Tellus (1978), 30, 201–218

Variational non-linear normal mode initialization

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(Manuscript received July 1; in final form October 12, 1977)

ABSTRACT

The non-linear normal mode initialization technique of Machenhauer (1977) and Baer (1977) has shown great potential in removing spurious high frequency gravity modes from primitive equation model integrations. In essence, this procedure orthogonalizes the time tendencies of the original fields to the time tendencies of the high frequency normal modes of the linearized model equations. In the present work this procedure is generalized by casting the problem in a variational framework. Thus a variational integral is minimized subject to strong constraints obtained from the theory of non-linear normal mode initialization. When applied to the shallow water equations, it is found that this variational approach includes most previous barotropic static initialization procedures as special cases. The procedure was tested under various conditions and the results displayed.

1. Introduction

Gravity wave oscillations have been a persistent problem in primitive equation models since their introduction in the late 1950's. These high frequency oscillations have excessive amplitude because the wind and mass fields cannot in general be prescribed initially with sufficient accuracy. In order to meet this problem, many initialization procedures have been developed over the years. In all of these procedures, the observed wind fields, mass fields or both are adjusted or balanced in some fashion so as to minimize gravity wave oscillations during the model integration.

The earliest procedures, the linear and non-linear-balance equations were developed because of the paucity of wind data at the time. The non-linear (forward) balance equation was not easy to use in practice because of ellipticity problems in tropical regions. It also produced only the rotational component of the winds. The increasing availability of wind data from aircraft and satellites, together with the ellipticity problem, eventually rendered obsolete the direct use of the balance equation for calculating winds.

Variational techniques, introduced by Sasaki (1958, 1969, 1970) and Stephens (1970) are static initialization methods which permit the mutual adjustment of wind and mass fields. In these techniques, the presumed accuracy of the original wind and mass fields is reflected in pre-specified weight functions. Variational methods minimize the adjustment of the original fields while satisfying exactly or approximately such constraints as the balance equation. Recently, baroclinic variational initialization has been attempted by including the hydrostatic equation as a further constraint (Phillips, 1977; Barker et al., 1977). These techniques make optimal use of the observed data, but when used with traditional constraints such as the non-linear balance equation, they cannot produce a divergent wind component and they cannot completely eliminate gravity oscillations from a primitive equation model.

One of the most promising initialization procedures of recent years has been the normal mode approach. In this procedure, the first step is to obtain the normal modes of a linearized version of the primitive equation model that is to be integrated. These normal modes can usually be identified as low frequency Rossby modes, or high frequency gravity modes, with computational modes also appearing if the model equations have been discretized. In what might be called "linear normal mode initialization", the original fields are adjusted in such a way as to orthogonalize them to undesirable gravity and computational modes. Flattery (1970, 1972) developed an operational objective analysis scheme for global data using the normal modes of the tidal equations. Williamson (1976) obtained the normal modes of a grid-point model of the shallow water equations and orthogonalized the original fields to the high frequency modes. This technique uses both wind and mass fields, produces a divergent wind component, but unfortunately does not completely eliminate gravity modes from model integrations. This is because the original fields are orthogonalized to the high frequency modes of the *linearized equations* only; during the time-integration the non-linear terms act as forcing terms to regenerate the high frequency modes. Moreover, the adjusted fields satisfy only a kind of linear balance, which ignores the non-linear relation between mass and wind found in the gradient wind relation; thus, the fit to the original data is not very good.

This problem was overcome by the "non-linear normal mode initialization" technique of Machenhauer (1977). Rather than orthogonalize the original fields to the high frequency modes, he orthogonalized their *time-tendencies* to the time-tendencies of the high frequency modes. This meant that the time tendencies of the high frequency modes were initially zero and would remain small, provided the non-linear terms did not change rapidly with time. In problems with low Rossby number, this was the case, and the technique worked very successfully. Independently, Baer (1977) with a quite different approach using a Rossby number expansion, arrived at basically the same result to second order. This type of procedure uses both wind and mass fields, produces a divergent wind component, and virtually eliminates gravity wave oscillations. This technique, can in principle, be used for any model or system of equations, for which the normal modes can be found.

It is the purpose of the present work to combine the non-linear normal mode initialization technique of Machenhauer (1977) and Baer (1977) with the variational procedure of Sasaki (1958). This combined procedure will allow for the mutual adjustment of the wind and mass fields based on the presumed accuracy of the observations, yield a divergent wind component and eliminate high frequency oscillations from the integration. It will also allow the user to adjust the wind to the mass or vice versa (within limits) as he chooses.

In the present article the technique is applied to the shallow water equations, but it can, in principle, be extended to more complicated systems. When the technique is applied to the shallow water equations, most previous barotropic (one-level) static initialization procedures will be seen to be special cases. This includes the forward and backward, linear and non-linear balance equations, variational methods employing the balance equations as constraints, and both linear and non-linear normal mode (barotropic) initialization techniques. It does not include baroclinic variational formulations such as that of Phillips (1977) or normal mode initializations not based on the barotropic equations.

In the following sections there will be a brief discussion of the normal modes of the shallow water equations followed by a résumé of the non-linear normal mode technique of Machenhauer (1977) and Baer (1977). The variational formulation will then be introduced and it will be demonstrated that the normal mode initialization is a special case. Other special cases will show the relationship between normal mode techniques and more conventional procedures such as the balance equation. Finally the technique will be applied to real data and its effectiveness will be tested in a model integration.

2. Normal modes of the linearized shallow water equations

The governing equations are the shallow water equations on the sphere, in the differentiated form used by Bourke (1972).

$$\frac{\partial}{\partial t}\nabla^2\psi + 2\Omega\mu\nabla^2\chi + \frac{2\Omega}{a^2}\left[\frac{\partial\psi}{\partial\lambda} + (1-\mu^2)\frac{\partial\chi}{\partial\mu}\right] = R_\psi, \tag{2.1}$$

$$\frac{\partial}{\partial t}\nabla^2\chi - 2\Omega\mu\nabla^2\psi + \frac{2\Omega}{a^2}\left[\frac{\partial\chi}{\partial\lambda} - (1-\mu^2)\frac{\partial\psi}{\partial\mu}\right] + \nabla^2\Phi = R_\chi, \tag{2.2}$$

$$\frac{\partial}{\partial t}\Phi + \bar{\Phi}\nabla^2\chi = R_\Phi, \tag{2.3}$$

where

$$R_\psi = -\frac{1}{a}\left[\frac{1}{1-\mu^2}\frac{\partial}{\partial\lambda}(U\nabla^2\psi) + \frac{\partial}{\partial\mu}(V\nabla^2\psi)\right], \tag{2.4}$$

$$R_\chi = \frac{1}{a}\left[\frac{1}{1-\mu^2}\frac{\partial}{\partial\lambda}(V\nabla^2\psi) - \frac{\partial}{\partial\mu}(U\nabla^2\psi)\right] - \tfrac{1}{2}\nabla^2\left(\frac{U^2+V^2}{1-\mu^2}\right), \tag{2.5}$$

$$R_\Phi = -\frac{1}{a}\left[\frac{1}{1-\mu^2}\frac{\partial}{\partial\lambda}(U\Phi) + \frac{\partial}{\partial\mu}(V\Phi)\right], \tag{2.6}$$

λ = longitude,
μ = $\sin\phi$, ϕ = latitude,
a = radius of earth,
g = gravitational constant,
Ω = angular velocity of earth,
ψ = stream function,
χ = velocity potential,
$\bar{\Phi}$ = horizontally averaged geopotential,
Φ = deviation from $\bar{\Phi}$,

$$U = -\frac{(1-\mu^2)}{a}\frac{\partial\psi}{\partial\mu} + \frac{1}{a}\frac{\partial\chi}{\partial\lambda},$$

$$V = \frac{1}{a}\frac{\partial\psi}{\partial\lambda} + \frac{(1-\mu^2)}{a}\frac{\partial\chi}{\partial\mu},$$

∇^2 = horizontal Laplacian operator in spherical coordinates.

The first step is to find the free normal modes of eqs. (2.1–2.3) linearized about a state of rest and with an equivalent depth $D = \bar{\Phi}/g$. This implies R_ψ, R_χ, R_Φ are set to zero, for the present. We follow basically the procedures of Kasahara (1976), differing mainly in the choice of dependent variables.

We assume that a particular normal mode, represented by the column vector (ψ, χ, Φ) has the following dependence on t, μ, λ.

$$\begin{bmatrix}\psi\\ \chi\\ \Phi\end{bmatrix} = \sum_{n=s}^{\infty}\begin{bmatrix}A_n^s\\ iB_n^s\\ 2\Omega C_n^s\end{bmatrix} P_n^s(\mu)\exp(is\lambda - 2\Omega i\sigma t), \tag{2.7}$$

where

s is the zonal (east–west) wavenumber,
$P_n^s(\mu)$ is the associated Legendre function of the first kind of degree s and order n,
σ is the normalized frequency of the normal mode, and
A_n^s, B_n^s, C_n^s are the real expansion coefficients of the Legendre Functions.

The Legendre Functions satisfy the following orthonormality condition

$$\int_{-1}^{1} P_n^s(\mu)P_m^s(\mu)\,d\mu = \delta_m^n, \tag{2.8}$$

where δ_m^n is the Kronecker delta function.

We insert the expansions (2.7) into eqs. (2.1–2.3) with $R_\psi = R_\chi = R_\Phi = 0$, multiply through by $P_m^r(\mu)\exp(-ir\lambda)$ and integrate over the sphere. Using the orthogonality properties of the complex trigonometric function $\exp(is\lambda)$, the Legendre orthogonality condition (2.8) and certain other properties of Legendre functions we obtain for each zonal wavenumber s,

$$-\frac{s}{n(n+1)}A_n^s + \frac{n-1}{n}\varepsilon_n^s B_{n-1}^s + \frac{n+2}{n+1}\varepsilon_{n+1}^s B_{n+1}^s = \sigma A_n^s, \tag{2.9}$$

$$\frac{n-1}{n}\varepsilon_n^s A_{n-1}^s + \frac{n+2}{n+1}\varepsilon_{n+1}^s A_{n+1}^s - \frac{s}{n(n+1)}B_n^s - C_n^s = \sigma B_n^s, \tag{2.10}$$

$$-n(n+1)kB_n^s = \sigma C_n^s, \tag{2.11}$$

where

$$\varepsilon_n^s = \sqrt{\frac{n^2-s^2}{4n^2-1}}, \qquad k = \frac{\bar{\Phi}}{4\Omega^2 a^2}.$$

Equations (2.9–2.11) define an infinite matrix problem, which in fact decouples into two separate matrix problems. There is first the symmetric problem with χ, Φ symmetric and ψ anti-symmetric with respect to the equator. Secondly, there is the anti-symmetric problem with χ, Φ anti-symmetric and ψ symmetric with respect to the equator. All modes are required in the global case, but in the hemispheric case only the symmetric modes (χ, Φ symmetric, ψ anti-symmetric) are required.

To render the problem tractable we truncate the infinite sum (2.7) at some finite value of $n = s + N$, for each zonal wavenumber s. This produces two algebraic eigenvalue problems (symmetric and anti-symmetric) with eigenvalue σ for each s. The method of obtaining the eigenvalues and eigenvectors closely follows Kasahara (1976) and uses standard algebraic eigenvalue methods.

For each zonal wavenumber s, and a given (even) N the symmetric and anti-symmetric matrixes are of order $3N/2$ and there are a total of $3N$ eigenvectors and eigenvalues. These eigenvectors are customarily divided into 3 classes on the basis of their eigenvalues. There are N high frequency eastward moving gravity modes, N high frequency westward moving gravity modes, and N low frequency westward moving Rossby modes. For large equivalent depths ($\bar{\Phi}$ large) there is a large separation between the Rossby wave frequencies and the westward gravity wave frequencies. This separation decreases as the equivalent depth decreases.

The meridional structure of the three modes (Rossby, eastward gravity and westward gravity) determined in this fashion can be seen in Longuet-Higgins (1968). These meridional structure functions are generally called Hough functions after Hough (1898).

In the case $s = 0$, there is a certain degeneracy and some of the modes (generally Rossby modes) have zero frequency, i.e. they are stationary. In this case there are no unique eigenvectors corresponding to these modes and their north–south structure cannot be determined by this method. For the symmetric case with $s = 0$, there are $N/2 + 2$ modes with $\sigma = 0$ but the remaining $N - 2$ gravity modes have non-zero frequency and their eigenstructure can be determined.

The horizontal structure of a particular free mode $\hat{\mathbf{Y}}_j^s(\mu, \lambda)$ with eigenfrequency σ_j^s is given by

$$\hat{\mathbf{Y}}_j^s(\mu, \lambda) = \begin{bmatrix} \hat{\psi}_j^s(\mu, \lambda) \\ \hat{\chi}_j^s(\mu, \lambda) \\ \hat{\Phi}_j^s(\mu, \lambda) \end{bmatrix} = \sum_{n=s}^{s+N} \begin{bmatrix} A_n^s(j) \\ iB_n^s(j) \\ 2\Omega C_n^s(j) \end{bmatrix} P_n^s(\mu) \exp(is\lambda), \tag{2.12}$$

where j assumes some ordering of the eigenmodes according to frequency and $0 \leqslant j \leqslant 3N$.

The functions $\hat{\psi}_j^s$, $\hat{\chi}_j^s$, $\hat{\Phi}_j^s$ are complex functions of latitude and longitude. We use the circumflex notation to indicate that these variables are known functions of λ, μ and to distinguish them from arbitrary fields ψ, χ, Φ which are denoted as follows:

$$\mathbf{Y} = \begin{bmatrix} \psi \\ \chi \\ \Phi \end{bmatrix}. \tag{2.13}$$

It is convenient to define the following integral operator

$$\langle \mathbf{Y} \cdot \hat{\mathbf{Y}}_j^{*s} \rangle = \frac{1}{4\pi} \int_{-1}^{1} \int_0^{2\pi} [\Phi \hat{\Phi}_j^{*s} - \bar{\Phi}(\hat{\psi}_j^{*s} \nabla^2 \psi + \hat{\chi}_j^{*s} \nabla^2 \chi)] \, d\mu \, d\lambda, \tag{2.14}$$

where the (*) indicates complex conjugation and the integral is taken over the sphere. It will be noted that $\langle \mathbf{Y} \cdot \mathbf{Y}^* \rangle$ defines the total energy (potential plus kinetic) for eqs. (2.1–2.3).

Since $\mathbf{Y}$ can be expanded in spherical harmonics

$$\mathbf{Y} = \sum_{s=-M}^{M} \sum_{n=s}^{s+N} \begin{bmatrix} \psi_n^s \\ i\chi_n^s \\ 2\Omega\Phi_n^s \end{bmatrix} P_n^s(\mu) \exp(is\lambda), \tag{2.15}$$

where ψ_n^s, χ_n^s, Φ_n^s are complex spherical harmonics expansion coefficients of the arbitrary fields ψ, χ, Φ then

$$\langle \mathbf{Y} \cdot \hat{\mathbf{Y}}_j^{*s} \rangle = 4\Omega^2 \sum_{n=s}^{s+N} [\Phi_n^s C_n^s(j) + n(n+1)k(\psi_n^s A_n^s(j) + \chi_n^s B_n^s(j))], \tag{2.16}$$

for $-M \leqslant s \leqslant M$.

The normal modes $\hat{\mathbf{Y}}_j^{*s}$ are orthogonal satisfying the following orthogonality condition

$$\langle \hat{\mathbf{Y}}_l^r \cdot \hat{\mathbf{Y}}_j^{*s} \rangle = 4\Omega^2 \delta_r^s \delta_j^l. \tag{2.17}$$

The magnitudes of the $A_n^s(j)$, $B_n^s(j)$, $C_n^s(j)$ are arbitrary when determined from (2.9–2.11) and thus they have been adjusted to satisfy the normalization implied by (2.17).

It is possible to expand an arbitrary field $\mathbf{Y}$ in terms of the normal modes $\hat{\mathbf{Y}}_j^s(\mu, \lambda)$ as well as in spherical harmonics. Thus

$$\mathbf{Y} = \sum_{s=-M}^{M} \sum_{j=1}^{3N} \gamma_j^s \hat{\mathbf{Y}}_j^s, \tag{2.18}$$

where the γ_j^s are the complex expansion coefficients of the free modes $\hat{\mathbf{Y}}_j^s$.

Unlike a spherical harmonic expansion (2.15), the free mode expansion (2.18) has the same expansion coefficient for each variable $(\hat{\psi}_j^s, \hat{\chi}_j^s, \hat{\Phi}_j^s)$. γ_j^s can be obtained from $\mathbf{Y}$, simply by

$$\gamma_j^s = \frac{1}{4\Omega^2} \langle \mathbf{Y} \cdot \hat{\mathbf{Y}}_j^{*s} \rangle. \tag{2.19}$$

3. Non-linear normal mode initialization

Having obtained the normal modes of eqs. (2.1–2.3) with $R_\psi = R_\chi = R_\Phi = 0$, the next step is to use them for initialization. We will first develop the equations for linear normal mode initialization and then introduce the non-linear techniques of Machenhauer (1977) and Baer (1977).

Suppose we have original fields of streamfunction ψ velocity potential χ and geopotential Φ, which we will denote by $\mathbf{Y}(0)$. In linear normal mode initialization we first calculate the amplitudes of the gravity modes in the original fields

$$\gamma_j^s(0) = \frac{1}{4\Omega^2} \langle \mathbf{Y}(0) \cdot \hat{\mathbf{Y}}_j^{*s} \rangle, \tag{3.1}$$

for $-M \leqslant s \leqslant M$, and $j \in [j_G]$, where we calculate $\gamma_j^s(0)$ for each mode in the set of modes $[j_G]$ we wish to suppress from the initial data and the subsequent model integration.

The next step is to simply remove these modes from the original data, producing adjusted fields $\mathbf{Y}(1)$

$$\mathbf{Y}(1) = \mathbf{Y}(0) - \sum_s \sum_j^G \gamma_j^s(0) \hat{\mathbf{Y}}_j^s \tag{3.2}$$

where $\sum_j^G$ indicates a sum over the modes $j \in [j_G]$.

If the procedure were used and then a model integrated in which R_ψ, R_χ, R_Φ were set to zero, then the high frequency modes would be filtered for all time. However, if the full non-linear eqs. (2.1–2.3) were integrated in a model, the high frequency oscillations would reappear because the $\hat{\mathbf{Y}}_j^s$ are not eigenmodes of the non-linear equations. This can be seen, for example, in Williamson (1976) who used essentially the procedure (3.1) and (3.2) for initialization and then ran with the full shallow water equations.

Ideally, one would like to obtain the normal modes of the fully non-linear eqs. (2.1–2.3) and then suppress these high frequency non-linear modes from the initial fields. This is, of course, not possible, but the procedure of Machenhauer (1977) does approach this ideal. Machenhauer hypothesized that if the terms R_ψ, R_χ, R_Φ did not change rapidly with time (as is the case for small Rossby number) then it might be possible to produce a better initialization by ensuring that the time tendencies of the high frequency linear modes were initially zero. In this case the initial amplitudes of the high frequency linear modes would be non-zero, but the time tendencies of the initialized fields would be orthogonal to the time tendencies of the high frequency linear modes.

In this case we expand ψ, χ, Φ in a normal mode expansion (2.18) and insert into eqs. (2.1–2.3). Using the fact that the $\hat{\mathbf{Y}}_j^s$ are eigenfunctions of the linear terms in (2.1–2.3) we get

$$\sum_s \sum_j \dot{\gamma}_j^s \nabla^2 \hat{\psi}_j^s = -\sum_s \sum_j 2\Omega i \sigma_j^s \gamma_j^s \nabla^2 \hat{\psi}_j^s + R_\psi, \tag{3.3}$$

$$\sum_s \sum_j \dot{\gamma}_j^s \nabla^2 \hat{\chi}_j^s = -\sum_s \sum_j 2\Omega i \sigma_j^s \gamma_j^s \nabla^2 \hat{\chi}_j^s + R_\chi, \tag{3.4}$$

$$\sum_s \sum_j \dot{\gamma}_j^s \hat{\Phi}_j^s = -\sum_s \sum_j 2\Omega i \sigma_j^s \gamma_j^s \hat{\Phi}_j^s + R_\Phi. \tag{3.5}$$

After multiplication of (3.3) by $\hat{\psi}_j^{*s}$ (3.4) by $\hat{\chi}_j^{*s}$ and (3.5) by $\hat{\Phi}_j^{*s}$, addition, integration over the sphere and application of the orthogonality condition (2.17) we obtain

$$\dot{\gamma}_j^s = -2\Omega i \sigma_j^s \gamma_j^s + \frac{1}{4\Omega^2} F(R_\psi, R_\chi, R_\Phi), \tag{3.6}$$

where

$$F(R_\psi, R_\chi, R_\Phi) = \frac{1}{4\pi} \int_{-1}^{1} \int_0^{2\pi} [R_\Phi \hat{\Phi}_j^{*s} - \bar{\Phi}(R_\psi \hat{\psi}_j^{*s} + R_\chi \hat{\chi}_j^{*s})] \, d\lambda \, d\mu.$$

If $\dot{\gamma}_j^s$ for a particular mode is set equal to zero then the time tendency of this mode will be initially zero in a model integration of the full shallow water eqs. (2.1–2.3). This can be ensured in (3.6) by calculating a new value of the amplitude of this mode $\gamma_j^s(1)$ from (3.6) with $\dot{\gamma}_j^s = 0$. Thus

$$\gamma_j^s(1) = \frac{1}{8\Omega^3 i\sigma_j^s} F(R_\psi, R_\chi, R_\Phi), \tag{3.7}$$

for $-M \leqslant s \leqslant M$, and $j \in [j_G]$.

The adjusted fields $\mathbf{Y}(1)$ in this case would be obtained by replacing the original amplitude $\gamma_j^s(0)$ by the desired values $\gamma_j^s(1)$ for $j \in [j_G]$;

$$\mathbf{Y}(1) = \mathbf{Y}(0) + \sum_s \sum_j^G [\gamma_j^s(1) - \gamma_j^s(0)]\hat{\mathbf{Y}}_j^s. \tag{3.8}$$

So far, R_ψ, R_χ, R_Φ have been calculated from $\mathbf{Y}(0)$, so that if we then proceed to use the $\mathbf{Y}(1)$ calculated above in the model integration to recalculate R_ψ, R_χ, R_Φ the balance will not be perfect and $\dot{\gamma}_j^s \neq 0$ for $j \in [j_G]$. This can be rectified by recalculating R_ψ, R_χ, R_Φ using $\mathbf{Y}(1)$ and thus producing $\gamma_j^s(2)$ in eq. (3.7) and $\mathbf{Y}(2)$ in eq. (3.8). This procedure is then repeated to obtain progressively better-balanced fields $\mathbf{Y}(2)$, $\mathbf{Y}(3)$, ... $\mathbf{Y}(F)$ until convergence. This procedure is very effective in removing spurious high frequency oscillations from integrations of the shallow water equations, as can be seen in Machenhauer (1977).

Baer (1977) and Baer and Tribbia (1977) developed a more general scheme employing a Rossby number expansion and two-timing techniques. Equations (3.1–3.2) are equivalent to their first order method and eqs. (3.7–3.8) to their second order method. Higher orders are also possible, but they involve higher time derivatives.

It might at first seem that this non-linear normal mode method cannot be improved or generalized. However we note that for each zonal wavenumber s, there are $2N$ gravity modes, but a total of $3N$ degrees of freedom for ψ, χ, Φ. This means that there are many possible ways that ψ, χ, Φ can be adjusted to satisfy (3.7–3.8) for each of the $2N$ gravity modes. This leads very naturally to a variational formulation in which we demand that the adjustments of ψ, χ, Φ be minimized subject to constraints (3.7–3.8) on the high frequency modes. This procedure, through the use of weights which express our confidence in the original fields, allows a considerable degree of control over the adjustment procedure. It will be shown that the non-linear normal mode initialization technique of Machenhauer (1977) is formally equivalent to the minimization of a variational integral with rather special weights.

4. Variational formulation

The basic procedure in developing a variational framework for non-linear normal mode initialization follows Sasaki (1958). We define first a variational integral which is a function of the adjustment in each of the dependent variables ψ, χ, Φ and of certain pre-specified, spatially varying weighting functions. The idea is to find the *minimum of this variational integral subject to the* constraints we wish to impose; thus obtaining adjusted ψ, χ, Φ fields which are optimally close to the original fields. Minimizing the variational integral can be shown to be equivalent to obtaining the solution of a series of differential equations called the Euler–Lagrange equations. In this case, we solve the Euler–Lagrange equations of the system by a spherical harmonic expansion, thus obtaining the adjustment to ψ, χ, Φ which minimizes the original variational integral.

The variational integral we wish to minimize is as follows

$$I(\Delta\mathbf{Y}, W_\mathbf{Y}) = \frac{a^2}{4\pi} \int_{-1}^{1} \int_0^{2\pi} [W_\psi(\nabla\Delta\psi)^2 + W_\chi(\nabla\Delta\chi)^2 + \frac{W_\Phi}{4\Omega^2}(\nabla\Delta\Phi)^2]\, d\psi\, d\mu, \tag{4.1}$$

where

$$\Delta\mathbf{Y} = \mathbf{Y}(1) - \mathbf{Y}(0) = \begin{bmatrix} \psi(1) - \psi(1) \\ \chi(1) - \chi(0) \\ \Phi(1) - \Phi(0) \end{bmatrix},$$

$$W_\mathbf{Y} = \begin{bmatrix} W_\psi(\mu, \lambda) \\ W_\chi(\mu, \lambda) \\ W_\Phi(\mu, \lambda) \end{bmatrix},$$

∇ = horizontal gradient operator in spherical coordinates.

$\Delta\mathbf{Y}$ expresses the adjustment in $\mathbf{Y}$ due to the minimization procedure where $\mathbf{Y}(0)$ represents the original fields and $\mathbf{Y}(1)$ the adjusted fields. The $W_{\mathbf{Y}}(\mu, \lambda)$ are the pre-specified spatially-dependent weighting functions mentioned earlier. These functions can be used to express our confidence in the original fields $\mathbf{Y}(0)$. Where W_ψ, W_χ or W_Φ are large, our confidence is high and only a small adjustment can take place. On the other hand if one of the weights is small, our confidence is low and a large adjustment is possible. The factor $1/4\Omega^2$ in the Φ term in (4.1) has been included for dimensional consistency. It will be noted that we have attempted to minimize the horizontal gradients of each of ψ, χ, Φ; this is geostrophically consistent at least as far as ψ, Φ are concerned, and is similar to the variational integral used by Stephens (1970).

The next step is to introduce the constraints we wish to impose. Essentially, we desire that the amplitude of a particular high frequency mode $[-M \leqslant s \leqslant M;\ j \in [j_G]]$ in the adjusted field be given by (3.7). This means that the adjustment $\Delta\mathbf{Y}$ must be such as to replace the original amplitude of the undesirable mode $-\gamma_j^s(0)$ by the desired value $\gamma_j^s(1)$. By making use of (2.19) and (2.14), the adjustments $\Delta\psi$, $\Delta\chi$, $\Delta\Phi$ can be shown to be related to $\gamma_j^s(1)$ and $\gamma_j^s(0)$ by

$$\frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}[\Delta\Phi\hat{\Phi}_j^{*s} - \bar{\Phi}(\Delta\psi\nabla^2\hat{\psi}_j^{*s} + \Delta\chi\nabla^2\hat{\chi}_j^{*s})]d\lambda\, d\mu = 4\Omega^2[\gamma_j^s(1) - \gamma_j^s(0)], \tag{4.2}$$

for $-M \leqslant s \leqslant M$, and $j \in [j_G]$.

For each zonal wavenumber s we will apply (4.2) for only those modes $j \in [j_G]$ which we wish to supress from the model integration. In (4.2) the only unknowns are $\Delta\psi$, $\Delta\chi$, $\Delta\Phi$. The horizontal eigenfunctions $\hat{\psi}_j^{*s}$, $\hat{\chi}_j^{*s}$, $\hat{\Phi}_j^{*s}$ are known and can be determined from (2.12). $\gamma_j^s(0)$ is a function of $\mathbf{Y}(0)$ and can be determined from (3.1). $\gamma_j^s(1)$ is also known and can be determined by calculating R_ψ, R_χ, R_Φ from $\mathbf{Y}(0)$ using (2.4–2.6) and then applying (3.6) and (3.7). It might be noted that the constraints imposed (4.2) are integral constraints, rather than the differential constraints which are usually imposed in variational initialization.

There are two possible directions in which we can proceed at this point. We can demand that the constraints (4.2) be satisfied exactly, in which case they will be called strong constraints. Alternatively, we can use a weak constraint formulation in which the constraints (4.2) are satisfied only approximately, according to a set of pre-specified weights. We have arbitrarily decided to choose strong constraints and are therefore forced to use the method of Lagrange's undetermined multipliers in order to obtain a solution (Craggs, 1973). To this end we create a new variational integral $J(\Delta\mathbf{Y}, W_{\mathbf{Y}})$ from $I(\Delta\mathbf{Y}, W_{\mathbf{Y}})$ and the Lagrange multipliers.

$$J(\Delta\mathbf{Y}, W_{\mathbf{Y}}) = I(\Delta\mathbf{Y}, W_{\mathbf{Y}}) + \sum_s \sum_j^G \alpha_j^s[\gamma_j^s(1) - \gamma_j^s(0)]. \tag{4.3}$$

The α_j^s, which are Lagrange's undetermined multipliers, are complex numbers, independent of latitude and longitude. It can be shown that the minimization of $J(\Delta\mathbf{Y}, W_{\mathbf{Y}})$ is equivalent to minimizing $I(\Delta\mathbf{Y}, W_{\mathbf{Y}})$ while exactly satisfying the constraints (4.2). It might be noted that in variational formulations which impose differential constraints, the Lagrange multipliers are functions of the independent spatial variables.

In order to make the minimization of $J(\Delta\mathbf{Y}, W_{\mathbf{Y}})$ more tractable we break up the weight functions $W_{\mathbf{Y}}$ in (4.1) into their zonal averages and deviations.

$$W_{\mathbf{Y}} = \overline{W}_{\mathbf{Y}}^{\lambda}(\mu) + W'_{\mathbf{Y}}(\mu, \lambda), \tag{4.4}$$

where

$$\overline{W}_{\mathbf{Y}}^{\lambda} = \frac{1}{2\pi}\int_0^{2\pi} W_{\mathbf{Y}}(\mu, \lambda)d\lambda.$$

It can be shown (Craggs, 1973, for example) that the minimization of a variational integral is equivalent to obtaining the solution to the Euler–Lagrange equations. These differential equations can be obtained from the variational integral $J(\Delta\mathbf{Y}, W_{\mathbf{Y}})$ by considering the individual variation of $\Delta\psi$, $\Delta\chi$ and $\Delta\Phi$, or by using standard formulas. The Euler–Lagrange equations for (4.3) are as follows:

$$2a^2\nabla\cdot\overline{W}_\psi^{\lambda}\nabla\Delta\psi + \frac{\bar{\Phi}}{4\Omega^2}\sum_s\sum_j^G \alpha_j^s\nabla^2\hat{\psi}_j^{*s} = -2a^2\nabla\cdot W'_\psi\nabla\Delta\psi, \tag{4.5}$$

$$2a^2\nabla\cdot\overline{W}_\chi^{\lambda}\nabla\Delta\chi + \frac{\bar{\Phi}}{4\Omega^2}\sum_s\sum_j^G \alpha_j^s\nabla^2\hat{\chi}_j^{*s} = -2a^2\nabla\cdot W'_\chi\nabla\Delta\chi, \tag{4.6}$$

$$\frac{a^2}{2\Omega^2}\nabla\cdot\overline{W}_\Phi\nabla\Delta\Phi-\frac{1}{4\Omega^2}\sum_s\sum_j^G\alpha_j^s\Phi_j^{*s}=-\frac{a^2}{2\Omega^2}\nabla\cdot W'_\Phi\nabla\Delta\Phi. \tag{4.7}$$

Equations (4.5–4.7) together with the constraints (4.2) constitute a system of differential equations which must be solved for the unknowns $\Delta\psi$, $\Delta\chi$, $\Delta\Phi$ and α_j^s. As mentioned earlier, the method of solution is by spherical harmonic expansion, using the normalization defined in eq. (2.15) thus

$$\Delta\mathbf{Y}=\begin{bmatrix}\Delta\psi\\ \Delta\chi\\ \Delta\Phi\end{bmatrix}=\sum_{s=-M}^{M}\sum_{n=s}^{s+N}\begin{bmatrix}\Delta\psi_n^s\\ i\Delta\chi_n^s\\ 2\Omega\Delta\Phi_n^s\end{bmatrix}P_n^s(\mu)\exp(is\lambda). \tag{4.8}$$

The $\Delta\psi_n^s$, $\Delta\chi_n^s$, $\Delta\Phi_n^s$ are complex expansion coefficients for the spectral harmonics. To illustrate the method of solution we will temporarily set W'_ψ, W'_χ, W'_Φ equal to zero. We then insert the expansions (4.8) into (4.5–4.7), multiply by an arbitrary spherical harmonic $P_m^r(\mu)\exp(-ir\lambda)$ and integrate over the sphere. The orthogonality condition for Legendre Functions (2.8) and the definition of $\hat{\psi}_j^s$, $\hat{\chi}_j^s$ and Φ_j^s (2.12) give

$$2\sum_{m=s}^{s+N}W_\psi(m,n,s)\Delta\psi_m^s+n(n+1)k\sum_j^G\alpha_j^sA_n^s(j)=0, \tag{4.9}$$

$$2\sum_{m=s}^{s+N}W_\chi(m,n,s)\Delta\chi_m^s+n(n+1)k\sum_j^G\alpha_j^sB_n^s(j)=0, \tag{4.10}$$

$$2\sum_{m=s}^{s+N}W_\Phi(m,n,s)\Delta\Phi_m^s+\sum_j^G\alpha_j^sC_n^s(j)=0, \tag{4.11}$$

for $-M\leqslant s\leqslant M$, and $s\leqslant n\leqslant s+N$ where

$$W_{\mathbf{Y}}(m,n,s)=\int_{-1}^{1}\overline{W}_{\mathbf{Y}}^{\lambda}(\mu)\left[(1-\mu^2)\frac{dP_m^s}{d\mu}\frac{dP_n^s}{d\mu}+\frac{s^2}{(1+\mu^2)}P_m^sP_n^s\right]d\mu.$$

In a similar manner the constraints (4.2) reduce to

$$\sum_{m=s}^{s+N}\left[\Delta\Phi_m^sC_m^s(j)+m(m+1)k(\Delta\psi_m^sA_m^s(j)+\Delta\chi_m^sB_m^s(j))\right]=\gamma_j^s(1)-\gamma_j^s(0) \tag{4.12}$$

for $-M\leqslant s\leqslant M$, and $j\in[j_G]$.

Equations (4.9–4.12) constitute a set of linear algebraic equations which are to be solved for the unknowns $\Delta\psi_m^s$, $\Delta\chi_m^s$, $\Delta\Phi_m^s$ and α_j^s, If the $\overline{W}_{\mathbf{Y}}^{\lambda}$ are symmetric with respect to the equator, then this set of equations decomposes into symmetric and anti-symmetric subsets. Let us suppose N even. Then for the symmetric case, for each value of s, there will be $N/2$ equations each of the form (4.9), (4.10), (4.11). In this case there are $N/2$ Rossby modes and N gravity modes. Suppose we decide to apply the method to all N gravity modes. Then there will be N constraints of the form (4.12). Thus for each value of s, eqs. (4.9–4.12) will define a matrix problem of order $5N/2$ for the symmetric case, and similarly for the anti-symmetric case. In a typical atmospheric forecast model using spherical harmonics, for example Daley et al. (1976), N is of order 30, so that the matrices to be inverted are of order 75. The left-hand side coefficient matrices defined in (4.9–4.12) are known and independent of data and can thus, in principle, be inverted once and for all.

In the case $s=0$, the matrix is singular because of the degeneracy of the eigenstructure (see Section 2). This can be remedied by removing 4 rows and columns from the matrix and then proceeding normally with the deflated matrix.

In the case where W'_ψ, W'_χ and W'_Φ on the right hand side of (4.5–4.7) are non-zero, an iterative procedure is used. In the first pass, the right hand sides are assumed zero and an initial guess $\Delta\psi$, $\Delta\chi$, $\Delta\Phi$ is obtained. The right hand sides of (4.5–4.7) are calculated from this guess and the procedure is repeated until convergence.

After we have solved (4.9–4.12) (together with an iteration if $W'_{\mathbf{Y}}\neq 0$) we have arrived at the same stage in the procedure as eq. (3.8), i.e. we have obtained an adjusted field $\mathbf{Y}(1)=\mathbf{Y}(0)+\Delta\mathbf{Y}$. where $I(\mathbf{Y}(1)-\mathbf{Y}(0),W_{\mathbf{Y}})$ is a minimum. At this point, there are two possibilities. Firstly, we can accept $\mathbf{Y}(1)$ as the initial field for subsequent model integration, in which case (4.1) would be minimized, but high frequency oscillations would not be completely removed from model integrations. The second option is to put the whole procedure in an iterative loop as described at the end of Section 3, thus obtaining progressively better balanced fields $\mathbf{Y}(2)$, $\mathbf{Y}(3)$, ... $\mathbf{Y}(F)$, and completely eliminating high frequency oscillations from subsequent model integrations. It should be noted that this latter procedure minimizes $I(\mathbf{Y}(k+1)-\mathbf{Y}(k),W_{\mathbf{Y}})$ for every iteration step k, but does not necessarily minimize $I(\mathbf{Y}(F) - \mathbf{Y}(0), W_{\mathbf{Y}})$; i.e. the total adjustment $\mathbf{Y}(F) - \mathbf{Y}(0)$ is not strictly a minimum. We choose the second option, generally obtaining satisfactory convergence in 2 to 3 scans (except in one case discussed below).

Case 1

It was stated at the end of Section 3, that the non-linear normal mode initialization technique of Machenhauer (1977) was equivalent to the minimization of a certain variational integral. We will now demonstrate that this is so. The variational integral that is to be minimized in this case is as follows:

$$H(\Delta\mathbf{Y}, W_{\mathbf{Y}}) = \frac{a^2k}{8\pi} \int_{-1}^{1} \int_{0}^{2\pi} \left[(\nabla\Delta\psi)^2 + (\nabla\Delta\chi)^2 + \frac{(\Delta\Phi)^2}{\bar{\Phi}} \right] d\psi\, d\mu, \tag{4.13}$$

subject to the same constraints (4.2) as before. It will be noted that $H(\Delta\mathbf{Y}, W_{\mathbf{Y}})$ is proportional to $\langle \Delta\mathbf{Y}, \Delta\mathbf{Y}^* \rangle$—the adjustment of the total energy.

If we derived the Euler–Lagrange equations of this system, substituted in a spherical harmonic expansion (4.8) we would obtain a series of equations similar to (4.9–4.11)

$$kn(n+1)[\Delta\psi_n^s + \sum_j^G \alpha_j^s A_n^s(j)] = 0, \tag{4.14}$$

$$kn(n+1)[\Delta\chi_n^s + \sum_j^G \alpha_j^s B_n^s(j)] = 0, \tag{4.15}$$

$$\Delta\Phi_n^s + \sum_j^G \alpha_j^s C_n^s(j) = 0. \tag{4.16}$$

Equation (4.12) would remain the same. If we multiply (4.14) by $A_n^s(j)$, eq. (4.15) by $B_n^s(j)$ and (4.16) by $C_n^s(j)$, sum over n, and use the orthogonality condition (2.17) we would find

$$\alpha_j^s = -[\gamma_j^s(1) - \gamma_j^s(0)] \tag{4.17}$$

upon application of the constraints (4.12). The insertion of (4.17) into (4.14–4.16) would give

$$\Delta\mathbf{Y} = \mathbf{Y}(1) - \mathbf{Y}(0) = \sum_s \sum_j^G [\gamma_j^s(1) - \gamma_j^s(0)], \tag{4.18}$$

which is the same result as (3.8).

This implies that the minimization of $H(\Delta\mathbf{Y}, W_{\mathbf{Y}})$ defined in (4.13), subject to the constraints (4.2) is equivalent to the non-linear normal mode technique of Machenhauer (1977) derived in Section 3. It is instructive to compare the minimization of $H(\Delta\mathbf{Y}, W_{\mathbf{Y}})$ with the minimization of $I(\Delta\mathbf{Y}, W_{\mathbf{Y}})$ defined in eq. (4.1). We note that in (4.1) the horizontal gradients of $\Delta\psi$, $\Delta\chi$, $\Delta\Phi$ are minimized which is geostrophically consistent; whereas eq. (4.13) minimizes the horizontal gradients of $\Delta\psi$ and $\Delta\chi$, but $\Delta\Phi$ appears in undifferentiated form. Thus, in effect, (4.13) is energetically consistent. This particular weighting gives a much smaller weight to small horizontal scales of Φ, and thus (4.13) (and the methods of Section 3), force the heights to the winds much more strongly than eq. (4.1).

Case 2

We go back to the original variational integral $I(\Delta\mathbf{Y}, W_{\mathbf{Y}})$ defined in eq. (4.1). If we choose $W_\chi = W_\Phi = 0$, $W_\psi = 1$ we will force χ, Φ to adjust without changing ψ. For each zonal wavenumber s in the symmetric case, there are only N degrees of freedom (those corresponding to χ, Φ) which can adjust. If the set $[j_G]$ contains all N gravity modes for each s, then (4.10) and (4.11) indicate that $\alpha_j^s = 0$ for $j \in [j_G]$.

If we consider the linear case—(i.e. $R_\psi = R_\chi = R_\Phi = 0$ and thus $\gamma_j^s(1) = 0$) we have an analogue to the linear reverse balance equation. In fact as $\bar{\Phi} \to \infty$, the velocity potential $\chi(1) \to 0$ and we will have exactly the linear reverse balance equation.

If, on the other hand, we calculate R_ψ, R_χ, R_Φ and $\gamma_j^s(1)$ and iterate the non-linear term as previously outlined we have an analogue to the non-linear reverse balance equation. Because, in this case, we produce both a geopotential and velocity potential from a streamfunction field we could call this a reverse non-linear tidal balance equation.

Case 3

We again choose the original variational integral $I(\Delta\mathbf{Y}, W_{\mathbf{Y}})$ defined in eq. (4.1), but choose $W_\psi = W_\chi = 0$, $W_\Phi = 1$ in an attempt to force winds to heights. There is a further degeneracy for $s = 0$ in this case and the left-hand side matrix defined in (4.9–4.12) must be further deflated by one row and column. The correct procedure for doing this can be found in Merilees (1968). As in case 2, $\alpha_j^s = 0$ for $[-M \leqslant s \leqslant M; j \in [j_G]]$.

If we consider the linear case ($R_\psi = R_\chi = R_\Phi = \gamma_j^s(1) = 0$) we have an analogue to the linear forward balance equation. As in Case 2, as $\tilde{\Phi} \to \infty$, $\chi(1) \to 0$ and we obtain exactly the forward linear balance equation.

In the non-linear case ($\gamma_j^s(1) \neq 0$), we must iterate the non-linear terms, producing an analogue to the non-linear forward balance equation. This procedure was found in practice to be non-convergent. The points of non-convergence were found to approximately coincide with the regions where the approximate ellipticity criteria for the classical non-linear forward balance equation ($\nabla^2\Phi + 2\Omega^2\mu^2 > 0$) was violated. These points generally occur in the tropical regions, which suggests that W_ψ in (4.1) must not be too small in the tropics.

5. Results

In order to test the variational initialization technique developed in Section 4 a number of experiments have been performed. These experiments demonstrate the effect of varying the analysis weights W_ψ, W_χ, W_Φ; the effect of varying the equivalent depth D; and the difference between linear and non-linear initialization. The effect of computational modes in model integrations run from the variationally adjusted fields was also investigated.

In all experiments the same original fields were used. These fields consisted of the 50 kPa northern hemisphere analysis of geopotential and windfields for 1200 GMT February 29, 1976 obtained from the Canadian operational objective analysis system (Rutherford, 1976). This objective analysis scheme is a multi-variate optimum interpolation procedure and thus the initial unadjusted windfields and geopotential fields are by no means independent. It was, however, thought that the original divergent wind component was very unrealistic and it was thus discarded. These original analyses were complete in the sense that they contained all the data available at the time. However, south of about 20° N, there was virtually no data, and the fields there were very artificial. The original stream function and geopotential fields can be seen in Fig. 5.

After the original fields $\mathbf{Y}(0)$ had been adjusted by the variational procedures of Section 4 they were used as initial conditions for an integration of the hemispheric barotropic (one-level) primitive equations spectral model of Bourke (1972). This model is a semi-implicit spherical harmonic model, which is completely compatible with eqs. (2.1–2.6). A rhomboidal spherical harmonic truncation (M $N = 30$) consistent with eq. (2.15) was used for all model integrations. In all cases the equivalent depth used in the adjustment process was the same as that used in the model integrations. The model has no spatial computational modes, but it does contain a temporal computational mode, because of the time centered differencing formulation. To

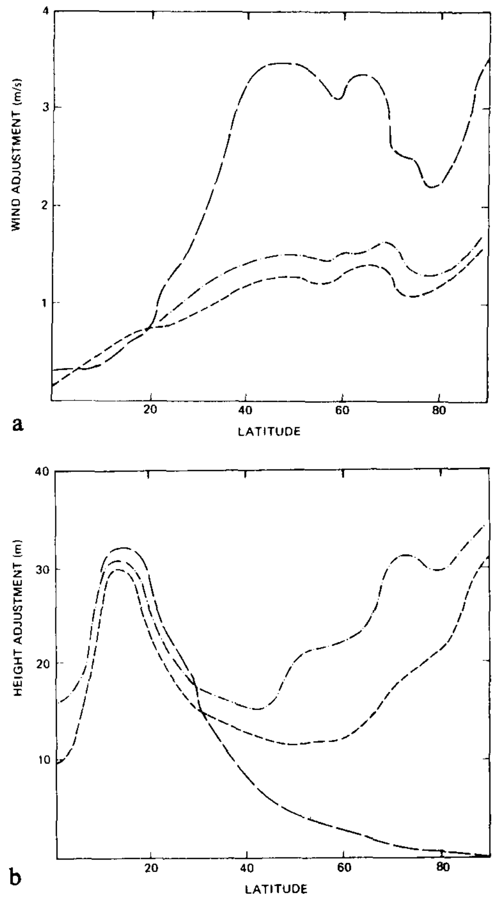

Fig. 1. (a) Wind adjustment (m/s) as a function of latitude for 10 km case. (– – –) non-linear adjustment $W_\psi = W_\chi = W_\Phi = 1$; (— · —) linear adjustment $W_\phi = W_\chi = W_\Phi = 1$; (——) non-linear adjustment $W_\psi = (1-\mu^2)^4$, $W_\chi = 1$, $W_\Phi = 1 - W_\psi$. (b) Same as 1a except for geopotential (m).

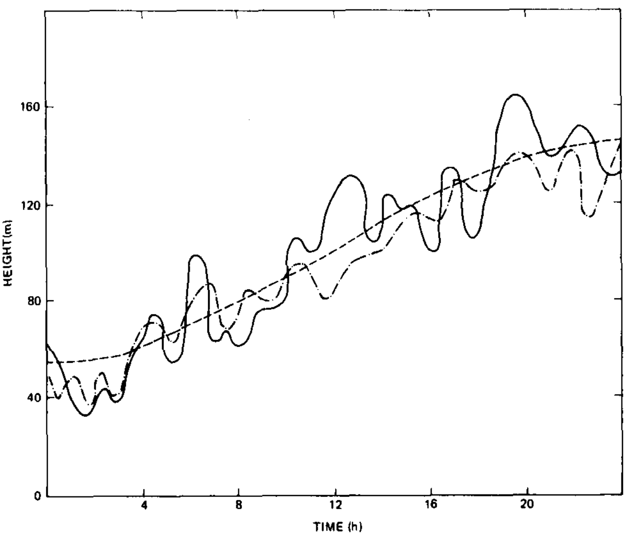

Fig. 2. Time evolution of point (46° N, 180° W) in 10 km model for 3 initial conditions. (——) unadjusted; (– – –) non-linear adjustment $W_\psi = W_\chi = W_\Phi = 1$; (— · —) linear adjustment $W_\psi = W_\chi = W_\Phi = 1$.

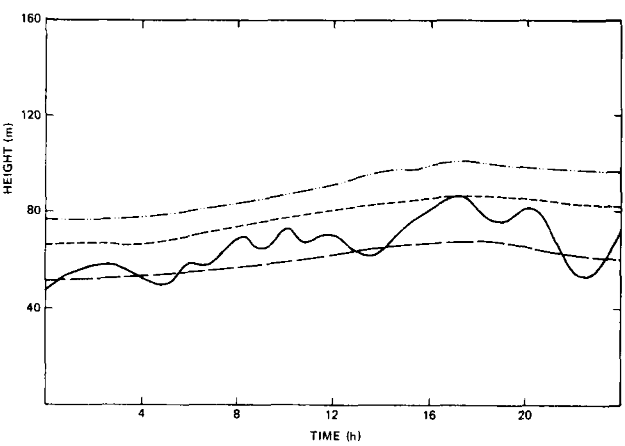

Fig. 3. Time evolution of point (60° N, 75° E) in 1.5 km model for 4 initial conditions. (——) unadjusted; (– – –) non-linear adjustment $W_\psi = W_\chi = W_\Phi = 1$, (——) non-linear adjustment $W_\psi = (1 - \mu^2)^4$, $W_\chi = 1$, $W_\Phi = 1 - W_\psi$ (— · · —) non-linear adjustment $W_\psi = 1$, $W_\chi = W_\Phi = 0$.

minimize the effect of this computational mode, most experiments were run with a very short time-step—3 minutes. The effect of a much longer time-step—30 minutes—will be seen later. The calculation of the terms R_ψ, R_χ, R_Φ in eqs. (2.4–2.6) was performed using the same algorithms for both the adjustment procedure and the model integration.

The first experiment compared variational linear and non-linear normal mode initialization. Linear variational adjustment is performed, simply by setting $\gamma_j^s(1) = 0$ in eq. (4.2). In this experiment the equivalent depth was 10 km and the weights in eq. (4.1) were $W_\psi = W_\chi = W_\Phi = 1$ everywhere. In Fig. 1a is plotted the zonally averaged adjustment of the windfield as a function of latitude. The non-linear

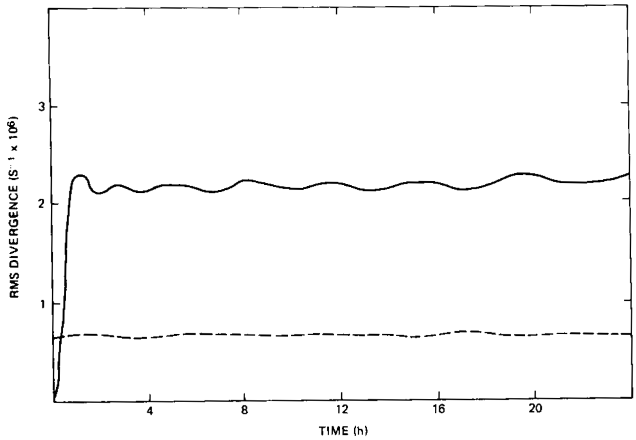

Fig. 4. Time evolution of hemispheric root mean square divergence (s^{-1}) for 2 initial conditions (——) unadjusted; (– – –) non-linear adjustment $W_\psi = W_\chi = W_\Phi = 1$.

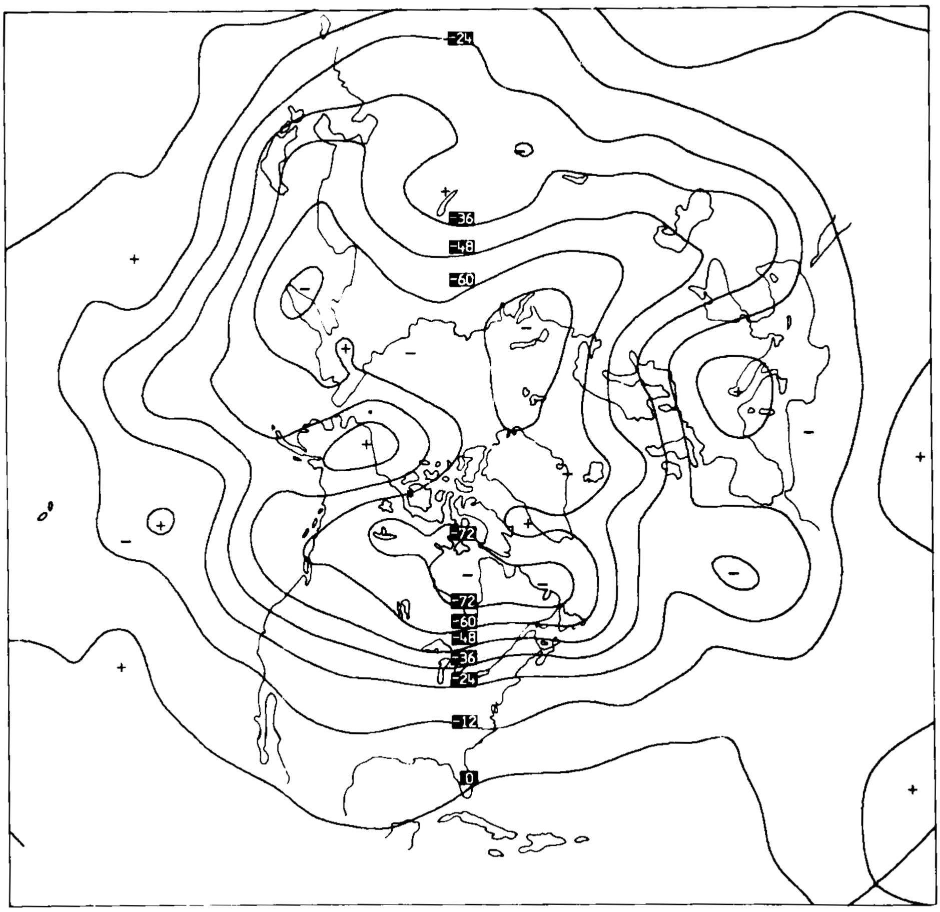

Fig. 5. (a) Initial unadjusted streamfunction (contours 12×10^6 m^2 s^{-1}) on a polar stereographic projection. The contour numbers in the black boxes refer to the contours which touch the lower right corner of the box.

adjustment is represented by the dashed (---) line and the linear adjustment by the dash-dot (—·—) line. In Fig. 1b the same results are shown for geopotential. In both linear and non-linear cases the wind adjustment is relatively small and the height adjustment large in the tropics. However in mid-latitudes both the geopotential adjustment and the wind adjustment are less severe in the non-linear case. Also shown in Fig. 1a and 1b is the non-linear adjustment for weights designed to force winds to heights at high latitude and heights to winds at low latitude. This case, indicated by long dashes (——) has weights of $W_\psi = (1 - \mu^2)^4$, $W_\chi = 1$, $W_\Phi = 1 - W_\psi$. There is very little difference in the tropics, in this case, but the effect at high latitudes is as expected.

The model described earlier was integrated for 24 hours from these initial conditions and the results compared with a model integration from unadjusted initial conditions. The equivalent depth of the model was 10 km as in the adjustment process. The time behaviour of the forecast fields was monitored by plotting the geopotential at a particular point (46° N, 180° W) in Fig. 2. Here we have plotted the time behaviour for unadjusted initial conditions—solid line (——) non-linearly adjusted initial conditions—short dashes (---) and linearly adjusted initial conditions—dash dot

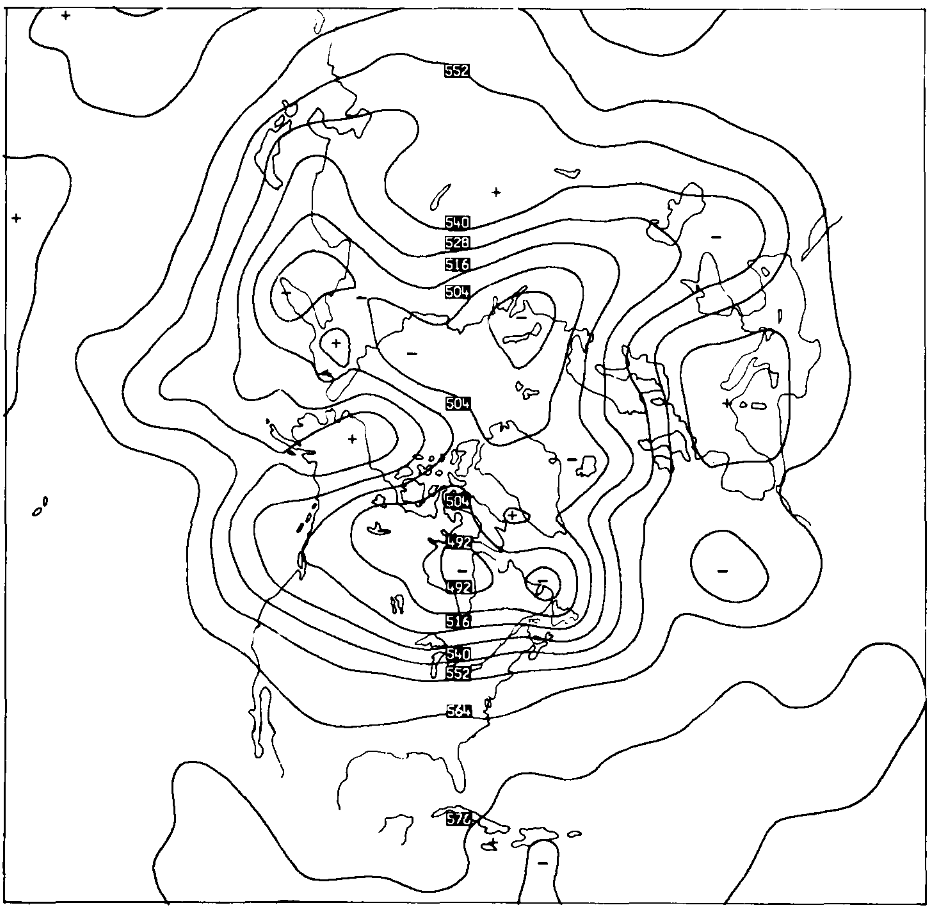

Fig. 5. (b) Initial unadjusted height fields (contours 12 dam).

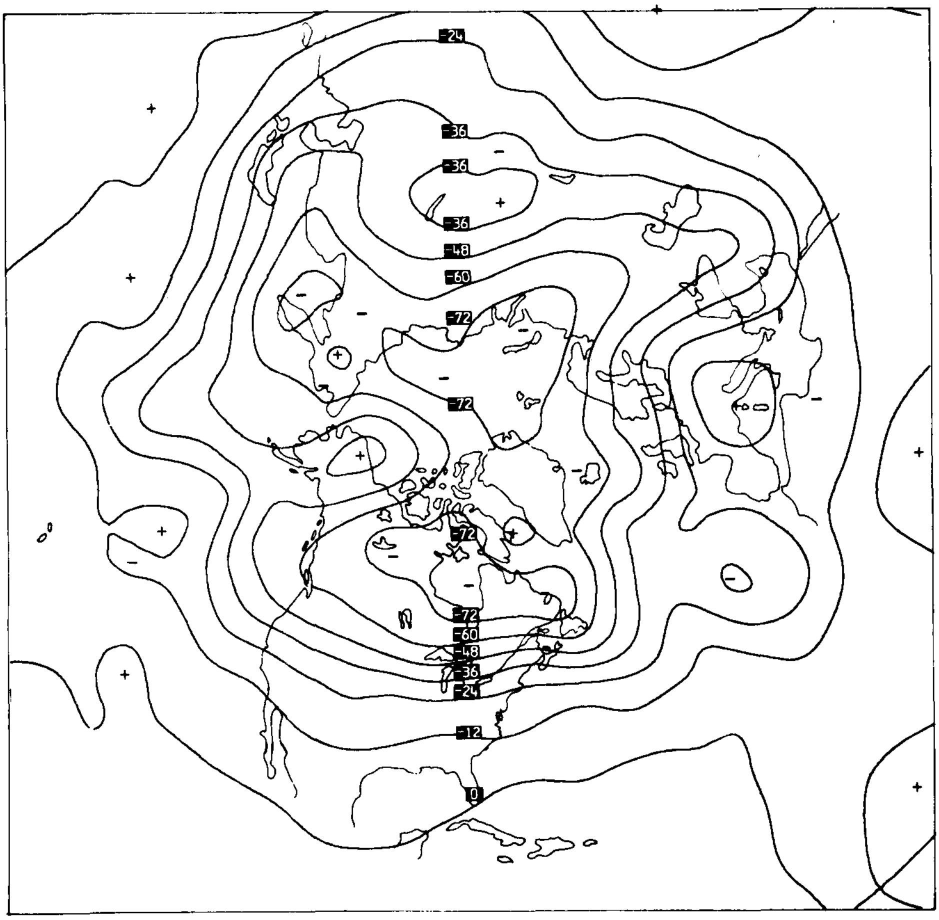

Fig. 6. (a) Non-linearly adjusted streamfunction $W_\psi = (1 - \mu^2)^4$, $W_\chi = 1$, $W_\Phi = 1 - W_\psi$. Same units as 5a.

(—·—). In the last two cases $W_\psi = W_\chi = W_\Phi = 1$. The high frequency gravity waves are apparent in the unadjusted forecast and also in the linearly adjusted forecast, but have completely disappeared from the non-linearly adjusted forecast.

Other experiments were performed with an equivalent depth of 1.5 km, to check whether there was any deterioration of the results. The adjustments in this case, not shown, are very similar to those of Fig. 1. We have, however, presented in Fig. 3, the time evolution of the geopotential field at a high latitude point (60° N, 75° E) for an equivalent depth of 1.5 km. In this case 4 different initial conditions were tested. The solid line (——) indicates unadjusted initial conditions. As one might expect, the gravity wave oscillations are of lower frequency than in the 10-km case. The remaining curves all have non-linear variationally adjusted initial conditions with different weighting. The dashed curve (– – –) is for the equal weight case $W_\psi = W_\chi = W_\Phi = 1$. The long dash curve (— — —) is for initial conditions which are forced to the winds at low latitudes and to the heights, at high latitudes $W_\psi = (1 - \mu^2)^4$, $W_\chi = 1$, $W_\Phi = 1 - W_\psi$. The dash dot curve (—··—) is for initial conditions which everywhere force heights to winds $W_\psi = 1$, $W_\chi = 0$, $W_\Phi = 0$. In all the non-linearly adjusted cases there are no high frequency oscillations, but the solutions at 24 hours, as at zero hours, are markedly different; demonstrating that the present forced adjustment procedure has altered the initial state of the Rossby modes.

In Fig. 4 is shown the hemispherically integrated root-mean-square divergence (Div $= \nabla^2\chi$) during

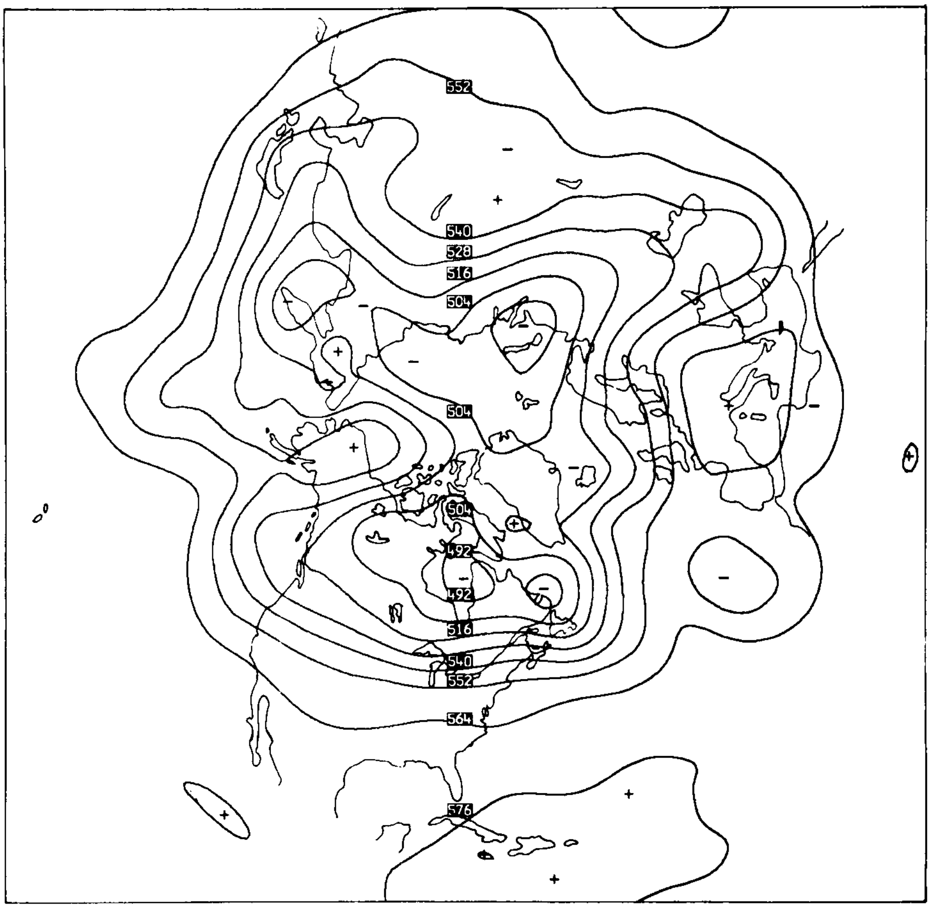

Fig. 6 (b) Non-linearly adjusted height fields—same weights as 6a.

the model integration for the case $W_\psi = W_\chi = W_\Phi = 1$, and equivalent depth of 1.5 km. The solid line (——) indicates the unadjusted case and the dashed (– – –) line the non-linearly adjusted case. In the unadjusted case there is no initial divergence, but it grows rapidly to a rms value of 2×10^{-6} s^{-1}. The non-linearly adjusted case has a small initial divergence, and this level of divergence is maintained throughout the integration. The difference between the two curves indicates the level of gravity wave activity during the unadjusted integration.

Figs. 5 and 6 give an indication of the adjustment that can be expected in the non-linear adjustment procedure. Figs. 5a and 5b show the initial unadjusted stream function and geopotential fields over the northern hemisphere on a polar stereographic projection. The initial divergence field is zero everywhere. Figs. 6a, 6b, 6c shows the non-linearly adjusted stream function, geopotential and divergence fields over the same area. In this case the equivalent depth was 1.5 km and the weights were $W_\psi = (1 - \mu^2)^4$, $W_\chi = 1$, $W_\Phi = 1 - W_\psi$; i.e. they forced winds to heights at high latitudes and heights to winds at low latitudes.

As mentioned earlier most of the model integrations were performed with a 3-minute timestep. However, the semi-implicit algorithm used in the model allows much longer timesteps to be taken. There is a small time computational mode present in model integrations, but its presence is aggravated by the long time steps. When the model is run with unadjusted initial conditions, the ensuing gravity wave oscillations completely

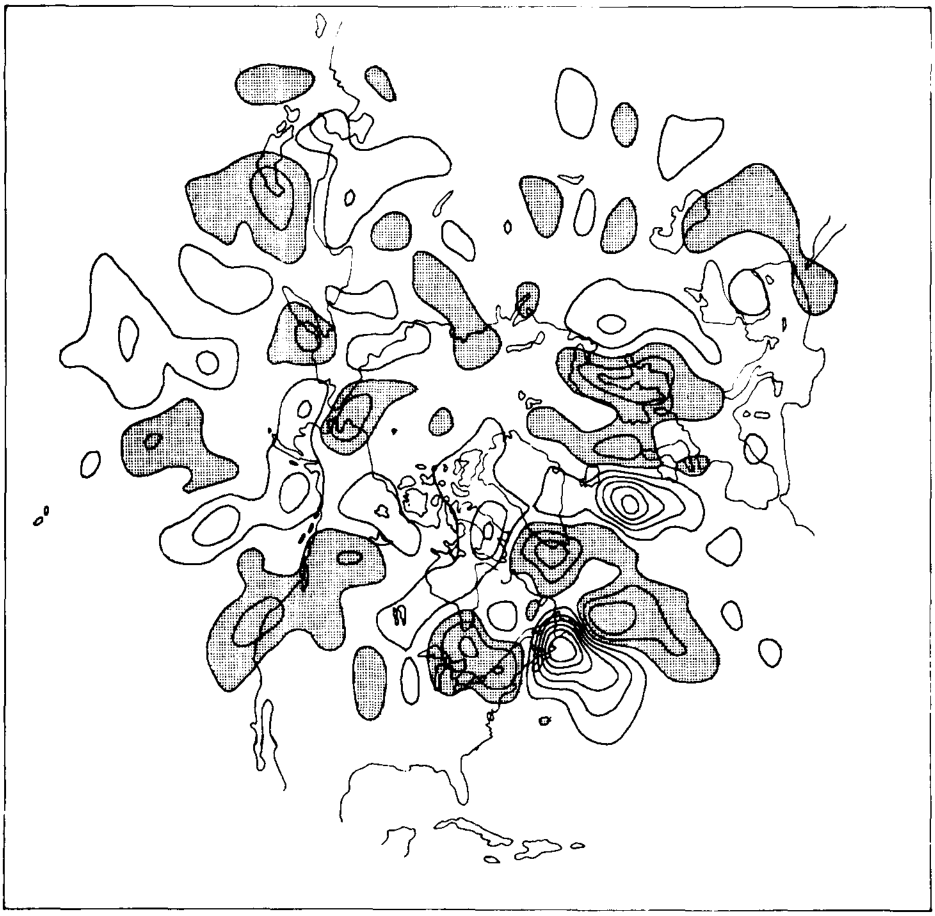

Fig. 6 (c) Non-linearly adjusted divergence field (contour $10^{-6}\ s^{-1}$). Same weights as 6a. The shaded area indicates divergence greater than $10^{-6}\ s^{-1}$.

swamp the computational mode. However, with non-linearly adjusted initial conditions and a long timestep, the computational mode can be detected. It was in fact very easy to remedy this problem with a smooth starting procedure and the long timestep model (30 minutes) produced virtually the same results as the short timestep model.

Finally, a word should be said about the efficiency of the method. Most of the computational time in the adjustment procedure goes into the calculation of the non-linear terms R_ψ, R_χ, R_Φ as it does in the model integrations. Convergence is obtained in two or three iterations, so it could be fairly said that the application of the procedure is roughly equivalent to the execution of 3–4 timesteps of the model.

6. Summary and conclusions

The non-linear normal mode initialization techniques of Machenhauer (1976) and Baer (1977) give some hope of being able to greatly reduce spurious gravity wave activity in primitive equation models. In the case of the shallow water equations, the problem can be said to be completely solved. The present variational generalization of the non-linear normal mode technique gives the user much greater control over the adjustment process and allows optimal use of the data. The variational formalism can also be used to show the relationship between previous static barotropic initialization methods and the non-linear normal mode technique.

The next step is to apply the technique to a baroclinic primitive equations model. First, however, it is necessary to find the normal modes of the model, which is likely to be much more difficult than in the barotropic case. Normal mode decomposition has already been successfully applied to the z coordinate NCAR GCM by Williamson and Dickinson (1976), in the absence of mountains. Normal mode decomposition of a pressure coordinate model (Flattery, 1967) is also possible in the absence of mountains and with a simplified lower boundary condition. Normal mode decomposition of a sigma coordinate model will be more difficult. However, once these difficulties are overcome the application of non-linear normal mode initialization techniques with the present variational formalization, will be relatively straightforward.

The ultimate goal of initialization is well worth striving for. Although primitive equation models have generally outperformed filtered models for forecasts greater than 24 hours, the very short range predictions of these models have been less satisfactory. Gravity wave oscillations and lack of good initial vertical motion fields have resulted in poor short-term precipitation forecasts. It is hoped, that when procedures such as the present method are applied to full baroclinic primitive equation models, it will be possible for those models to be competitive even at very short range.

7. Acknowledgements

The author wishes to acknowledge helpful suggestions and criticism from Drs. Richard Asselin, Helen Warn and Andrew Staniforth. The author is also grateful for useful discussions with Drs. Bennert Machenhauer, Ferdinand Baer and Akira Kasahara. Mrs. Claudette Thibeault assisted with computer output and Mrs. Anne-Marie Driard typed the manuscript.

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ВАРИАЦИОННАЯ ИНИЦИАЛИЗАЦИЯ С ПОМОЩЬЮ НЕЛИНЕЙНЫХ НОРМАЛЬНЫХ МОД

Показано, что техника инициализации с помощью нелинейных нормальных мод, предложнная Махенхауэром (1977) и Баером (1977), обладает большими возможностями для удаления ложных высокочастотных гравитационных колебаний, возникающих при интегрировании примитивных уравнений. По существу, эта техника ортогонализирует тенденции начальных полей по отношению к тенденциям высокочастотных нормальных мод линеаризованных модельных уравнений. В настоящей работе эта процедура обобщается на вариационной основе. Ищется минимум вариационного интеграла, удовлетворяющего сильным ограничениям, полученным из теории инициализации с помощью нелинейных нормальных мод. Применение указанной процедуры к уравнениям мелкой воды показывает, что этот вариационный подход включает в себя большинство предыдущих баротропных статических методов инициализации как частные случаи. Процедура проверяется при различных условиях и приводятся результаты этой проверки.