6 = 1.34. \quad (4.32)$$

Equations (4.1), (4.2), (4.3), (4.7), (4.8), (4.9), (4.10), (4.14) and (4.26) form a closed system for the second moments occurring in the equations of Section 3. The constants appearing in the equations of Section 4 are specified by (4.18), (4.19), (4.20), (4.23) and (4.32).

5. Conclusions

A system of equations has been derived in order to predict the first moments of the dependent variables in moist, deep convection. Those second moments which arise in the equations for the first moments are modelled by means of an eddy diffusivity. The eddy diffusivity is proportional to the turbulent kinetic energy and to a characteristic time scale (Kolmogorov, 1942). An additional partial differential equation is needed to determine the turbulent energy, but the time scale is governed by an algebraic function of the mean strain rate and the Richardson number. The system models the behaviour of the turbulence covariances in all stability conditions without introducing many additional differential equations. In particular, it is shown in Part II that, despite the use of an eddy diffusivity, the model equations can represent the deepening of well-mixed layers.

The air in cloud is assumed to be precisely saturated in the present model: any water in excess of the saturation value is assumed to condense instantaneously. This implies that the algebraic equation for the cloud water specific humidity involves a Heaviside step function. Consequently the average value of the cloud water specific humidity depends upon the probability distribution of the total water (vapour plus liquid) specific humidity. Consistent with observations in cumulus cloud by Warner (1955), the total water is approximated by a binary switching process.

In Part II of this work it is demonstrated that a finite element method can be used to solve numerically the system of model equations.

REFERENCES

Batchelor, G. K. 1953. *Theory of homogeneous turbulence.* Cambridge Univ. Press.

Betts, A. K. 1973. Non-precipitating cumulus convection and its parameterization. *Q. J. R. Meteorol. Soc. 99*, 178.

Businger, J. A., Wyngaard, J. C., Isumi, Y. and Bradley,

E. F. 1971. Flux-profile relationships in the atmospheric surface layer. *J. Atmos. Sci. 28*, 181.

Dutton, J. A. and Fichtl, G. H. 1969. Approximate equations of motion for gases and liquids. *J. Atmos. Sci. 26*, 241.

Kolmogorov, A. N. 1942. Equations of turbulent motion of an incompressible turbulent fluid. *Izv. Akad. Nauk SSSR Ser. Phys. VI*, No. 1–2, 56.

Launder, B. E. and Spalding, D. B. 1972. *Mathematical models of turbulence*. Academic Press.

Leavitt, E. and Paulson, C. A. 1975. Statistics of surface layer turbulence over the tropical ocean. *J. Phys. Ocean. 5*, 143.

Manton, M. J. 1978. On the solution of initial value problems by finite element methods. Submitted for publication.

Manton, M. J. and Cotton, W. R. 1977. *Formulation of approximate equations for modeling moist deep convection on the mesoscale*. Dept. Atmos. Sci., Colorado State Univ., Atmos. Sci. Paper No. 266.

Mellor, G. L. 1973. Analytic prediction of the properties of stratified planetary surface layers. *J. Atmos. Sci. 30*, 1061.

Mellor, G. L. and Yamada, T. 1974. A hierarchy of turbulence closure models for planetary boundary layers. *J. Atmos. Sci. 31*, 1791.

Monin, A. S. and Yaglom, A. M. 1971. *Statistical fluid mechanics*, Vol. 1. M.I.T. Press.

Smedman-Högström, A.-S. 1973. Temperature and humidity spectra in the atmospheric surface layer. *Boundary-Layer Meteorol. 3*, 329.

Squires, P. 1958. The spatial variation of liquid water and droplet concentration in cumuli. *Tellus 10*, 372.

Warner, J. 1955. The water content of cumuliform cloud. *Tellus 7*, 449.

Warner, J. 1967. Some observations on the orographic cloud of the island of Hawaii and on trade wind cumuli nearby. *Tellus 19*, 456.

Warner, J. 1968. The supersaturation in natural clouds. *J. Rech. Atmos. 3*, 233.

Warner, J. and Squires, P. 1958. Liquid water content and the adiabatic model of cumulus development. *Tellus 10*, 390.

Wyngaard, J. C. and Coté, O. R. 1971. The budgets of turbulent kinetic energy and temperature variance in the atmospheric surface layer. *J. Atmos. Sci. 28*, 190.

Wyngaard, J. C. and Coté, O. R. 1974. The evolution of a convective planetary boundary layer—a higher-order-closure model study. *Boundary-Layer Meteorol. 7*, 289.

Yamada, T. and Mellor, G. L. 1975. A simulation of the Wangara atmospheric boundary layer data. *J. Atmos. Sci. 32*, 2309.

КОНЕЧНО ЭЛЕМЕНТНАЯ МОДЕЛЬ ВЛАЖНОГО ПОГРАНИЧНОГО СЛОЯ АТМОСФЕРЫ: ЧАСТЬ I — УРАВНЕНИЯ МОДЕЛИ

Для предсказания первых моментов зависимых переменных во влажной глубокой конвекции выводится система уравнений. Замыкание нелинейной системы достигается принятием ковариаций зависимых величин пропорциональными турбулентной вязкости. Турбулентная вязкость пропорциональна турбулентной кинетической энергии и характерному масштабу времени турбулентности. Хотя масштаб времени является простой алгебраической функцией средней скорости деформации и числа Ричардсона, для определения турбулентной энергии требуется дополнительное дифференциальное уравнение в частных производных. Предполагается, что водяные капли в облаках образуются мгновенно, когда воздух становится насыщенным. Поэтому средняя водность облаков зависит от распределения вероятности полного количества влаги (пар плюс жидкость). Для оценки средней водности облаков полное влагосодержание аппроксимируется в виде двойного альтернирующего процесса. В части II работы описан метод конечных элементов, который используется для численного счета системы уравнений.