SHORTER CONTRIBUTION

Geostrophic Departures in the Jet Stream

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Abstract

The effect of gradients in the cross-current eddy motions on the geostrophic balance is illustrated by means of an integrated model. It is suggested that gradients of the Reynolds stress associated with such eddy motions in the atmosphere and in the oceans may lead to sensible average geostrophic departures which appear as a normal inertial effect in eddying currents.

In the past many investigators have devoted attention to the study of the degree to which actual wind currents may be measured by the geostrophic approximation. Some of these studies have been directed toward an examination of effects which might be expected on rational grounds, while others are entirely empirical in nature. In view of the fact that research workers interested in the large-scale currents in the atmosphere are usually forced to resort to wind estimates obtained from pressure data, the nature of the approximations involved in this procedure is of much practical concern. The purpose of this discourse is to bring attention to one systematic effect which might be expected on a rational basis in those regions of the atmosphere where the circulations attain their greatest intensities, although lesser manifestations of the same kind might be found to exist at other levels in the atmosphere as well as in the oceans.

The effect to be discussed cannot be regarded as any new concept in the field of hydrodynamics, having long ago been recognized in the treatment of such topics as the turbulent flow of fluids in channels (see for example GOLDSTEIN 1938). However, it appears that the meteorological applications of the concepts involved should be emphasized.

Although the subject is not dependent upon the existence of any particular integrated model of flow, we shall find it convenient to introduce the basic arguments through the examination of one such specific model. In terms of cartesian coordinates in which x positive is taken eastward and y positive northward, let us study the two-dimensional flow of a fluid which satisfies the following equations of motion and continuity:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = fv - \frac{1}{\varrho}\frac{\partial p}{\partial x} \\ &\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -fu - \frac{1}{\varrho}\frac{\partial p}{\partial y} \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \end{aligned}\right\} \qquad (1)$$

Here u is the eastward and v the northward particle velocity, p is pressure, t time, f the Coriolis parameter and ϱ density. We assume that ϱ is constant and uniform and that

$$f = f_0 + \beta y \qquad (2)$$

where f_0 and β are constants.

The three relations (1) form a closed system of partial differential equations for the determination of the three dependent variables u, v, p in terms of the independent variables x, y, t. It has been shown

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by NEAMTAN (1946) that this system of simultaneous equations possesses an integral in which

$$u = U + V\cos(\alpha y+\varepsilon) - Al\sin k(x-ct)\cdot\cos l y, \tag{3}$$

$$v = Ak\cos k(x-ct)\cdot\sin l y, \tag{4}$$

provided that

$$\alpha = \sqrt{k^2+l^2}\,;\; c = U - \beta/(k^2+l^2) \tag{5}$$

and $U, V, A, k, l, \varepsilon$ are arbitrary constants which we take as being real.

This (particular) solution is completed by writing that

$$\begin{aligned}\frac{p}{\varrho} = &-\frac{1}{2}(u^2+v^2) - Alc\sin k(x-ct)\cdot\cos l y \\ &-\frac{1}{2}A^2\alpha^2\sin^2 k(x-ct)\cdot\sin^2 l y + \\ &+ A\sin k(x-ct)\cdot\sin l y\,[f_0+\beta y + \\ &\qquad + V\alpha\sin(\alpha y+\varepsilon)] \\ &-f_0 U y - \frac{V}{\alpha}(f_0+\beta y)\sin(\alpha y+\varepsilon) - \\ &\qquad -\frac{1}{2}\beta U y^2 \\ &- V\left(\frac{\beta}{\alpha^2}-U\right)\cos(\alpha y+\varepsilon) + \\ &\qquad + \frac{1}{2}V^2\cos^2(\alpha y+\varepsilon) \\ &+ F(t),\end{aligned} \tag{6}$$

obtained through the integration of the first two equations of (1) with the aid of (3), (4) and (5). The last term in (6) is an arbitrary function of time alone and may ordinarily be taken to be an appropriate constant without affecting the other characteristics of the solution. In view of the form of the continuity equation contained in (1), it is possible to use a stream function ψ for the velocity components, of the form

$$\psi = -Uy - \frac{V}{\alpha}\sin(\alpha y+\varepsilon) + A\sin k(x-ct)\cdot\sin l y. \tag{7}$$

It is apparent that the motion consists of a basic zonal flow on which periodic disturbances are superposed.

Letting u_g represent the geostrophic wind toward the east, we write that

$$u_d \equiv u - u_g, \tag{8}$$

so that u_d is the nongeostrophic part of u. Also, introducing the space average of a quantity ( ) over a wave length L in the eastward direction by the relation

$$\overline{(\ )} \equiv \frac{1}{L}\int_L (\ )\,dx, \tag{9}$$

we have that

$$\bar{u}_d = \bar{u} - \bar{u}_g. \tag{10}$$

Either by using the geostrophic formula with (6) and comparing with (3), or more directly by substitution in the second equation of (1) from (3) and (4) and averaging, we find that

$$\bar{u}_d = -\frac{A^2k^2l}{f}\sin l y\cdot\cos l y. \tag{11}$$

It is thus seen that $\bar{u}$ contains a systematic nongeostrophic component which varies in a periodic fashion with the independent variable y alone.

Since v vanishes for $y = O$ and for $y = \pi/l$, it will tend to clearness if we visualize the flow as contained in a channel between two parallel walls at these ordinates. The departure $\bar{u}_d$ then vanishes according to (11) at the two walls and also at the center, $y = \pi/2\,l$. In the southern half of the channel, between $y = O$ and $y = \pi/2\,l$, $\bar{u}_d$ is negative while in the northern half it is positive. On the average over the whole channel the conditions are such that the flow is geostrophic.

By way of physical explanation of this phenomenon we might note that the minimum value of the Reynolds stress $-\overline{\varrho uv}$ is found at $y = \pi/2\,l$. This is interpreted as a net transfer of positive y-momentum from the southern half to the northern half. This drain must be made good in the lower half by an excess of the mean northward pressure gradient force over and above the requirements needed to balance meridional Coriolis forces on the fluid in this region. The reverse process takes place in the upper half.

Viewed somewhat differently the process may be explained physically by noting that individual particles enter the southern half with negative y-momentum and leave with positive y-momentum. They must therefore be accelerating northward on the average while in the southern half and southward while in the northern half. From this one may infer that $\bar{u}_d$ cannot be zero everywhere, unless indeed the motion is purely zonal.

It is to be noted that the results described do not depend upon the profile of $\bar{u}$ but only upon that of $\overline{\varrho v v}$. We may, for example, place a maximum or »jet» of $\bar{u}$ in the southern half at $y = \pi/4\, l$ by taking an appropriate value U together with $V > 0$ and proper values of α, ε.

Much of what has been said follows directly if we write the second equation of (1) in the form of a continuity equation for y-momentum with the aid of the third and average so that

$$\frac{\partial \overline{\varrho v}}{\partial t} + \frac{\partial \overline{\varrho u v}}{\partial x} + \frac{\partial \overline{\varrho v v}}{\partial y} = -f\overline{\varrho u} - \frac{\partial \bar{p}}{\partial y}. \quad (12)$$

The results stated above follow immediately, since the first two terms vanish. Moreover we see that similar results are to be expected in any two dimensional model with a quasi-steady regime of turbulence or eddy motion. For all such cases the mean geostrophic deviation is determined by the cross-stream gradient of the Reynolds stress $-\overline{\varrho v v}$.

The direct application of the model described to the atmosphere would lead to the presence of a westward directed mean geostrophic departure equatorward from the latitude at which $\overline{\varrho v v}$ is a maximum in each hemisphere, and an eastward mean departure poleward. However, the effect is in actuality superposed on other sources of geostrophic departures such as the curvature of the latitude circles which produces a westward departure even in the case of purely zonal flow. Another factor is that for the actual atmosphere even in the simple cartesian equation (12) an additional vertical term must be added so that, taking z positive upward and w as the upward particle velocity, and neglecting horizontal density variations in the x-direction,

$$\bar{\varrho}\, \bar{u}_d = -\frac{1}{f}\left(\frac{\partial \overline{\varrho v v}}{\partial y} + \frac{\partial \overline{\varrho v w}}{\partial z}\right). \quad (13)$$

It is difficult to estimate the effect of this vertical term, although if (13) be integrated also with respect to height through the whole atmosphere, the total contribution of this added term must result from the presence of a net meridional surface stress at a given latitude. There is not much *a priori* reason to expect that such net meridional surface stresses around latitude circles are of importance.

The latitude at which $\overline{\varrho v v}$ reaches a maximum is probably displaced farther poleward in each hemisphere as compared with the latitude at which $\bar{u}$ reaches its maximum, although observational studies should be made concerning this point. Furthermore, there is no reason to suppose that this quantity tends to zero as the pole is approached. For purposes of orientation let it be supposed that $\overline{v v}$ increases by 10^7 cm^2/sec^2 for an increase of 2 000 km in y. Neglecting horizontal density variations, the first member on the right hand side of (13) then gives a contribution of -6.7 meters per second to $\bar{u}_d$, if $f = 0.75 \times 10^{-4}$ (corresponding to about 30° N latitude). This may give some notion of correct orders of magnitude in the vicinity of the jet stream. At lower levels the contribution may be one fifth or one tenth as large.

In an interesting paper published recently, LOEWE and RADOK (1950) have presented a meridional cross section of the atmosphere in the southern hemisphere through the jet stream. These writers make a comparison of geostrophic versus actual wind measurements in the jet. The actual winds show velocities much lower than those computed from the isobaric contours. Although many questions might arise concerning the representativeness of the data, it may also be that a part of the discrepancy is due to the manifestation of the phenomenon here described. It is to be noted that local climatological time averages such as those used by LOEWE and RADOK would tend to approximate space averages except for the possible presence of standing semipermanent disturbances.

Finally it should be mentioned that the considerations used here may be of some consequence in the charting of mean ocean currents by means of dynamic velocity computations, in those cases where there exists a gradient in the cross-current eddy motion. This may, for example, be significant in the case of such currents as the Gulf Stream.

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