On the Sensitivity of the Wind Field Relative to Pressure Variations

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Abstract

It is assumed that the motion is frictionless and adiabatic but not necessarily horizontal. A geostrophic approximation of second order is introduced, and a measure of the sensitivity of the wind field relative to pressure variations is obtained. It is shown that the sensitivity depends essentially upon the lateral shear of the geostrophic wind and the tangential and orthogonal curvature of the geostrophic streamlines. In particular, when the geostrophic wind is large, the effect of the orthogonal curvature is predominant.

1. *Notations.* — The following symbols will be used.

$V = (u, v, w)$ = wind velocity.
$\dot{V} = (\dot{u}, \dot{v}, \dot{w})$ = acceleration.
$\ddot{V} = (\ddot{u}, \ddot{v}, \ddot{w})$ = derivative of acceleration.
$V_g = (u_g, v_g)$ = geostrophic wind.
K_i = tangential curvature of geostrophic streamlines, positive or negative according as the curvature is cyclonic or anticyclonic.
K_n = orthogonal curvature of geostrophic streamlines, positive or negative according as the streamlines converge or diverge.
E = mean radius of the figure of the earth.
g = acceleration of gravity.
Φ = angle of latitude.
Ω = angular speed of the earth's rotation.
$f = 2\Omega \sin \Phi$, $f' = 2\Omega \cos \Phi$
p = atmospheric pressure.
α = specific volume.
T = temperature.
c_p = specific heat at constant pressure.
Z_Θ = height above the reference level of an isentropic surface.
ψ = the Montgomery acceleration potential for an isentropic surface (1937).

$$\psi = C_p T + gZ_\Theta$$

In particular, if the y-axis is chosen along the direction of the horizontal pressure force

$$\frac{\partial \psi}{\partial x} = 0, \quad v_g = 0, \quad u_g = V_g = -\frac{1}{f}\frac{\partial \psi}{\partial y} \tag{2}$$

and also

$$\frac{\partial^2 \psi}{\partial x^2} = -K_i \frac{\partial \psi}{\partial y},$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = K_n \frac{\partial \psi}{\partial y}, \quad \frac{\partial^2 \psi}{\partial y^2} = -f\frac{\partial V_g}{\partial y} \tag{3}$$

Furthermore, the following auxiliary notations will be introduced

$$I_x = -\frac{1}{f^2}\frac{\partial^2 \psi}{\partial x \partial t}, \quad I_y = -\frac{1}{f^2}\frac{\partial^2 \psi}{\partial y \partial t} \tag{4}$$

and may be identified with the components of the so-called isallobaric wind, in the meaning defined by BRUNT and DOUGLAS (1928).

2. *The Equations of Motion.* — Choosing the x-axis toward the east and the y-axis toward the north, the equations of frictionless motion may be written

$$\dot{u} - fv + f'w = X$$
$$\dot{v} + fu = Y$$

where X and Y represent the horizontal components of the pressure force.

Differentiating the above equations with respect to time and eliminating $\dot{u}$ and $\dot{v}$, we obtain

$$\left.\begin{aligned} \ddot{u} + f^2u - v\dot{f} + w\dot{f}' + f'\dot{w} &= \dot{X} + fY \\ \ddot{v} + f^2v + u\dot{f} + wff' &= \dot{Y} - fX \end{aligned}\right\} \quad (5)$$

Since no further differentiation will be performed we may leave out of consideration certain small terms the magnitudes of which do not exceed about one or two percent of the major terms.

Now, $\dot{f} = vf'/E$, and $\dot{f}' = -vf/E$. Except near the equator and near the poles, f and f' are about equal and of the order of 10^{-4} sec.$^{-1}$, while E is $6.4 \cdot 10^6$ m. In large scale currents, u and v are of the order 10 m/sec., w of the order 10^{-1} m/sec., and $\dot{w} = 10^{-5}$ m/sec.^{2} or less. It follows then that the magnitude of the terms $(\dot{f}v - \dot{f}'w - f'\dot{w})$ and $(\dot{f}u + ff'w)$ are about one or two percent of those of the terms f^2u and f^2v, respectively, and may therefore be omitted. The differentiated equations of motion, therefore, reduce to

$$\left.\begin{aligned} \ddot{u} + f^2u &= \dot{X} + fY \\ \dot{v} + f^2v &= \dot{Y} - fX \end{aligned}\right\} \quad (6)$$

We may now choose the system of coordinates such that the positive y-axis coincides with the horizontal pressure force, for the only terms in (5) affected by this rotation are those involving f', which have been shown to be negligible. With this choice of coordinate axes, the horizontal pressure force may be expressed in either of the following forms

$$\left.\begin{aligned} X &= -\frac{\partial\psi}{\partial x} = -fv_g = 0 \\ Y &= -\frac{\partial\psi}{\partial y} = fu_g = fV_g \end{aligned}\right\} \quad (7)$$

Now, if the motion is adiabatic, then without assuming horizontal motion

$$\left.\begin{aligned} \dot{X} &= -\frac{\partial^2\psi}{\partial x\partial t} - \frac{\partial^2\psi}{\partial x^2}u - \frac{\partial^2\psi}{\partial x\partial y}v \\ \dot{Y} &= -\frac{\partial^2\psi}{\partial y\partial t} - \frac{\partial^2\psi}{\partial x\partial y}u - \frac{\partial^2\psi}{\partial y^2}v \end{aligned}\right\} \quad (8)$$

Substituting from equations (7) and (8) in (6), and using the notations defined in section 1, the differentiated equations of motion may be written

$$\left.\begin{aligned} \ddot{u} &= Au + Bv + E \\ \ddot{v} &= Cu + Dv + F \end{aligned}\right\} \quad (9)$$

where the coefficients have the following meaning:

$$A = -f(f + V_g K_i); \qquad B = C = fV_g K_n;$$

$$D = -f\left(f - \frac{\partial V_g}{\partial y}\right); \qquad E = f^2(V_g + I_x);$$

$$F = f^2 I_y$$

If the corresponding coefficients are computed for the isobars in a level surface, additional terms appear, expressing the effect of the isosteric-isobaric solenoids. Except in frontal regions, where the degree of baroclinity is appreciable, the difference between the isentropic and isobaric coefficients is small, and for most purposes the two sets may be used interchangeably.

From a formal point of view, equations (9) are particularly well suited for analyses of the oscillations and stability of air currents. In the first approximation one may then assume that A, B, C, and D (which are second space derivatives of the pressure force) are constants. However, the coefficients E and F ($F = v_g + I_y$, with $v_g = 0$) are essentially first space derivatives of the pressure force, and in order to obtain solutions to (9) it is necessary to introduce assumptions as to the relation betwen E, F and u, v. Since any such assumption, capable of a clear physical interpretation, is difficult to formulate in a manner consistent with a complete set of equations, solutions to (9) will not be sought. Instead, an assumption will be made concerning the acceleration, so as to obtain an approximation to the instantaneous relation between the wind in a general pressure field.

3. *Geostrophic Approximation of Second Order.* — It is well known that the assumption of geostrophic balance gives a fair approximation to the true wind in the large-scale pressure systems. It is then assumed that the acceleration is small in comparison with the Coriolis acceleration, viz.,

$$|\dot{V}| \ll |2\,\Omega\, x\, V| \quad (10)$$

If this is to hold not only at a given moment but all the time, it is reasonable to assume that

$$|\ddot{V}| \ll \left| \frac{d}{dt}(2\,\Omega \times V)H \right| \qquad (11)$$

We shall refer to (10) and (11) as geostrophic approximations of first and second order, respectively. Since neither of the above assumptions can be justified by reference to mathematical argument, their usefulness must be determined by comparison of derived results with observation.

In the following it will be assumed that the motion is sufficiently geostrophic for the terms containing second derivatives of the velocity to be neglected in comparison with the derivative of the Coriolis force.

4. *Sensitivity of the Wind Field.* — Returning now to equation (9) and introducing the second order geostrophic approximation, we obtain (since $B = C$) the following expressions for the velocity components

$$\left.\begin{aligned} u &= \frac{CF - DE}{AD - C^2} \\ v &= \frac{CE - AF}{AD - C^2} \end{aligned}\right\} \qquad (12)$$

Now, since the x-axis is chosen along the geostrophic wind, v is the cross-isobar component of the actual wind. In balanced motion this component vanishes (i.e., $CE = AF$). However, for any value of $(CE - AF)$ different from zero, the cross-isobar component and the tangential acceleration increases as the denominator decreases. Hence, with some justification, the quantity

$$S = \frac{1}{AD - C^2} = \frac{1}{\left(1 + \frac{1}{f}V_g K_i\right)\left(1 - \frac{1}{f}\frac{\partial V_g}{\partial y}\right) - \frac{1}{f^2}V_g^2 K_n^2} \qquad (13)$$

may be taken as a measure of the sensitivity with which the wind reacts relative to pressure variations.

It should be emphasized, however, that (12) and (13) represent an approximation, the accuracy of which depends upon the smallness of $\ddot{u}$ and $\ddot{v}$ as compared with uf^2 and vf^2, respectively.

It is of interest to note that the above expression for the sensitivity of the wind field is related (inversely) to the degree of dynamic stability. SOLBERG (1936) considered a zonal geostrophic current (i. e. $K_i = K_n = 0$) and found the degree of dynamic stability expressed by $\left(1 - \frac{1}{f}\frac{\partial V_g}{\partial y}\right)$, which agrees with (13). VAN MIEGHEM (1944) considered a symmetrical circular vortex (i. e., $K_n = 0$) and found the degree of stability expressed by

$$1 - \frac{1}{f}\frac{\partial V_g}{\partial y} + \frac{1}{f}V_g K_i$$

which (on the same assumptions) differs from (13) by the absence of the quadratic term $\frac{1}{f^2}V_g K_i \frac{\partial V_g}{\partial y}$.

5. *Types of Pressure Fields.* — Substituting the definitions of the coefficients A, B, etc. given with equation (9) into (12), we obtain for the velocity components

$$u = \frac{(V_g + I_x)\left(1 - \frac{1}{f}\frac{\partial V_g}{\partial y}\right) + I_y \frac{1}{f} V_g K_n}{\left(1 + \frac{1}{f}V_g K_i\right)\left(1 - \frac{1}{f}\frac{\partial V_g}{\partial y}\right) - \frac{1}{f^2}V_g^2 K_n^2}$$

$$v = \frac{I_y\left(1 + \frac{1}{f}V_g K_i\right) - (V_g + I_x)\frac{1}{f}V_g K_n}{\left(1 + \frac{1}{f}V_g K_i\right)\left(1 - \frac{1}{f}\frac{\partial V_g}{\partial y}\right) - \frac{1}{f^2}V_g^2 K_n^2} \qquad (14)$$

It will be seen that the deviation of the wind from the geostrophic wind depends partly upon the isallobaric gradient (here represented by I_x and I_y), and partly upon the lateral shear of the geostrophic wind and the tangential and orthogonal curvatures of the streamlines of the geostrophic wind (i. e. the curvatures of the lines p = constant in the barotropic case, and ψ = constant in the baroclinic case).

(1). *Stationary geostrophic wind.* — In this case $I_x = I_y = 0$, and the angle (β) of deflection of the actual wind from the geostrophic wind is given by

$$\tan\beta = \frac{V_g K_n}{f - \dfrac{\partial V_g}{\partial y}} \qquad (15)$$

As will be seen from Table 1, the magnitude of β may be appreciable. At such low geostrophic velocities as occur near the earth's surface, β is normally less than the frictional deflection and is, therefore, difficult to detect. In the upper troposphere in middle latitudes where geostrophic winds in excess of 50 m/sec. are a normal occurrence, very large angles of deflection will occur unless the isobars are almost strictly parallel.

Table 1

Magnitude of the Deviation of the Direction of the Actual Wind from that of the Geostrophic Wind (in degrees) for Various Values of V_g (in m/sec.) and K_n (in km) in Latitude 45°, computed from Eq. 15.

	$\frac{\partial V_g}{\partial y} = \frac{1}{2} f$			$\frac{\partial V_g}{\partial y} = 0$			$\frac{\partial V_g}{\partial y} = -f$		
V_g / $\pm K_n$[1]	10	40	80	10	40	80	10	40	80
400	27	65	76	14	45	65	7	27	45
800	14	45	65	7	27	45	3	14	27
1 600	7	27	45	3	14	27	2	7	14

It will be seen from (13) that the sensitivity decreases as $V_g^2 K_n^2$ increases.[1] Hence, the strong winds present in the upper part of the troposphere are extremely sensitive relative to isobar convergence and divergence. It is often observed that appreciable developments in the pressure field occur at and below regions of such convergence in the upper troposphere. Normally, appreciable isobar convergence, or divergence, associated with strong winds result in pressure variations, with the result that the angle β is modified by the isallobaric gradient. The figures in Table 1 suffice, however, to demonstrate the importance of K_n as far as the sensitivity of the wind field is concerned. Since the acceleration tangential to the isobar is equal to $f v \equiv f u \tan\beta$, it follows that regions of appreciable isobar convergence or divergence are characterized by large accelerations.[2]

(2). *Straight and parallel isobars without geostrophic shear.* — In this case $K_i = K_n = \partial V_g/\partial y = 0$, and (14) reduce to

$$u = V_g + I_x, \; v = I_y \qquad (16)$$

which is the well-known BRUNT-DOUGLAS equation (1928), showing that for this type of pressure field the actual wind is equal to the geostrophic wind supplemented by a vector which is proportional to the isallobaric gradient. It will be seen that in this simple type of pressure field $S = 1$.

(3). *Straight and parallel isobars with geostrophic shear.* — In this case (14) reduce to

$$u = V_g + I_x, \; v = I_y / \left(- \frac{1}{f} \frac{\partial V_g}{\partial y} \right) \qquad (17)$$

It will be seen that the component along the isobar is the same as in the foregoing case, while the cross-isobar component is larger or smaller according as the geostrophic shear is anticyclonic (> 0) or cyclonic (< 0). It follows then that *the wind field is more sensitive to pressure changes when the geostrophic shear is anticyclonic than it is when the shear is cyclonic.*

(4). *Curved concentric isobars.* — In this case $K_n = 0$, and (14) reduce to

$$\left. \begin{aligned} u &= (V_g + I_x) / \left(1 + \frac{1}{f} V_g K_i \right); \\ v &= I_y / \left(1 - \frac{1}{f} \frac{\partial V_g}{\partial y} \right) \end{aligned} \right\} \qquad (18)$$

and the accelerational components are

$$\dot{u} = \frac{f I_y}{1 - \dfrac{1}{f} \dfrac{\partial V_g}{\partial y}}; \quad \dot{v} = \frac{V_g^2 K_i - f I_x}{1 + \dfrac{1}{f} V_g K_i} \qquad (19)$$

It will be seen that *the motion is more sensitive to pressure variations when the geostrophic shear and curvature are anticyclonic than when they are cyclonic.*

[1] The abnormal case, when either of the two parentheses in (13) is negative, will not be considered since the wind could not then be predominantly geostrophic.

[2] SCHERHAG (1948), who places much emphasis on the isobar convergence and divergence, interprets these features as convergence and divergence of the actual motion.

The relation between the wind and the above-mentioned simple pressure fields is shown diagrammatically in fig. 1. In diagram *A*, there is no geostrophic shear and the geostrophic departure is determined exclusively by the isallobaric gradient, according to BRUNT and DOUGLAS equation. In diagram *B* the geostrophic departure is amplified through anticyclonic shear, and in *C* it is reduced through cyclonic shear. Diagrams *D* and *E* show the conditions around a cyclone and an anticyclone moving from the left to the right. Here, the effects of shear and curvature combine to give a large variety of geostrophic departures. It will be seen that the winds are rather insensitive to pressure changes along the periphery of the anticyclone and in the central part of the cyclone, while the reserve is true along the periphery of the cyclone and near the centre of the anticyclone.

(5). *Converging and diverging isobars.* — In this case the relation between the wind and the pressure field is given by (14) and since both u and v depend upon all the characteristics of the pressure field, the relationship is not readily represented in a simple diagram. It may be noted, however, that the cross-isobar components of the wind are determined primarily by the isallobaric gradient and the orthogonal curvature. These components are, however, amplified or reduced according as the sensitivity factor is larger or smaller than unity. It will be seen from (13) that the effect of isobar convergence or divergence (i. e., K_n) is always to increase the sensitivity factor. Particularly in the upper part of the troposphere in middle latitudes, where the geostrophic winds are very strong, the sensitivity of the wind field may be greatly increased even through moderate amount of isobar convergence and divergence.

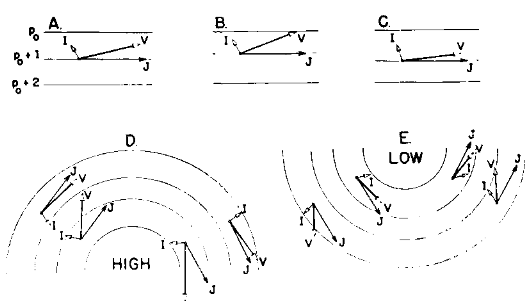

Fig. 1. Illustrating the relation between the actual wind (**V**), the geostrophic wind (**J**) and the isallobaric wind in relation to isobar shear and curvature.

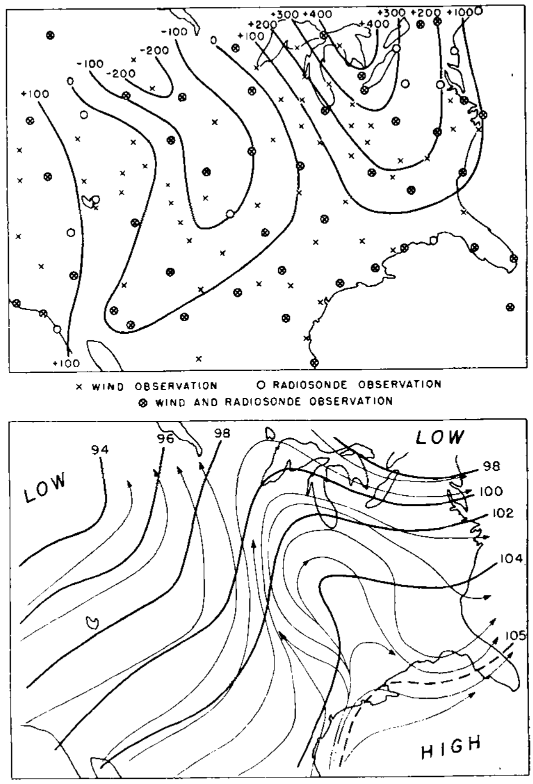

Fig 2. Upper chart: 12-hour tendency of the height of the 700 mb. surface, April 3, 1948, 0930 to 2130 G. M. T.
Lower chart: streamlines of the actual wind and contours of the 700 mb. surface, April 3, 1948, 1530 G. M. T.

Figure 2 shows, as an example[1], the streamlines of the actual wind in relation to the pressure force and the isallobaric field. An examination of the charts will show that eq. (14) is capable of accounting for all essential features of the deviation of the actual wind from the geostrophic wind.

The extent to which the sensitivity factor (eq. 13) will prove useful as a synoptic indication of development in the pressure field remains to be determined. It is hoped that a synoptic investigation by A. F. GUSTAFSON and L. C. STARRETT, which will be completed shortly, will clarify certain aspects of this problem.

[1] The streamline chart is reproduced with the kind permission of Major A. F. GUSTAFSON, U.S. Air Force, from his unpublished report on a detailed analysis of the wind and pressure fields over the USA during the International Aerological Days April 1948.

REFERENCES.

BRUNT, D. and DOUGLAS, C. K. M. 1928: On the modification of the strophic balance; *Mem. Roy. Met. Soc.* **3**, 22.

MONTGOMERY, R. B. 1937: A suggested method for representing gradient flow in isentropic surfaces. *Bull. Am. Met. Soc.* **18**. pp. 10—12.

SOLBERG, H. 1936: Le mouvement d'inertie de l'atmoshpère stable et son rôle dans la théorie des cyclones. *Mem. Met. Ass. U. G. G. I.*

VAN MIEGHEM, J. 1944: Perturbation d'un courant atmosphérique zonal. *Classe de Science de l'Ac. de Belgique.*