Adiabatic Perturbation Equations for a Zonal Atmospheric Current

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Abstract

By a proper choice of coordinates and variables, a simple and quasi-symmetrical system of differential equations is obtained for the infinitesimal adiabatic perturbations of the most general continuous zonal air-flow. The coordinate surfaces are those of the orthogonal system built up with the isentropic surfaces and the meridians, and the variables are the displacements of the fluid in the directions of the coordinate lines, and the local disturbance of the pressure. From these perturbation equations a differential equation for the pressure disturbance alone is derived in the case of a wave, and it is then shown that this wave equation can be always greatly simplified if account is taken of the order of magnitude of the various parameters. Simplified expressions are also given for the components of the vibration (fluid displacement). More particular equations and formulae are then derived for short-period, middle-period and long-period waves successively, this classification being made according to the value of the orbital frequency compared to the coefficient of hydrostatic stability and Coriolis parameter, and special cases are also considered where the wave equation reduces to one with constant coefficients (elastic, gravitational, Rossby's waves). Finally some incorrect recently published results are discussed.

1. Definition and meteorological significance of the perturbation problem

It is well known that when a fluid motion may be considered as a *disturbed motion*, that is, resulting of an infinitesimal perturbation superimposed upon a given *undisturbed motion*, its differential equations can be linearized, in other words replaced with a system of linear equations, the so-called *perturbation equations*, the resolution of which is of course easier than that of the motion equations themselves, at least when the undisturbed motion is a simple one.

However, when considered from this viewpoint, the perturbation problem (resolution of the perturbation equations under given conditions) might look to be one of only mathematical interest, and the large place which it takes in modern meteorology would not be justified. It seems therefore necessary to give here a brief discussion about this topic: what is the practical significance of the perturbation problem in meteorology?

During the past two centuries, the perturbation method was used by several of the greatest mathematicians or physicists when they built the now classical theories of such phenomena as sound-waves (NEWTON, LAPLACE, POISSON, STOKES, RAYLEIGH), ocean's tides (LAPLACE), gravitational water-waves (LAGRANGE, CAUCHY, POISSON, STOKES, GREEN, AIRY, KELVIN), capillary-gravitational waves (KELVIN). Nobody can deny the great success of these theories for the explanation of important observed facts, but it must also be noted that most of them only concern very simple problems: the fluid in the undisturbed state is stagnant or in uniform motion, and possesses no internal stratification, so that the coefficients of the perturbation equations are generally constant, and the perturbations are essentially stable.

The first example of unstable waves was given by HELMHOLTZ 1868 when he considered the oscillations of the system formed by two homogeneous currents moving side by side with uniform but different velocities (shearing-gravitational waves). In this case, the shear at the common boundary is the only factor of instability, and the only stabilizing factor is gravity, which operates mainly on long waves. From this it follows that all the waves are unstable the wave-length of which is smaller than a critical value, whence HELMHOLTZ concluded that a turbulent layer of transition must develop between the two currents, the width of this layer being comparable to the critical wave-length (in modern terminology, the system is said to be "dynamically unstable", or to possess a "dynamic instability"). HELMHOLTZ applied his theory to stationary billow-clouds, by assuming that the surface of discontinuity is the upper boundary of an atmospheric inversion.

In the case of air over water, this problem was generalized by KELVIN 1871, by introducing capillarity as an additional stabilizing factor, and this author made an application to the theory of the formation of water-waves by wind. Since capillarity mainly operates on short waves, and gravity on long waves, it is found that the system is dynamically unstable only if the wind exceeds some critical value, and if the wind increases from this value, there is a limited spectral-interval of unstable waves, the width of which increases from zero. We have here a simple example of an "instability criterium" (condition for the undisturbed motion to be dynamically unstable), which may be physically interpreted as the condition for the motion to change from laminar to turbulent type in the vicinity of the surface of water.

If the theory be correct, the critical velocity would be that of the slightest wind which is able to raise waves on a smooth water-surface, and the theory would also give the wave-length of these first-appearing waves. As a matter of fact, it is actually observed that there is a definite wind-velocity at which the surface of water begins to be ruffled, but this velocity is much smaller than KELVIN's critical value (about six times smaller). The explanation of this disagreement, as pointed out by JEFFREYS 1925, is probably that a turbulent air-layer begins to form on the lee-side of each wave at a wind-velocity which is much smaller than the critical value, thus increasing the effect of wind-shear as a factor of instability. But there is still no complete theory of wind-waves.

It has been noticed for a long time that many of the atmospheric perturbations have the same character as gravitational or shearing-gravitational waves, but this analogy remained a purely formal one as long as the general structure of the atmosphere was not known.

The earliest theory concerning atmospheric waves was perhaps HELMHOLTZ's already mentioned solution of the billow-clouds problem, and a quite similar theory is that proposed by KÜTTNER 1939 for mountain lee-waves (in both cases an inversion boundary is assumed to exist).

But by far the most interesting surface-wave theory applicable to the atmosphere appeared after J. BJERKNES and his co-workers discovered the atmospheric fronts in 1919, and were able to describe the cyclones of Northern Europe as wavy disturbances growing and progressing along a quasi-permanent temperature-discontinuity, the so-called polar-front surface. This interpretation could not fail to suggest a close analogy between cyclone-waves and wind-waves for instance, and therefore it was natural to consider the theoretical problem of the oscillations of the system formed by two air currents having uniform but different temperatures and velocities, separated by the polar-front surface and bounded downwards by a horizontal surface at sea level. Unfortunately this problem is much more difficult than KELVIN's wind-wave problem, specially because of the slope of the polar-front surface, which cannot be neglected at the scale of cyclone-waves. Tentative approximate solutions were given by SOLBERG 1928 and then by V. BJERKNES and SOLBERG 1933, and the general results were quite similar to KELVIN's ones concerning wind-waves: owing to the fact that there are two stabilizing factors, namely the gravity, operating now mainly on the shortest waves (the very short waves, of the scale of billow-clouds, are disregarded in this problem), and the earth's rotation, which on the contrary mainly operates on the longest waves, the system is dynamically stable as long as the wind shear is smaller than a critical value, and when this value is just exceeded a dominant unstable wave appears, which may be interpreted as the cyclone wave.

This remarkable result, following the great push of enthusiasm created by J. BJERKNES' discovery, largely contributed to the success of the Norwegian meteorological school. However, when considered objectively, this result is in reality more a disapproval than a justification of J. BJERKNES' fundamental idea that cyclones are the result of the conflict between two air masses of different temperatures: in V. BJERKNES-SOLBERG's waves, a temperature difference is indeed a stabilizing factor, the only factor of instability being the wind shear. On the other hand, it very soon appeared that the polar-front surface is not a continuous surface, that it never exists out of cyclones, and that many extra-tropical cyclones, for instance those of the mediterranean area, do not show any front at their early stage. It was therefore obvious, in the years preceding the last world war, that the problem of cyclone formation had to be completely reexamined.

In presence of this failure, the modern meteorologists were forced to abandon the idea that atmospheric perturbations are necessarily related to discontinuities, and they were led to consider, as factors of stability and instability, those arising from the internal stratification and wind-shear of the atmospheric layers. The simplest cases being obviously those where there is no internal wind-shear, one must expect to find here the first successful attempts of this recent stage of theoretical meteorology. Let us mention specially the beautiful results arrived at by LAMB 1932, TAYLOR 1936 and PEKERIS 1937, concerning the atmospheric tides, ROSSBY's well-known theory of long waves in the westerlies, and also, as provisional theories, HAURWITZ's results about billow-clouds, and the recent researches on mountain waves.

But it is clear that, if the assumption of no shear is maintained, there can be no hope to arrive at an even approximate theory for atmospheric perturbations of the scale of cyclones, since these perturbations are always associated with important horizontal temperature-gradients, and these in their turn are associated with vertical wind-shears. This consideration explains why the cyclone problem has become practically identified with the problem of unstable waves in a baroclinic atmospheric current, and why so much importance is attached in present days to the derivation of instability criteria for such currents. Unfortunately, the results obtained up to now in this connection are rather discordant: according to a first group of meteorologists (SOLBERG 1936, HOEILAND 1939 and 1941, FJOERTOFT 1941 and 1946, VAN MIEGHEM 1944 and 1946), it is possible to derive, at every point of a baroclinic air flow, an instability criterium which only involves the *local* values of hydrodynamical parameters, while according to others (JAW 1946, CHARNEY 1947, EADY 1949), on the contrary, the *boundary conditions* play a dominant rôle, and for the current values of the wind-shear all the short waves are unstable as far as the vertical acceleration is neglected (the very short waves are stable, but they are unimportant in the problem), in other words the system is practically always dynamically-unstable. (In order to reconcile this result with the existence of a dominant wave-length in the turbulent motion, Eady imagines to introduce the postulate that, among the numerous unstable waves, it is the wave of maximum growth-rate which must predominate.)

What may be now the conclusion of this brief historical survey? The failure of the perturbation method when applied to shearing or baroclinic systems is probably to be attributed, not to the method itself, but well to the incorrectness of the assumptions made by the authors of theories, concerning the hydrodynamical conditions imposed to the motion: in the wind-wave problem, for instance, the motion must be taken as turbulent instead of laminar as in TAYLOR's theory, and in all of the recent researches about baroclinic currents, the simplifying assumptions made in course of the computations are generally not permissible, as we shall prove it for some of them at the end of this paper. It is therefore to be expected that, once the perturbation problem is expressed in correct analytical form, and the mathematical derivation are correctly made, the same success will be obtained for baroclinic as for stagnant fluids.

However it is obvious that the perturbation method will be always quite unable to explain and forecast the growth of perturbations inside the turbulent regions which result from dynamic instability. This is in fact a very serious limitation of the meteorological significance of the perturbation problem, and it may be that most of cyclonic perturbations, for instance, have nothing to do with dynamic instability (this is strongly suggested by J. BJERKNES' original theory). However there is certainly a great deal of meteorological features such as long waves, mountain waves, maintainance of turbulence (cf. ROSSBY 1947), etc., which will be largely explained by using perturbation method.

2. General scope of the present paper

In the preceding section we have mentioned that, because of the great difficulties encountered in the mathematical derivations, most of the authors of recent researches about the perturbations of baroclinic currents have been led to introduce several simplifying assumptions in the course of the computations, and that these assumptions are generally not permissible. Since the most interesting problems are probably those related to zonal currents, we have found it important to give a system of perturbation equations for the most general continuous zonal stream, and to show that these equations can be practically always written in a relatively simple form if proper variables and coordinates are used, when the parameters are taken with the usual order of magnitude observed in the lower (or meteorological) atmosphere (up to 30 km, say). This is the main scope of this paper. In the last sections, we also give some applications to simple cases, and a brief discussion about some incorrect recently published results.

3. Definition and general properties of symbols

a) A_j, abridged notation for $\partial A/\partial j$

D/Dt, individual derivative with respect to time t

$\bar{A}$, value of A in undisturbed motion

∂A, local disturbance of A, that is, variation of A, from undisturbed to disturbed motion, at fixed time and point

DA, individual disturbance of A, that is, variation of A at fixed time and fluid-element (the computation rules for differentials hold for the operators ∂ and D separately)

(A), order of magnitude of A

$A \sim B$ (or $B \sim A$); $A \gg B$ (or $B \ll A$); $A \gtrsim B$ (or $B \lesssim A$), means that $(A) = (B)$; $(A) > (B)$; $(A) \gg (B)$, respectively

$A \# B$ (or $B \# A$), means that $A - B \ll A$

b) $\mathbf{M}$, position vector of the current point M defined by the space coordinates q_1, q_2, q_3 ($D\mathbf{M}$ is the displacement of the fluid-element from undisturbed to disturbed motion)

Dx_1, Dx_2, Dx_3, components of $D\mathbf{M}$ along x_1, x_2, x_3, directions of the three coordinate lines

$\partial/\partial x_1$, $\partial/\partial x_2$, $\partial/\partial x_3$, partial derivatives in the directions x_1, x_2, x_3

$$D\bar{A} = A_{q1}\, Dq_1 + A_{q2}\, Dq_2 + A_{q3}\, Dq_3 = \\ = A_{x1}\, Dx_1 + A_{x2}\, Dx_2 + A_{x3}\, Dx_3 \quad (3.1)$$

$$DA = \partial A + D\bar{A} \quad (3.2)$$

$$\partial\,(\partial A/\partial j) = (\partial/\partial j)\,(\partial A);\ D(DA/Dt) = \\ = (D/Dt)\,(DA) \quad (3.3)$$

c) i, imaginary unit; $\log A$, Neperian logarithm of A; $\mathbf{A} \cdot \mathbf{B}$, salar product; $\mathbf{A} \times \mathbf{B}$, vectorial product

d) L, r, h, cylindrical absolute coordinates relative to earth's axis: L, absolute longitude; r, distance to earth's axis; h, distance above equatorial plane

x, y, z, axes pointing eastwards, northwards, upwards, respectively

x, Y, Z, direct system of rectangular axes such that Y is directed northwards parallel to isentropic surface

$\mathbf{x}$, unit vector of x-direction

e) g, relative gravity (that is, ordinary gravity); E, potential of absolute gravity; p, pressure; q, density; T, absolute temperature; Θ, potential temperature; $m = c_v/c_p$, ratio of specific heats;

$a = m\, q/p$, coefficient of piezotropy for adiabatic changes

$s = \log\,(p^m/q) = \log \Theta + \text{const}$, a linear function of specific entropy

$s' = \log\,[(1/p^m)\sqrt{q/r}] = \frac{1}{2}\,[\log\,(1/rp^m) - s]$; $s'' = s' + \log \sqrt{s_z}$

$N_0 = \sqrt{g s_z}$, coefficient of hydrostatic stability

f) k; N; $P = 2\pi/N$, wave-number, orbital angular-frequency and orbital period of a sine wave in x-direction, respectively

g) $D'x,\ D'y,\ D'z,\ D'Y,\ D'Z = \sqrt{qr}\,(Dx,\ Dy,\ Dz,\ DY,\ DZ)$

$\partial' p = \sqrt{r/q}\ \partial p$; $\partial'' p = \partial' p/\sqrt{s_z}$

h) φ, latitude; $\tan \varepsilon = s_y/s_z$, slope of the isentropic surface

$\Phi = \varphi + \varepsilon$, angle of Y with equatorial plane

$1/R = \varphi_y$, meridian curvature of the level surface

ω, absolute angular velocity about earth's axis

$\mathcal{W}$, vector of magnitude ω parallel to earth's axis

ω_0, earth's angular velocity

$u = r\,(\omega - \omega_0)$, eastward velocity relative to earth

$f = 2\,\omega_0 \sin\varphi$, Coriolis parameter

$F = 2\,\omega \sin\varphi$, vertical component of $2\,\mathcal{W}$

$F_1 = 2\,\omega \sin\Phi$; $F_2 = 2\,\omega\cos\Phi$, components of $2\,\mathcal{W}$ along Z and Y, respectively

$N_1^2 = -(p_Z/q)\,s_Z + F_2 r\,\omega_Z$

$\varepsilon' = F_2^2/(N_1^2 - N^2)$; $F' = F_1 + \varepsilon' r\,\omega_Y$

$F_1' = F_1 - r\,\omega_Y$, component of absolute vorticity normal to the isentropic surface

$G_1 = F_1 F_1' - N^2$; $G' = G_1 + \varepsilon'\,(F_1' r\,\omega_Y - N^2)$

$G = F\,(F - u_Y) - N^2$

$B = (1/G)\,[F^2\,(F - u_Y)_Y + F_Y N^2] + (F/s_z)\,(aFu_z + s_{zY})$

4. Specification of the motions

The undisturbed motion is assumed to be a zonal flow of *limpid* and *unviscid* air, defined by *continuous* meridian distributions of ω and of two of the parameters p, q, T, Θ, a, s defined in (3.e), these three distributions being related together by the motion equations, given in next section.

The perturbation is simply assumed to be adiabatic and laminar.

5. Vectorial perturbation equations

If use is made of the cylindrical coordinates L, r, h defined in (3.d), the mechanical equations for an unviscid fluid motion may be written as follows:

$$\left.\begin{aligned} &\frac{D}{Dt}\left(r^2\frac{DL}{Dt}\right) + \frac{p_L}{q} = 0 \\ &\frac{D^2 r}{Dt^2} - r\left(\frac{DL}{Dt}\right)^2 + \frac{p_r}{q} + E_r = 0 \\ &\frac{D^2 h}{Dt^2} + \frac{p_h}{q} + E_h = 0 \end{aligned}\right\} \quad (5.1)$$

Applied to the undisturbed motion, they give

$$p_L = 0;\ (p_r/q) + E_r = r\omega^2;\ (p_h/q) + E_h = 0$$

If we differentiate the last two of these relations with respect to h and r respectively, and then subtract, we get

$$(1/q^2)\,(q_r p_h - q_h p_r) = 2\,\omega\, r\,\omega_h \quad (5.2)$$

or, introducing the parameter s defined in (3.e),

$$(1/q)\,(s_h p_r - s_r p_h) = 2\,\omega\, r\,\omega_h \quad (5.3)$$

and these relations are two equivalent forms of the so-called geostrophic equilibrium condition. It follows from (5.2) that the undisturbed motion is barotropic if, and only if, ω is independent of h.

Instead of (5.3) we may write, in vectorial form,

$$\nabla s \times (\nabla p/q) = (2\,\mathcal{W}\cdot r\nabla\omega)\,\mathbf{x} \quad (5.4)$$

The mechanical perturbation equations are readily deduced from (5.1) by applying the operator ∂ to the three equations of this system. We thus get, using (3.3),

$$\left.\begin{aligned} &\partial\,\frac{D}{Dt}\left(r^2\frac{DL}{Dt}\right) + \frac{1}{q}\frac{\partial}{\partial L}\partial p = 0 \\ &\partial\,\frac{D^2 r}{Dt^2} - r\,\partial\left(\frac{DL}{Dt}\right)^2 + \frac{1}{q}\frac{\partial}{\partial r}\partial p - p_r\frac{\partial q}{q^2} = 0 \\ &\partial\,\frac{D^2 h}{Dt^2} + \frac{1}{q}\frac{\partial}{\partial h}\partial p - p_h\frac{\partial q}{q^2} = 0 \end{aligned}\right\}$$

But in the undisturbed motion we have

$$\frac{D}{Dt}\left(r^2\frac{DL}{Dt}\right) = 0;\ \frac{D^2 r}{Dt^2} = 0;\ \frac{D^2 h}{Dt^2} = 0;\ \frac{DL}{Dt} = \omega$$

whence, using (3.2) and (3.3), and since $rDL = Dx$,

$$\partial \frac{D}{Dt}\left(r^2 \frac{DL}{Dt}\right) = D\frac{D}{Dt}\left(r^2\frac{DL}{Dt}\right) = r^2 D\frac{D^2L}{Dt^2} +$$

$$+ \omega D\frac{Dr^2}{Dt} = r^2\frac{D^2}{Dt^2}DL + 2\,\omega\, r\frac{D}{Dt}Dr =$$

$$= r\left(\frac{D^2}{Dt^2}Dx + 2\,\omega\frac{D}{Dt}Dr\right);$$

$$\partial\frac{D^2r}{Dt^2} = D\frac{D^2r}{Dt^2} = \frac{D^2}{Dt^2}Dr;\ \partial\frac{D^2h}{Dt^2} = \frac{D^2}{Dt^2}Dh;$$

$$r\partial\left(\frac{DL}{Dt}\right)^2 = 2\,\omega\, r\,\partial\frac{DL}{Dt} =$$

$$= 2\,\omega\, r\left(D\frac{DL}{Dt} - D\overline{\omega}\right) = 2\,\omega\left(\frac{D}{Dt}Dx - rD\overline{\omega}\right).$$

We thus arrive at the system

$$\left.\begin{aligned}
&\frac{D^2}{Dt^2}Dx + 2\,\omega\frac{D}{Dt}Dr + \frac{1}{q}\frac{\partial}{\partial x}\partial p = 0\\
&\frac{D^2}{Dt^2}Dr - 2\,\omega\left(\frac{D}{Dt}Dx - r\,D\overline{\omega}\right) + \frac{1}{q}\frac{\partial}{\partial r}\partial p -\\
&\qquad - \frac{p_r}{q}\frac{\partial q}{q} = 0\\
&\frac{D^2}{Dt^2}Dh + \frac{1}{q}\frac{\partial}{\partial h}\partial p - \frac{p_h}{q}\frac{\partial q}{q} = 0
\end{aligned}\right\}$$

which is equivalent to the vectorial equation

$$\frac{D^2}{Dt^2}\mathbf{DM} + 2\,\omega\times\left[\frac{D}{Dt}\mathbf{DM} - (rD\overline{\omega})\,\mathbf{x}\right] +$$

$$+ \frac{1}{q}\nabla\partial p - \frac{\nabla p}{q}\frac{\partial q}{q} = 0 \qquad (5.5)$$

Let us now consider the continuity equation and the thermal equation. For the motion itself they may be written respectively, if the air is treated as a perfect gas,

$$\operatorname{div}\frac{\mathbf{DM}}{Dt} + \frac{1}{q}\frac{Dq}{Dt} = 0;\ \frac{Ds}{Dt} = 0$$

and the corresponding perturbation equations may be written at once from analogy, if a proper choice is made of the correspondence between the fluid elements in the undisturbed and in the disturbed motion, as follows:

$$\operatorname{div}\,\mathbf{DM} + \frac{Dq}{q} = 0;\ Ds = 0 \qquad (5.6)$$

From the last one of these equations we get, according to the definition of s,

$$\frac{\partial q}{q} = m\frac{\partial p}{p} + D\overline{s};\ \frac{Dq}{q} = \frac{a}{q}\,\partial p + m\frac{D\overline{p}}{p}$$

and if we substitute in (5.5) and the first equation (5.6), we get the desired system of vectorial perturbation equations as follows:

$$\left.\begin{aligned}
&\frac{D^2}{Dt^2}\mathbf{DM} + 2\,\omega\times\left[\frac{D}{Dt}\mathbf{DM} - (rD\overline{\omega})\,\mathbf{x}\right] -\\
&\quad - \frac{\nabla p}{q}D\overline{s} + \frac{1}{q}\left(\nabla - m\frac{\nabla p}{p}\right)\partial p = 0 \qquad (5.7.1)\\
&\operatorname{div}\,\mathbf{DM} + m\frac{D\overline{p}}{p} + \frac{a}{q}\,\partial p = 0 \qquad (5.7.2)
\end{aligned}\right\}$$

6. Perturbation equations referred to isentropic coordinates

Once a coordinate system is chosen, it is easy to derive from (5.7) a set of scalar equations in which the unknowns are ∂p and Dx_1, Dx_2, Dx_3, projections of $\mathbf{DM}$ on the directions of the coordinate lines: these equations are obtained by substituting to (5.7.1) its projections on these three directions, expressing $D\overline{\omega}$, $D\overline{s}$ and $D\overline{p}$ in terms of Dx_1, Dx_2 and Dx_3 by means of (3.1), and likewise for div $\mathbf{DM}$ by means of Gauss' divergence theorem for instance. We thus get a system of four differential equations which are obviously linear and homogeneous with respect to the four unknowns, and therefore form a complete system of perturbation equations.

In the present case of an adiabatic motion, it is found that the simplest equations are obtained if reference is made, not to the usual coordinate lines formed by parallels, meridian level-lines and verticals, i.e. tangent at each point to the x, y, z-axes, but to the similar system tangent to the x, Y, Z-axes (see (3.d) for definitions). A coordinate system corresponding to such coordinate lines may be conveniently called an *isentropic system*.

If isentropic coordinates are used, we have simply $D\bar{s} = s_Z DZ$ since $s_Y = 0$, and the system of perturbation equations equivalent to (5.7) may be written as follows:

$$\left.\begin{aligned}
&\frac{D^2}{Dt^2}Dx - F_1\frac{D}{Dt}DY + F_2\frac{D}{Dt}DZ + \\
&\qquad + \frac{1}{q}\frac{\partial}{\partial x}\partial p = 0, \\
&F_1\frac{D}{Dt}Dx + \left(\frac{D^2}{Dt^2} - F_1 r\omega_Y\right)DY + \\
&\quad + \left(-\frac{p_Y}{q}s_Z - F_1 r\omega_Z\right)DZ + \\
&\qquad + \frac{1}{q}\left(\frac{\partial}{\partial Y} - m\frac{p_Y}{p}\right)\partial p = 0, \\
&-F_2\frac{D}{Dt}Dx + F_2 r\omega_Y DY + \\
&+\left(\frac{D^2}{Dt^2} + N_1^2\right)DZ + \frac{1}{q}\left(\frac{\partial}{\partial Z} - m\frac{p_Z}{p}\right)\partial p = 0, \\
&\frac{\partial}{\partial x}Dx + \left(\frac{\partial}{\partial Y} + m\frac{p_Y}{p} + \frac{r_Y}{r} - \Phi_Z\right)DY + \\
&+\left(\frac{\partial}{\partial Z} + m\frac{p_Z}{p} + \frac{r_Z}{r} + \Phi_Y\right)DZ + \frac{a}{q}\partial p = 0.
\end{aligned}\right\} \qquad (6.1)$$

(see (3.h) for symbols).

The quantities Φ_Z and Φ_Y which appear in the last equation are equal to the curvatures of the coordinate lines tangent to Y and Z respectively, and the following relations can be established:

$$\Phi_Z = \frac{s_{ZY}}{s_Z};\ \varphi_Y = \frac{\cos\varepsilon}{R} - \frac{g_y}{g}\sin\varepsilon \qquad (6.2)$$
$$(\Phi_Y = \varphi_Y + \varepsilon_Y)$$

On the other hand, the geostrophic equilibrium condition (5.4) takes the form

$$-\frac{p_Y}{q}s_Z = F_1 r\omega_Z + F_2 r\omega_Y \qquad (6.3)$$

whence we conclude that the coefficient of DZ in the second equation (6.1) is equal to the coefficient of DY in the third one, and a glance at the other coefficients of the system reveals that most of their terms are either symmetrical or anti-symmetrical with res, ect to the principal diagonal. (The same result holds with any other meridian coordinates.) The symmetry is however incomplete because of the presence of the parameters $1/q$, r_Y/r, r_Z/r, Φ_Z and Φ_Y, but it is easy to get rid of the first three of them, by using as new unknowns the quantities $D'x$, $D'Y$, $D'Z$, $\partial' p$ defined in (3.g). We thus get the following system, equivalent to (6.1) and obviously simpler:

$$\left.\begin{aligned}
&\frac{D^2}{Dt^2}D'x - F_1\frac{D}{Dt}D'Y + F_2\frac{D}{Dt}D'Z + \\
&\qquad + \frac{\partial}{\partial x}\partial' p = 0, \\
&F_1\frac{D}{Dt}D'x + \left(\frac{D^2}{Dt^2} - F_1 r\omega_Y\right)D'Y + \\
&\quad + F_2 r\omega_Y D'Z + \left(\frac{\partial}{\partial Y} + s'_Y\right)\partial' p = 0, \\
&\qquad -F_2\frac{D}{Dt}D'x + F_2 r\omega_Y D'Y + \\
&+\left(\frac{D^2}{Dt^2} + N_1^2\right)D'Z + \left(\frac{\partial}{\partial Z} + s'_Z\right)\partial' p = 0, \\
&\frac{\partial}{\partial x}D'x + \left(\frac{\partial}{\partial Y} - s'_Y - \Phi_Z\right)D'Y + \\
&+\left(\frac{\partial}{\partial Z} - s'_Z + \Phi_Y\right)D'Z + a\,\partial' p = 0.
\end{aligned}\right\} \qquad (6.4)$$

where s' is the parameter defined in (3.e).

It is even possible to go further, and to get rid also of the term Φ_Z by taking as unknowns the quantities

$$D'x/\sqrt{s_Z};\ D'Y/\sqrt{s_Z};\ \sqrt{s_Z}\ D'Z;\ \partial' p/\sqrt{s_Z}$$

The symmetry of the corresponding system of equations is practically perfect, since Φ_Y is always small compared to s'_Z. However, in view of the following developments, it is more convenient to retain the system (6.4) itself, which will be therefore our *fundamental system of perturbation equations.*

7. Zonal waves and wave equation

A system of differential equations such as (6.4), the coefficients of which are independent of L and t, is satisfied by solutions of the form

$$(D'x,\ D'Y,\ D'Z,\ \partial'p) = (D'x_0,\ D'Y_0,\ D'Z_0,\ \partial'p_0)\ \exp\ i\,(nL - n't)$$

where n is a real positive integer, n' a real or complex constant and $D'x_0$, $D'Y_0$, $D'Z_0$, $\partial'p_0$ are functions of the meridian coordinates only. The corresponding perturbations may be conveniently referred to as *zonal waves*, for which we may speak of a wave-number k and an orbital angular-frequency N, defined by

$$k = n/r;\ N = n\omega - n' \qquad (7.1)$$

so that both of these parameters are, as a rule, functions of the meridian coordinates.

In order to get the perturbation equations for such waves, we only have to make in (6.4) the substitutions

$$\partial/\partial x = i\,k;\ D/Dt = i\,N \qquad (7.2)$$

If this is done, we observe that in the first three equations no differential operator figures in the coefficients of $D'x$, $D'Y$ and $D'Z$. We can therefore easily solve these three equations for $D'x$, $D'Y$ and $D'Z$, and substituting the results in the last equation we get a differential equation for $\partial'p$ only, which we shall refer to as the *wave equation* of the perturbations.

In the wave equation, the differential operators $\partial/\partial x$ and D/Dt may be reintroduced in place of ik and iN respectively if these parameters have been first removed from the denominator, and the differential equation obtained in this fashion may be more convenient than the wave equation itself to give the solution of a perturbation problem when the boundary conditions are not simple. However in many cases the solution is more readily obtained by using the waves as elementary solutions, according to Fourier's general method.

8. Simplification of the wave equation

The wave equation generally appears with a rather complex form, but it happens that it can always be greatly simplified if account is taken of the order of magnitude of the parameters. It is important to note here that *the simplification must be made only once the wave equation is completely derived*, and not in the course of this derivation. We shall see at the end of the paper to what kind of errors one may be led by the non-observation of this fundamental rule.

We must first observe that, as a rule, the scale of space-distribution of $\partial'p$ and of all its partial derivatives is smaller than, or at most comparable to, that of any parameter Q involved in the wave equation, which we shall refer to as a *fundamental parameter*. Otherwise the portion of atmosphere under consideration would appear as a surface of discontinuity for Q, as far as the perturbation is concerned, and this would not agree with our assumption that the motion is everywhere continuous.

If P denotes the variable $\partial'p$ or any of its partial derivatives with respect to Y or Z, we may thus write, as a rule,

$$Q_Y P \precsim Q\frac{\partial}{\partial Y}P;\ Q_Z P \precsim Q\frac{\partial}{\partial Z}P \qquad (8.1)$$

whence

$$\frac{\partial}{\partial Y}Q\,P \sim Q\frac{\partial}{\partial Y}P;\frac{\partial}{\partial Z}Q\,P \sim Q\frac{\partial}{\partial Z}P \qquad (8.2)$$

In other words, *the order of magnitude of a term of the wave equation is not changed by an arbitrary permutation between the parameters and the differential operators.*

The relations (8.1) and (8.2) still hold, as a rule, if Q is replaced with a function Q' of fundamental parameters, provided that Q' is not simply the difference $Q_1 - Q_0$ between two approximately equal quantities ($Q_1 \# Q_0$), or a function of such a difference. Therefore if a fundamental parameter may be put in the form

$$Q = Q_0 + Q',\ \text{with}\ Q' \ll Q_0\ (\text{or}\ Q \# Q_0)$$

we can write

$$\frac{\partial}{\partial Y}Q'\,P \sim Q'\frac{\partial}{\partial Y}P \ll Q_0\frac{\partial}{\partial Y}P \sim \frac{\partial}{\partial Y}Q_0\,P$$

and likewise with the operator $\partial/\partial Z$. We thus get

$$\frac{\partial}{\partial Y} Q\, P \# \frac{\partial}{\partial Y} Q_0\, P; \quad \frac{\partial}{\partial Z} Q\, P \# \frac{\partial}{\partial Z} Q_0\, P \quad (8.3)$$

a formula which is very convenient to simplify the coefficients of the wave equation.

Considering now two arbitrary fundamental parameters Q_1 and Q_2, we can easily verify, by assuming successively $Q_1 \gtrsim Q_2$ and $Q_1 \lesssim Q_2$, that, as a rule,

$$Q_1 + Q_2 \sim Q_1 - Q_2 \quad (8.4)$$

$$Q_1\, Q_2 \gtrsim Q_1^2 \pm Q_2^2 \sim (Q_1 \pm Q_2)^2 \quad (8.5)$$

and these relations still hold if Q_1 and Q_2 are replaced with arbitrary symbolic functions of fundamental parameters and differential operators applied to the variable $\partial' p$, so that we have, for instance,

$$\left(\frac{\partial}{\partial Y} + Q_1\right) P \sim \left(\frac{\partial}{\partial Y} - Q_1\right) P,$$

$$Q_1 Q_2 \frac{\partial}{\partial Y} P \lesssim \left(Q_1^2 \pm Q_2^2 \frac{\partial^2}{\partial Y^2}\right) P,$$

$$Q_1 Q_2 \frac{\partial^2}{\partial Y \partial Z} P \lesssim \left(Q_1^2 \frac{\partial^2}{\partial Y^2} \pm Q_2^2 \frac{\partial^2}{\partial Z^2}\right) P, \text{ etc.}$$

By these formulae the different terms of the wave equation can be compared. In particular, it follows from (8.5) that

$$\lambda\, Q_1 Q_2 \ll Q_1^2 \pm Q_2^2, \text{ provided } \lambda \ll 1 \quad (8.6)$$

another important formula which will be largely used to simplify the wave equation.

9. General approximate wave equation

If we solve the first three equations (6.4) for $D'x$, $D'Y$ and $D'Z$ we get, in the case of a wave, the following symmetrical expressions:

$$\left.\begin{aligned}
D'x &= \frac{i}{G'}\left[-\left(1 + \frac{F' r\, \omega_Y}{N^2}\right) k + \right. \\
&\left. + \frac{F'}{N}\left(\frac{\partial}{\partial Y} + s'_Y\right) + \frac{F_2 N}{N_1^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right)\right] \partial' p, \\
D'Y &= \frac{1}{G'}\left[\frac{F'}{N} k - (1 + \varepsilon')\left(\frac{\partial}{\partial Y} + s'_Y\right) - \right. \\
&\left. - \frac{F_2 F_1'}{N_1^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right)\right] \partial' p, \\
D'Z &= \frac{1}{G'}\left[\frac{F_2 N}{N_1^2 - N^2} k - \frac{F_2 F_1'}{N_1^2 - N^2} \cdot \right. \\
&\left. \cdot \left(\frac{\partial}{\partial Y} + s'_Y\right) - \frac{G_1}{N_1^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right)\right] \partial' p.
\end{aligned}\right\} \quad (9.1)$$

(see (3.h) for symbols).

If we substitute from (9.1) in the last equation (6.4) and observe that, according to (7.1), we have

$$k_Y / k = - r_Y / r; \quad N_Y = k r\, \omega_Y \quad (9.2)$$

we obtain, for the wave equation,

$$(A_1 + A_2 + A_3 + A_4)\, \partial' p = 0 \quad (9.3)$$

with

$$\left.\begin{aligned}
A_1 &= -\frac{k^2}{G'} + \left(\frac{\partial}{\partial Y} - s'_Y - \frac{s_{ZY}}{s_Z}\right) \frac{1 + \varepsilon'}{G} \cdot \\
&\cdot \left(\frac{\partial}{\partial Y} + s'_Y\right) + \left(\frac{\partial}{\partial Z} - s'_Z + \Phi_Y\right) \cdot \\
&\cdot \frac{G_1}{G'(N_1^2 - N^2)}\left(\frac{\partial}{\partial Z} + s'_Z\right), \\
A_2 &= \frac{F' k}{G' N}\left(\frac{\partial}{\partial Y} + s'_Y\right) - \left(\frac{\partial}{\partial Y} - s'_Y - \frac{s_{ZY}}{s_Z}\right) \cdot \\
&\cdot \frac{F' k}{G' N} - \frac{1}{G'} \frac{F' r\, \omega_Y}{N^2} k^2 = \\
&= \frac{F' k}{G' N}\left(\frac{G'_Y}{G'} - \frac{F'_Y}{F'} + 2 s'_Y + \frac{r_Y}{r} + \frac{s_{ZY}}{s_Z}\right), \\
A_3 &= \frac{F_2 N k}{G'(N_1^2 - N^2)}\left(\frac{\partial}{\partial Z} + s'_Z\right) - \\
&- \left(\frac{\partial}{\partial Z} - s'_Z + \Phi_Y\right) \frac{F_2 N k}{G'(N_1^2 - N^2)} \\
A_4 &= \left(\frac{\partial}{\partial Y} - s'_Y - \frac{s_{ZY}}{s_Z}\right) \frac{F_2 F_1'}{G'(N_1^2 - N^2)} \cdot \\
&\cdot \left(\frac{\partial}{\partial Z} - s'_Z\right) - \left(\frac{\partial}{\partial Z} - s'_Z + \Phi_Y\right) \cdot \\
&\cdot \frac{F_2 F_1'}{G'(N_1^2 - N^2)}\left(\frac{\partial}{\partial Y} + s'_Y\right)
\end{aligned}\right\}$$

Let us note here the important result that the term in k^2/N^2 is exactly cancelled out in the expression for A_2 in virtue of (9.2), at the same time as the terms in $\partial/\partial Y$ cancel out each other.

Let us now refer to the current values of the parameters, as observed in the meteorological atmosphere. We have in cgs units

$$\left.\begin{array}{l} s_Z \sim 10^{-7}; -p_Z/q \# g \# 10^3;\ F_1 \# f \gtrsim 10^{-4}; \\ F_2 \gtrsim 10^{-4};\ a \sim 10^{-9};\ r\omega_Y \text{ and } u_Y \gtrsim f;\ r\omega_Z \text{ and} \\ u_Z \gtrsim 10^{-3};\ u \gtrsim 10^4;\ 1/R \sim 10^{-9};\ g_y/g \gtrsim 10^{-11} \end{array}\right\} \tag{9.4.1}$$

whence

$$\left.\begin{array}{l} F_1' \# F - u_Y \sim f;\ s_y \gtrsim 10^{-10};\ \varepsilon \gtrsim 10 f \ll 1; \\ \varepsilon' \ll 1;\ s_Z' \sim 10^{-6};\ s_z \# s_Z;\ N_1 \# N_0 \sim 10^{-2}; \\ F' \# F_1 \# F;\ G' \# G_1 \# G;\ \varphi_Y \ll s_Z' \end{array}\right\} \tag{9.4.2}$$

Using these relations, we can prove that, in the wave equation,

i) G_1 and F_1 may be substituted everywhere to G' and F' respectively, and ε' may be omitted in the second term of A_1. This is obvious in virtue of (8.3) where one makes successively $Q_0 = G_1$ and $Q = G'$; $Q_0 = F_1$ and $Q = F'$; $Q_0 = 1$ and $Q = 1 + \varepsilon'$.

ii) G_{1Y} and F_{1Y} may be likewise substituted to G'_Y and F'_Y in the second expression for A_2.

We have to prove that we may neglect the two quantities

$$a_1 = \frac{1}{G'}\frac{F'k}{G'N}(G' - G_1)_Y;$$

$$a_2 = \frac{1}{G'}\frac{k}{N}(F' - F_1)_Y$$

From the definitions of G' and F' we get, since $kr\,\omega_Y = N_Y \gtrsim N\,(\partial/\partial Y)$,

$$a_1 \lesssim \frac{1}{G'}\frac{F'k}{G'N}(G' - G_1)\frac{\partial}{\partial Y} \lesssim \frac{\varepsilon'}{G'}\cdot\left(\frac{F'F_1'}{G'}\frac{\partial^2}{\partial Y^2} - \frac{F'N}{G'}k\frac{\partial}{\partial Y}\right);$$

$$a_2 \lesssim \frac{1}{G'}\frac{k}{N}(F' - F_1)\frac{\partial}{\partial Y} \lesssim \frac{\varepsilon'}{G'}\frac{\partial^2}{\partial Y^2}$$

whence, since $F'F_1' \lesssim G'$; $F'N \lesssim G'$; $\varepsilon' \ll 1$, and according to (8.5),

$$a_1 \ll \frac{1}{G'}\left(\frac{\partial^2}{\partial Y^2} - k\frac{\partial}{\partial Y}\right) \lesssim \frac{1}{G'}\left(k^2 + \frac{\partial^2}{\partial Y^2}\right) \lesssim A_1;$$

$$a_2 \ll \frac{1}{G'}\frac{\partial^2}{\partial Y^2} \lesssim A_1$$

iii) Φ_Y may be omitted everywhere.

Owing to the relations $\Phi_Y = \varphi_Y + \varepsilon_Y$; $\varphi_Y \ll s_Z'$; $\varepsilon_Y \gtrsim \varepsilon\ (\partial/\partial Y)$, we have only to prove that we may neglect the three quantities

$$\frac{\varepsilon}{N_1^2 - N^2}\frac{\partial}{\partial Y}\left(\frac{\partial}{\partial Z} + s_Z'\right);\ \frac{F_2 N\varepsilon}{G'(N_1^2 - N^2)}k\frac{\partial}{\partial Y};$$

$$\frac{F_2 F_1'\varepsilon}{G'(N_1^2 - N^2)}\frac{\partial}{\partial Y}\left(\frac{\partial}{\partial Y} + s_Y'\right)$$

Since we have, by (8.5),

$$A_1 \gtrsim \frac{1}{G'}k^2 + \frac{1}{G'}\left(\frac{\partial}{\partial Y} \pm s_Y'\right)^2 + \frac{1}{N_1^2 - N^2}\left(\frac{\partial}{\partial Z} \pm s_Z'\right)^2$$

it follows from (8.6) that these terms are negligible compared to A_1 provided

$$\varepsilon\sqrt{\frac{G'}{N_1^2 - N^2}} \ll 1;\ \varepsilon\frac{F_2 N}{N_1^2 - N^2} \ll 1;\ \varepsilon\frac{F_2 F_1'}{N_1^2 - N^2} \ll 1$$

and these three conditions are actually satisfied.

iv) The remaining terms of A_3 and A_4 are negligible compared to A_1. Since $s_{ZY}/s_Z \gtrsim \partial/\partial Y$, this is readily proved by putting in (8.6), successively,

$$Q_1 = \frac{1}{\sqrt{G'}}k;\ Q_2 = \frac{1}{\sqrt{N_1^2 - N^2}}\left(\frac{\partial}{\partial Z} \pm s_Z'\right);\ \lambda = \frac{F_2 N}{\sqrt{G'(N_1^2 - N^2)}}$$

$$Q_1 = \frac{1}{\sqrt{G'}}\left(\frac{\partial}{\partial Y} \pm s_Y'\right);\ Q_2 = \frac{1}{\sqrt{N_1^2 - N^2}}\cdot\left(\frac{\partial}{\partial Z} \pm s_Z'\right);\ \lambda = \frac{F_2 F_1'}{\sqrt{G'(N_1^2 - N^2)}}$$

for in both cases we have $\lambda \ll 1$.

v) N_0 may be substituted to N_1 in the expression for A_1. Let us put

$$C_1 = \left(\frac{\partial}{\partial Z} - s_Z'\right)\frac{1}{N_1^2 - N^2}\left(\frac{\partial}{\partial Z} + s_Z'\right);$$

$$C_0 = \left(\frac{\partial}{\partial Z} - s_Z'\right)\frac{1}{N_0^2 - N^2}\left(\frac{\partial}{\partial Z} + s_Z'\right);$$

$$\frac{1}{N_1^2 - N^2} = \frac{1 + \varepsilon_1}{N_0^2 - N^2}\ (\varepsilon_1 \ll 1).$$

If we refer to the definitions of N_0 and N_1 and to equations of motion, we find that

$$\varepsilon_1 = \varepsilon_2 + \mu\,(\omega^2 - \omega_0^2), \text{ with}$$

$$\mu = \frac{s_Z r \cos\Phi}{N_1^2 - N^2};\ \varepsilon_2 = -\frac{F_2 r \omega_Z}{N_1^2 - N^2}$$

whence, by (8.2),

$$C_1 - C_0 \sim \mu\left(\frac{\partial}{\partial Z} - s'_Z\right)\frac{\omega^2 - \omega_0^2}{N_0^2 - N^2}\cdot$$
$$\cdot\left(\frac{\partial}{\partial Z} + s'_Z\right) + \varepsilon_2 C_0 = \varepsilon_1 C_0 +$$
$$+ \mu\,\frac{(\omega^2 - \omega_0^2)_Z}{N_0^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right) = \varepsilon_1 C_0 -$$
$$-\varepsilon_2 \frac{s_Z}{s'_Z}\,\frac{s'_Z}{N_0^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right) \gtrless \varepsilon_1 C_0 -$$
$$-\varepsilon_2 \frac{s_Z}{s'_Z}\, C_0 \ll C_0$$

in other words, $C_1 \not\approx C_0$.

We thus arrive at the following simplified wave equation:

$$\left[\frac{k^2}{G_1} - \left(\frac{\partial}{\partial Y} - s'_Y - \frac{s_{ZY}}{s_Z}\right)\frac{1}{G_1}\left(\frac{\partial}{\partial Y} + s'_Y\right) - \right.$$
$$- \left(\frac{\partial}{\partial Z} - s'_Z\right)\frac{1}{N_0^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right) + a -$$
$$\left. - \frac{k}{G_1 N} B_1\right]\partial' p = 0 \qquad (9.5)$$

with

$$B_1 = F_1\left(\frac{G_{1Y}}{G_1} - \frac{F_{1Y}}{F_1} + 2 s'_Y + \frac{r_Y}{r} + \frac{s_{ZY}}{s_Z}\right)$$

This equation can be transformed in several ways.

We can first prove, by the same methods used for above simplifications, that G_1, F_1, s_Z, s'_Z and $\partial/\partial Z$ may be replaced everywhere with G, F, s_z, s'_z and $\partial/\partial z$, respectively. That means that we may disregard the small departure of Z-direction from the vertical, as we could expect it from the smallness of all the horizontal gradients compared to the vertical ones, and that furthermore $r\omega_Y$ may be replaced with u_Y.

On the other hand, if we refer to the definition of s' and to relation (6.3) we get, since $s_Y = 0$,

$$2 s'_Y + \frac{r_Y}{r} = -m\frac{p_Y}{p} = -a\frac{p_Y}{q} =$$
$$= \frac{a}{s_Z}\left(F_1 r\omega_Z + F_2 r\omega_Y\right)$$

and, according to (9.2),

$$G_Y = F(F - u_Y)_Y + F_Y(F - u_Y) - 2 N k r\omega_Y$$

whence we get, after a few obvious simplifications have been made, the following form for the wave equation:

$$\left[\left(1 + \frac{2Fu_Y}{G}\right)\frac{k^2}{G} - \left(\frac{\partial}{\partial Y} - s'_Y - \frac{s_{zY}}{s_z}\right)\frac{1}{G}\cdot\right.$$
$$\cdot\left(\frac{\partial}{\partial Y} + s'_Y\right) - \left(\frac{\partial}{\partial z} - s'_z\right)\frac{1}{N_0^2 - N^2}\cdot$$
$$\left.\cdot\left(\frac{\partial}{\partial z} + s'_z\right) + a - \frac{k}{GN} B\right]\partial' p = 0$$

(9.6)

where B is the quantity defined at the end of (3.h).

Another practically equivalent form is obtained by introducing the variable $\partial'' p$ and the parameter s'' [see (3.f) and (3.e)], as follows:

$$\left[\left(1 + \frac{2Fu_Y}{G}\right)\frac{k^2}{G} - \left(\frac{\partial}{\partial Y} - s''_Y\right)\frac{1}{G}\cdot\right.$$
$$\cdot\left(\frac{\partial}{\partial Y} + s''_Y\right) - \frac{1}{N_0^2}\left(\frac{\partial}{\partial z} - s''_z\right)\frac{N_0^2}{N_0^2 - N^2}\cdot$$
$$\left.\cdot\left(\frac{\partial}{\partial z} + s''_z\right) + a - \frac{k}{GN} B\right]\partial'' p = 0$$

(9.7)

[In (9.6) and (9.7) $r\omega_Y$ may be substituted to u_Y, and the same substitution may be made in the expressions for G and B.]

We observe that *the vector 2 $\mathcal{W}$ only figures by its vertical component* in the wave equation (9.6) or (9.7), in agreement with the current assertion that the horizontal component of earth's rotation may be always neglected in atmospheric motions.

Let us now compare the orders of magnitude of the different terms of the wave equation (9.5). We have, when dealing with operators applied to $\partial' p$,

$$\frac{k^2}{G_1} - \left(\frac{\partial}{\partial Y} - s'_Y - \frac{s_{ZY}}{s_Z}\right)\frac{1}{G_1}\left(\frac{\partial}{\partial Y} + s'_Y\right) \sim$$

$$\sim \frac{1}{G}\left[k^2 + \left(\frac{\partial}{\partial Y} + s'_Y\right)^2\right],$$

$$\left(\frac{\partial}{\partial Z} - s'_Z\right)\frac{1}{N_0^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right) \sim$$

$$\sim \frac{1}{N_0^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right)^2,$$

$$a \lesssim \frac{s_Z'^2}{N_0^2} \lesssim \frac{1}{N_0^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right)^2,$$

$$\frac{k}{G_1 N} B_1 \lesssim \frac{F}{N}\frac{1}{G} k \left(\frac{\partial}{\partial Y} + s'_Y\right) \lesssim$$

$$\lesssim \frac{F}{N}\frac{1}{G}\left[k^2 + \left(\frac{\partial}{\partial Y} + s'_Y\right)^2\right],$$

whence, by (9.5),

$$\frac{1}{N_0^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right)^2 \lesssim \left(1 + \frac{F}{N}\right)\frac{1}{G}\cdot$$

$$\cdot\left[k^2 + \left(\frac{\partial}{\partial Y} + s'_Y\right)^2\right] \lesssim \left(1 + \frac{F}{N}\right)^2 \cdot$$

$$\cdot \frac{1}{G}\left(k + \frac{\partial}{\partial Y} + s'_Y\right)^2$$

and therefore

$$\left(\frac{\partial}{\partial Z} + s'_Z\right)\partial' p \lesssim \sqrt{\frac{N_0^2 - N^2}{G}}\left(1 + \frac{F}{N}\right)\cdot$$

$$\cdot\left(k + \frac{\partial}{\partial Y} + s'_Y\right)\partial' p \tag{9.8}$$

This formula will be very useful in the next sections.

10. General approximate formulae for the vibration components

In the case of a wave, the vector **DM** may be conveniently called the *vibration*, and its components are given as functions of $\partial' p$ by formulae (9.1). Let us now show that these formulae can be simplified like the wave equation, *as far as the magnitude and direction of the vibration is concerned*. For this purpose, we must observe that any term of the expression for $D'x$, $D'Y$ or $D'Z$ may be omitted if it is small compared to any one of these three components.

i) The last terms of $D'x$ and $D'Y$ are negligible compared to the other terms of these components.

This is readily seen by using relation (9.8), whence we deduce that

$$\frac{F_2 N}{N_1^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right)\partial' p \lesssim \frac{F_2 N}{\sqrt{G(N_0^2 - N^2)}}\cdot$$

$$\cdot\left(1 + \frac{F'}{N}\right)\left(k + \frac{\partial}{\partial Y} + s'_Y\right)\partial' p \ll$$

$$\ll \left(1 + \frac{F'}{N}\right)\left(k + \frac{\partial}{\partial Y} + s'_Y\right)\partial' p$$

$$\frac{F_2 F_1'}{N_1^2 - N^2}\left(\frac{\partial}{\partial Z} + s'_Z\right)\partial' p \lesssim \frac{F_2 F_1'}{\sqrt{G(N_0^2 - N^2)}}\cdot$$

$$\cdot\left(1 + \frac{F'}{N}\right)\left(k + \frac{\partial}{\partial Y} + s'_Y\right)\partial' p \ll$$

$$\ll \left(1 + \frac{F'}{N}\right)\left(k + \frac{\partial}{\partial Y} + s'_Y\right)\partial' p$$

ii) The first two terms of $D'Z$ are negligible compared to the first term of $D'x$ and the second term of $D'Y$, respectively.

This results from the fact that

$$\frac{F_2 N}{N_1^2 - N^2} \ll 1 \text{ and } \frac{F_2 F_1'}{N_1^2 - N^2} \ll 1$$

iii) $\partial/\partial Z$ and s'_z may be replaced with $\partial/\partial z$ and s'_Z, respectively, in the last term of $D'Z$.

Similar proof as above.

iv) $D'y$ and $D'z$ may be computed by the same formulae as $D'Y$ and $D'Z$, respectively.

This is obvious from the relations

$$D'y = D'Y\cos\varepsilon + D'Z\sin\varepsilon;$$
$$D'z = D'Z\cos\varepsilon - D'Y\sin\varepsilon$$

Finally, if we simplify the terms of (9.1) according to (9.4), we arrive at the following approximate formulae:

$$\left.\begin{aligned} D'x &= \frac{i}{G}\left[-\left(1+\frac{Fr\omega_Y}{N^2}\right)k + \right. \\ &\quad \left. + \frac{F}{N}\left(\frac{\partial}{\partial Y}+s_Y'\right)\right]\partial' p \\ D'Y \text{ or } D'y &= \frac{1}{G}\left[\frac{F}{N}k - \left(\frac{\partial}{\partial Y}+s_Y'\right)\right]\partial' p \\ D'Z \text{ or } D'z &= -\frac{1}{N_0^2-N^2}\left(\frac{\partial}{\partial Z}+s_Z'\right)\cdot \\ &\quad \cdot\partial' p \text{ or } -\frac{1}{N_0^2-N^2}\left(\frac{\partial}{\partial_z}+s_z'\right)\partial' p \end{aligned}\right\} \qquad (10.1)$$

We observe that, just as in the wave equation, the vector $2\,\mathcal{W}$ only figures by its vertical component.

11. Classification of waves according to orbital period P

Up to this point, we have made no assumption about N. We shall now see that further simplifications appear if this variable is assumed to have a definite order of magnitude compared to the parameters N_0 and f.

Since we have always $N_0/f \gtrsim 10^2$, it is obvious that we are always in only one of the three following cases:

i) $N \gtrsim N_0 \gg f$ ($P < 2$ hours), short-period waves

ii) $N \ll N_0$ and $\gtrsim f$ (2 hours $< P < 5$ pendulum days), middle-period waves

iii) $N \ll f \ll N_0$ ($P > 5$ pendulum days), long-period waves

According to these definitions, the width of the middle-period waves interval in the frequency spectrum becomes larger and larger as latitude decreases, and at the equator there are no long-period waves at all.

i) Short-period waves

From the assumption $N \gg f$, it can be proved, by the same method used in preceding sections, that the wave equation reduces to

$$\left[\frac{k^2}{N^2} - \left(\frac{\partial}{\partial y}-s_y'\right)\frac{1}{N^2}\left(\frac{\partial}{\partial y}+s_y'\right) - \right.$$
$$\left. -\left(\frac{\partial}{\partial z}-s_z'\right)\frac{1}{N^2-N_0^2}\left(\frac{\partial}{\partial z}+s_z'\right)-a\right]\partial' p = 0 \qquad (11.1)$$

On the other hand, the expressions (10.1) for the vibration components may be replaced with

$$D'x = \frac{i}{N^2}\,k\,\partial' p;\; D'y = \frac{1}{N^2}\left(\frac{\partial}{\partial y}+s_y'\right)\partial p;$$
$$D'z = \frac{1}{N^2-N_0^2}\left(\frac{\partial}{\partial z}+s_z'\right)\partial' p \qquad (11.2)$$

We can see that now neither the vector $\mathcal{W}$ nor its ascendant figure in the formulae, so that we can obtain the same formulae by assuming from the beginning that $\omega = 0$.

ii) Middle-period waves

If $N \ll N_0$, we can simplify the wave equation (9.6) or (9.7) by omitting N^2 beside N_0^2, but a more interesting result concerns the vibration components.

The relation (9.8) gives

$$\frac{1}{N_0^2-N^2}\left(\frac{\partial}{\partial Z}+s_Z'\right)\partial' p \lesssim \frac{1}{N_0\sqrt{G}}\cdot$$
$$\cdot\left(1+\frac{F}{N}\right)\left(k+\frac{\partial}{\partial Y}+s_Y'\right)\partial' p \ll \frac{1}{G}\cdot$$
$$\cdot\left(1+\frac{F}{N}\right)\left(k+\frac{\partial}{\partial Y}+s_Y'\right)\partial' p$$

whence we conclude, by (10.1), that

$$D'z \ll D'x \text{ or } D'y$$

In other words, the vertical component of the motion is negligible compared to its horizontal components. This is in agreement with the observed fact that if the scale of an atmospheric motion is such that its vertical acceleration is negligible (an assumption which corresponds to the condition $N \ll N_0$) this motion is always mainly horizontal.

Let us observe here that, in low and middle latitudes, some of the middle-period waves satisfy both of the conditions $N \ll N_0$ and

$N \gg f$, so that the special properties of the short-period waves also belong to these waves.

iii) Long-period waves

For these waves, we have of course the same fundamental property as for the middle-period ones, namely the predominance of the horizontal components of the motion, but further simplifications result from the condition $N \ll f$:

In first place the expressions for G and B reduce to

$$G = F(F - u_Y);$$

$$B = F\left[\frac{(F - u_Y)_Y}{F - u_Y} + \frac{aFu_z + s_{zY}}{s_z}\right]$$

and secondly the horizontal components of the vibration may be computed by the approximate formulae

$$D'x = \frac{i}{F - u_Y}\frac{I}{N}\left[-\frac{r\omega_Y}{N}k + \left(\frac{\partial}{\partial Y} + s'_Y\right)\right]\partial'p =$$

$$= \frac{i}{F - u_Y}\left(\frac{\partial}{\partial Y} + s'_Y\right)\frac{1}{N}\partial'p;$$

$$D'y = \frac{1}{F - u_Y}\frac{k}{N}\partial'p$$

12. Wave equation with constant coefficients

a) Let us first consider the case of a short-period wave, with

$$\left.\begin{array}{l}\omega = \text{const, whence } N = \text{const} \\ N_0 = \text{const};\ \partial/\partial y \gtrsim 10^{-8};\ \partial/\partial z \gtrsim 10^{-5}\end{array}\right\} \quad (12.1)$$

We thus have $s_y' \ll \partial/\partial y$; $s_z' \ll \partial/\partial z$

so that the wave equation (11.1) reduces to

$$\left(\frac{k^2}{N^2} - \frac{1}{N^2}\frac{\partial^2}{\partial y^2} - \frac{1}{N^2 - N_0^2}\frac{\partial^2}{\partial z^2} - a\right)\partial'p = 0 \quad (12.2)$$

and all its coefficients may be considered as constants, as well as r and R, since their scale of space-distribution is, according to (12.1), large compared to that of the perturbation. We may therefore take, as particular solutions, the pseudo-plane waves defined by

$$\partial'p = \partial'p_1 \exp.\ i\ (k\ x + k'y + k''z - n't) \quad (12.3)$$

where $x = rL$; $y = R\varphi$; z is the altitude, and $\partial'p_1$, k', k'' are three constants.

From (12.2) it follows that k, k', k'' and N must be connected together by the relation

$$(k^2 + k'^2)/N^2 + k''^2/(N^2 - N_0^2) - a = 0$$

or

$$a\,N^4 - (K^2 + a\,N_0^2)\,N^2 + N_0^2\,K^2\sin^2\psi = 0$$

where $K = \sqrt{k^2 + k'^2 + k''^2}$ is the geometrical wave-number, and ψ is the slope angle of the wave surface ($\tan\psi = k''/K$).

a.1) If k' and k'' are real (internal waves with uniform amplitude), it results from (12.1) that $K \gtrsim 10^{-5}$, whence $a\ N_0^2 \ll K^2$ and $\sqrt{a}N_0 \sin\psi \ll K$, so that the above relation may be written as well

$$(a\,N^2 - K^2)\ (N^2 - N_0^2\ \sin^2\psi) = 0$$

We thus get two types of waves:

i) If $aN^2 - K^2 = 0$, the orbital phase-velocity $U = |N/k|$ has the uniform value $U = 1/\sqrt{a}$, the sound velocity, and since $N \gg N_0$ the relations (11.2) give us

$$\frac{Dx}{k} = \frac{Dy}{k'} = \frac{Dz}{k''}$$

a proportion which means that the vibration is *longitudinal*. The corresponding waves are obviously nothing but the classical *elastic waves*.

ii) If $N^2 - N_0^2\ \sin^2\psi = 0$, whence $|N| = N_0 \sin\psi$, we get from (11.2)

$$kDx + k'Dy + k''Dz = 0$$

in other words the vibration is *transversal*. We have proposed the name *internal gravity-waves* for the corresponding waves.

a.2) If k'' is a pure imaginary number, k' being real, the waves are no longer internal waves with uniform amplitude, they are surface waves with an amplitude which decreases exponentially upwards or downwards. For instance if we assume $N \gg N_0$, and $aN^2 \ll 10^{-10}$, we get $k''^2 = -(k^2 + k'^2)$, and the corresponding waves are of the type of the classical gravitational waves observed near the surface of deep water for instance.

b) Let us then consider the case of a long-period wave, with

$$\left.\begin{array}{l}\omega = \text{const, whence again } N = \text{const}\\ N_0 = \text{const; } k \gtrsim 10^{-8};\ \partial/\partial y \sim 10^{-9}\end{array}\right\} (12.4)$$

Since ω = const and $s_z \neq$ const, we have

$$G = F^2;\ B \neq F_Y \neq f_y$$

and the wave equation (9.6) reduces to

$$\left(k^2 - \frac{F^2}{N_0^2}\frac{\partial^2}{\partial z^2} - \frac{k}{N} f_y\right) \partial' p = 0$$

If the study of the motion is restricted to a narrow middle-latitude zone of say 20° or 30° in width, all the coefficients of this equation may be taken as constants, as well as r and R. We may therefore again consider particular solutions of the form (12.3), with now $k' = 0$, and

$$k^2 + (F^2/N_0^2)\, k''^2 - (k/N) f_y = 0$$

If we consider the orbital phase-velocity $U_x = -(N/k)$ in the x-direction, we get from this relation

$$U_x = -f_y/[k^2 + (F^2/N_0^2)\, k''^2] \qquad (12.4)$$

and if, as a result of proper boundary conditions, the term in k'' is small compared to the term in k, this dispersion equation reduces to Rossby's formula for long atmospheric waves.

c) If in (b) we replace the condition ω = const with the two following ones, which are practically compatible:

absolute vertical vorticity $= F - u_y =$ const;
$u_z = 0$

we have $F - u_Y =$ const, whence $B = 0$, and the wave equation becomes

$$\left[\frac{F + u_Y}{F} k^2 - \left(\frac{F - u_Y}{N_0}\right)^2 \frac{\partial^2}{\partial z^2}\right] \partial' p = 0$$

If the study of the motion is again restricted to a narrow zone, the coefficients of this equation may be taken as constants, and we may again consider solutions of the form (12.3), with $k' = 0$ and

$$k'' = \pm i \sqrt{\frac{F + u_Y}{F}}\ \frac{N_0}{F - u_Y}\, k$$

13. Considerations about some incorrect results recently published

a) Some authors have considered that, since the large-scale motions are always mainly horizontal, it is permissible, in the case of long-period waves, to disregard the vertical motion in the perturbation equations, a procedure which obviously facilitates the derivation of the wave equation. It may be useful to show how far the wave equation obtained in this fashion is incorrect.

If we neglect the vertical motion, we are led to replace the perturbation equations (6.4) with the following approximate ones, referred to x, y, z-axes:

$$\left.\begin{array}{l}\dfrac{D^2}{Dt^2} D'x - F\dfrac{D}{Dt} D'y + \dfrac{\partial}{\partial x}\partial' p = 0,\\[2ex] F\dfrac{D}{Dt} D'x + \left(\dfrac{D^2}{Dt^2} - Fr\omega_y + fus_y\right) D'y + \\[2ex] \qquad + \left(\dfrac{\partial}{\partial y} + s_y'\right)\partial' p = 0,\\[2ex] \dfrac{\partial}{\partial x} D'x + \left(\dfrac{\partial}{\partial y} - s'_y\right) D'y + a\partial' p = 0.\end{array}\right|$$

whence, by eliminating $D'x$ and $D'y$, we easily get the wave equation

$$\left[\left(1 - \frac{fus_y}{N^2}\right)\frac{k^2}{G_0} - \left(\frac{\partial}{\partial y} - s_y'\right)\frac{1}{G_0}\cdot \right.$$
$$\left. \cdot\left(\frac{\partial}{\partial y} + s_y'\right) + a - \frac{k}{G_0 N} B_0\right] \partial' p = 0 \qquad (13.1)$$

with

$$G_0 = G + fus_y$$

$$B_0 = F\left(\frac{G_{0y}}{G_0} - \frac{F_y}{F} + 2 s_y' + \frac{r_y}{r}\right) \neq$$
$$\neq F\left(\frac{G_{0y}}{G_0} - \frac{F_y}{F} + afu - s_y\right)$$

Let us now consider the special case where

$$\omega_y = 0, \text{ whence } N_y = 0, \text{ and}$$

$$N^2 \ll F^2;\ k \gtrsim 10^{-8};\ \partial/\partial y \sim 10^{-9}$$

With these assumptions, (13.1) reduces to the following dispersion equation:

$$\left(1 - \frac{fus_y}{N^2}\right) k^2 - \frac{k}{N} B_0 = 0$$

with now

$$B_0 \# f_y + f(afu - s_y)$$

This is a second-degree equation in N, the roots of which are imaginary if

$$fus_y < 0 \text{ and } k^2 > \frac{B_0^2}{4\,|fus_y|}$$

so that the above derivation leads to unstable waves in a westerly current if the potential temperature decreases towards the pole. It is clear that this result, which has been given as an explanation for the origin of some of the extra-tropical atmospheric waves, is due to the presence, in wave equation (13.1), of the term in k^2/N^2, which has not been cancelled out in the derivation of this equation, precisely because the vertical component of the motion has been neglected. Since such a term does not exist in the correct wave equation, as already emphasized, we must conclude that the existence of unstable waves in the westerlies cannot be justified in this simple fashion.

By this example, we also understand to what kind of errors may lead the application of the method under discussion.

b) Another simplification which is frequently made is that, according to what is assumed in the so-called parcel method in the theory of hydrostatic stability, the local change of pressure ∂p is negligible, so that one may put $\partial p = 0$ (or $\partial' p = 0$) in the wave equations.

If we do that in equations (6.4), and if we only retain the first three of these equations, we get, by eliminating the unknowns $D'x$, $D'Y$, $D'Z$, the following condition in the case of a wave:

$$G \# F(F - r\omega_Y) - N^2 = 0 \quad (13.2)$$

which determines the value of N at any point.

The corresponding waves, in the assumption that they can exist, are nothing but the so-called *isobaric waves*, so often considered in theoretical meteorology, and the condition (13.2) shows that these waves would be stable or unstable, according as $r\omega_Y$ is smaller or larger than F. If for instance $\omega = \text{const}$, we get $N = \pm F$, and if $\omega = \omega_0$ the orbital frequency must be equal to the Coriolis parameter f.

But it is easy to see that such waves cannot exist except for particular undisturbed motions. For N is already connected to ω by the relation $N = n\omega - n'$, which is compatible with (13.2) only for a special space-distribution of ω. Furthermore, whatever may be this special distribution, the isobaric waves are necessarily stable, for $n\omega - n'$ can be a pure imaginary everywhere only if ω is a constant, and we have already seen that in this case N is real.

We can therefore conclude that the consideration of isobaric waves is only of poor interest in the case of a distorted current and cannot explain the apparition of unstable waves.

14. Conclusion

This paper was written with the intent, not to give the solution of a particular perturbation-problem related to a zonal air-current, but only to secure the first step in the way to such a solution, namely the derivation of the wave equation. The perturbation problem is thus reduced to a purely mathematical one: the resolution of the wave equation under proper boundary conditions.

This resolution is certainly not an easy task for most of practical problems, but it is clear that it will be always possible to manage it, by using modern computers if necessary, and we have proved in the preceding section that, in fact, most of the disagreements between the results obtained by various authors are to be imputed to incorrect derivations of the wave equation.

We think it therefore not superfluous, in view of generalizations to non-zonal or non-adiabatic motions for instance, to emphasize here that, when deriving a differential equation, *the simplifications resulting from the order of magnitude of the parameters must be made only at the very end of the derivation, and not in the course of the computations.* According to this principle, the approximate meteorological laws generally applied to large-scale motions (vanishing of vertical acceleration or velocity, vorticity theorem or conservation of vertical vorticity, etc.), though very useful to give the distributions of some of the hydrodynamical quantities when some others are known, must never be used as starting relations in a sequence of computations where differentiations are involved.

REFERENCES

BJERKNES, V. and collaborators, 1933: Chap. 14, Physikalische Hydrodynamik, Jul. Springer, Berlin 1933; Hydrodynamique Physique, Presses Univ. de France, Paris 1934.

CHARNEY, J. G., 1947: The Dynamics of Long Waves in a Baroclinic Westerly Current, *J. Met.*, **4**, No 5.

EADY, E. T., 1949: Long Waves and Cyclone Waves, *Tellus*, **1**, No 3.

FJOERTOFT, R., 1941: On the Deepening of a Polar Front Cyclone, *Met. Ann. Oslo*, No 1.

FJOERTOFT, R., 1946: On the Frontogenesis and Cyclogenesis in the Atmosphere, Part I: On the Stability of the Stationary Circular Vortex, *Geof. Publ.*, **16**, No 5.

HELMHOLTZ, 1868: Ueber discontinuierliche Flüssigkeitsbewegungen, *Berl. Monatsber.*, April; *Phil. Mag.*, Nov.

HOEILAND, E., 1939: On the Interpretation and Application of the Circulation Theorems of V. Bjerknes, *Archiv for mathem. og naturv., Oslo*, **42**, No 5.

—, 1941: On the Stability of the Circular Vortex, *Avh. utgitt av Det Norske Vid.-Ak., Oslo, Matem. Nat. Klass, Oslo*, No 11.

JAW, J., 1946: The Formation of the Semi-Permanent Centers of Action in Relation to the Horizontal Solenoidal Field, *J. Met.*, **3**, No 4.

JEFFREYS, H., 1925: On the Formation of Water Waves by Wind, *Proc. Roy. Soc.* A, **107**.

KELVIN (THOMSON, W.), 1871: Hydrokinetic Solutions and Observations, *Phil. Mag.*, **42**; *Nature*, **5**.

KÜTTNER, J., 1939: Moazagotl und Föhnwelle, *Beitr. Phys. fr. Atm.*, **25**.

LAMB, H., 1932: Atmospheric Waves, Chap. 10, §§ 309—316, Hydrodynamics, Cambr. Univ. Press 1932.

VAN MIEGHEM, J., 1944: Forme intrinsèque du critère d'instabilité dynamique de E. Kleinschmidt. *Bull. Sc. Ac. Roy. Belg.*, 5e série, **30**, p. 19; Interprétations énergétiques du critère d'instabilité de Kleinschmidt, ibid, **31**, p. 345; Perturbations d'un courant atmosphérique permanent zonal. *Mém. Inst. Mét. Belg.*, **18**, p. 1.

—, 1946: Sur la Stabilité du Courant Géostrophique, *La Météorologie*, Paris; Contribution à l'Etude de la Cyclogénèse, *Mém. Inst. Mét. Belg.*, **23**.

PEKERIS, C. L., 1937: Atmospheric Oscillations, *Proc. Roy. Soc.* A, **158**.

ROSSBY, C.-G., 1947: On the Distribution of Angular Velocity in a Gaseous Envelope under the Influence of Large Scale Mixing Processes, *Bull. Am. Met. Soc.*, **28**.

SOLBERG, H., 1928: Integrationen der Atmosphärischen Störungsgleichungen, *Geof. Publ.*, **5**, No 9.

—, 1936: Le Mouvement d'Inertie de l'Atmosphere Stable et son Rôle dans la Théorie des Cyclones, UGGI, Ass. Met., Edinburgh 1936.

TAYLOR, G. I., 1936: The Oscillations of the Atmosphere, *Proc. Roy. Soc.* A, **156**.