Tellus (1977), *29*, 502–511

Solutions of the balance equation with minimum correction of the mass field

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(Manuscript received December 29, 1976; in final form March 14, 1977)

ABSTRACT

If the balance equation is solved for the stream function by conventional methods, the mass field has to satisfy a condition of ellipticity. In this paper we present a method to find a stream function when this condition is not satisfied. The method is an iterative method presented by Fjørtoft (1962). It is shown that if this method converges, it gives a solution where the residual of the balance equation is a minimum.

The method is tried on a height analysis of 300 mb which contains rather large hyperbolic areas. The convergence of the method is very slow, and a solution using a conventional method is used as a first guess. From the solution found we find a height field in balance with this stream function by solving the balance equation for the height field. It is shown that this field is a better approximation to the original height field than an elliptic field found after correction by a standard method. The new height field does not satisfy the condition of ellipticity with respect to the balance equation.

1. Introduction

For successful short-range forecasting with primitive equation models it is necessary to adjust the initial data so that the mass field is in balance with the wind field. Such an adjustment is called initialization and many different schemes of initialization are in use today.

Some schemes use the primitive equations with a heavy damping of the gravity waves to obtain the balanced data for the model integrations. These schemes are often referred to as dynamic balancing or initialization schemes. Several examples of such approaches are found in the Proceedings of the JOC study group conference on four-dimensional data assimilation, Paris (1975). In other initialization schemes the required balance between wind and mass fields is obtained from diagnostic expressions derived from approximations of the vorticity equation and the divergence equation. Such initialization is called static balancing. An account of how to obtain initial data from such diagnostic models is given by Phillips (1960). A variational procedure where the diagnostic models are imposed as a constraint has recently been tried by Haltiner et al. (1975).

An approach which may be placed in between the two categories mentioned above is the higher order balanced systems as solved by Pedersen & Grønskei (1969) and by Lystad (1976). These authors find a balance between all the dependent variables including the time derivatives of the stream function and the thickness, using iterative methods. Their methods have not yet been tried in connection with initialization of primitive equation models.

Static schemes usually applied solve the balance equation to find a stream function which is in balance with the geopotential field. This equation consists of the dominant terms of the divergence equation. It is useful not only for initialization of numerical models, but also to estimate the non-divergent wind when only the height field is known. The balance equation, which is of the Monge Ampère type (see e.g. Courant Hilbert, 1962), is nonlinear in the stream function and is usually solved as an elliptic boundary value problem. The ellipticity condition, discussed, e.g. by Charney (1955) and Bolin (1955), does not allow larger anticyclonic geostrophic vorticity than half the value of the Coriolis parameter. The condition for dynamic stability in the atmosphere, which prescribes the absolute vorticity to be positive, is less restrictive. The balance equation is linear in the height field and no similar condition need be satisfied to find a solution for this field when the stream function is known.

When the balance equation is solved for the stream function, the height field is adjusted so that the condition of ellipticity is satisfied. Almost every hemispheric map of the height at 2–300 mb shows areas which are non-elliptic, Paegle & Paegle (1975). Although many of these hyperbolic areas are a result of inadequate data coverage and analysis techniques, there is a significant percentage of hyperbolic data in the atmosphere. These areas are most often found in subtropical high pressure regions and south of the tropospheric jet stream. In connection with jets, lateral geostrophic shear with values up to the Coriolis parameter on the anticyclonic flank was first demonstrated by Palmen (1948) and Palmen & Nagler (1948). If the current is straight in the whole integration domain, the nonlinear terms disappear from the balance equation and there is no need for correcting the height field. In a more general case, when the vorticity in some areas is mostly due to horizontal shear, correction of the height is undesired although the balance equation is hyperbolic. Also, in other areas where the analysis is well defined from the data, it is undesirable to change the height just because of a mathematical constraint.

In this paper we present a method to find the stream function which is in minimum unbalance with the height field, using ideas originating from Fjørtoft. The height field is not corrected in advance. Once the optimal stream function has been found, a height field in full balance with this field is found from the reversal of the balance equation. We first use an iterative method to find an optimal stream function and then, by reversing the balance equation, find a new height field which is in full balance with the stream function. This new height field differs from the original only in the hyperbolic areas and the ellipticity condition is not satisfied. The method is applied to a geopotential field of 300 mb and the result is compared with a solution coming from a traditional method where the height field is corrected in advance.

2. An optimal solution without correcting the geopotential

2.1. *Minimizing the residual*

The balance equation which we are trying to solve may be written

$$L(\psi) \equiv \nabla \cdot f \nabla \psi + 2J(\psi_x, \psi_y) - \nabla^2 \varphi \qquad (2.1)$$

ψ is given on the boundary. The ellipticity condition is:

$$\frac{1}{f}\nabla^2 \varphi > -\frac{f}{2} + \frac{1}{f}\nabla f \cdot \nabla \psi \qquad (2.2)$$

The variables are:

ψ, stream function;
φ, geopotential;
f, Coriolis parameter.

The indexes x, y mean partial derivatives with respect to the free variables x and y.

The condition is sufficient and necessary in order to solve the balance equation with respect to ψ as an elliptic boundary value problem. Solutions also exist, however, when the ellipticity condition is not satisfied. The case of a linear stream with a great horizontal shear is an example of a solution of (2.1) which does not satisfy the condition (2.2). This example is, however, very special. In a more general situation there exist smaller hyperbolic areas within greater elliptic areas. Although we should expect that the problem has a solution in most cases, we do not know if it is unique and if it can be found by standard methods.

Nevertheless, we will try to find a value of ψ which makes the mean absolute residual as small as possible in the hyperbolic areas and which satisfies the equation in the rest of the integration domain. This solution of ψ we will define as the optimal solution. It is not necessarily in exact balance with the geopotential field, and so the balance equation has to be reversed to find the relevant φ.

According to Fjørtoft (personal communication) the problem can be stated in this way:

Given $\nabla^2\varphi$ everywhere in the interior of the area. Find $\psi(x, y)$ so that

$$\iint L^2(\psi)\,dx\,dy = \text{Minimum}$$

ψ given on the boundary.

$L(\psi)$ is the residual of the balance equation. To find the minimum of the integral we use the method of variation of parameters (see Appendix 1). As a result of the calculations we get that the integral is a minimum if ψ satisfies the condition:

$$\nabla \cdot (\mathscr{S}(\psi) \cdot \nabla L(\psi)) = 0 \tag{2.3}$$

$\mathscr{S}$ is a symmetric tensor:

$$\mathscr{S} = \begin{pmatrix} f + 2\dfrac{\partial^2 \psi}{\partial y^2}, & -2\dfrac{\partial^2 \psi}{\partial x\, \partial y} \\ -2\dfrac{\partial^2 \psi}{\partial x\, \partial y}, & f + 2\dfrac{\partial^2 \psi}{\partial x^2} \end{pmatrix}$$

If the balance equation is elliptic, the solution to this minimization problem should satisfy the balance equation $L(\psi) = 0$. The condition (2.3) may also be satisfied if $L(\psi) \neq 0$. The problem is how to find ψ in that case.

2.2. *An iterative procedure*

In the case of solving an elliptic equation it is possible to find a relaxation coefficient κ which makes the iterative scheme

$$\psi^{n+1} = \psi^n + \kappa L(\psi^n) \quad n = 0,1,\ldots \tag{2.4}$$

converge.

Fjørtoft (1962) has suggested that a modified iterative method could be used to solve equations which are hyperbolic in some areas. The method is simply to change the sign of κ for each iteration. We will try this method also in connection with this non-linear equation, and see if it converges towards a solution of the minimum condition (2.3).

Changing the sign of κ gives for iteration $n + 2$:

$$\psi^{n+2} = \psi^{n+1} - \kappa L(\psi^{n+1}) \tag{2.5}$$

Addition of the two iteration eqs. (2.4) and (2.5) gives:

$$\psi^{n+2} - \psi^n = -\kappa(L(\psi^{n+1}) - L(\psi^n)) \tag{2.6}$$

If this scheme converges, i.e. $\psi^{n+2} - \psi^n \to 0$ as $n \to \infty$, the change in the residual from one iteration to the next will become arbitrarily small, i.e. $L(\psi^{n+1}) - L(\psi^n) \to 0$ as $n \to \infty$. This does not imply $L(\psi^n) \to 0$, but if the difference $\psi^{n+1} - \psi^n$ is small, we may linearize and obtain:

$$L(\psi^{n+1}) - L(\psi^n) \approx \delta_n L(\psi^n) \tag{2.7}$$

where $\delta_n L$ is the variation of L due to a small change of ψ at iteration number n. Using the definition of L, this variation can be written (see Appendix, eq. (A7)):

$$\begin{aligned} \delta_n L &= \nabla \cdot \left(\mathscr{S}\left(\frac{\psi^{n+1} + \psi^n}{2} \right) \cdot \nabla \delta_n \psi^n \right) \\ &= \kappa \nabla \cdot \left(\mathscr{S}\left(\frac{\psi^{n+1} + \psi^n}{2} \right) \cdot \nabla L(\psi^n) \right) \end{aligned} \tag{2.8}$$

The last expression is due to eq. (2.4). When this is placed back in eq. (2.6) for the difference of the residuals, we get:

$$\psi^{n+2} - \psi^n = -\kappa^2 \nabla \cdot \left(\mathscr{S}\left(\frac{\psi^{n+1} + \psi^n}{2} \right) \cdot \nabla L(\psi^n) \right) \tag{2.9}$$

This shows that if the iteration procedure converges in the sense that

$$|\psi^{n+2} - \psi^n| \to 0 \quad \text{as} \quad n \to \infty$$

it follows that the limit value $(\psi^{n+1} + \psi^n)/2$ also satisfies the condition for minimum absolute residual in the balance equation.

2.3. *An example*

The gradient wind equation is a special case of the balance equation. It is valid only when the accelerations are centripetal, i.e. when the height contours are circular. Also this simple equation has not a real solution in all cases. We present this example because of the analogy of the mathematical methods.

The gradient wind equation may be written using the dimensionless rotation angle ω:

$$L(\omega) \equiv \omega^2 + \omega - \omega_g = 0 \tag{2.10}$$

where ω_g is the geostrophic value. This equation has two solutions

$$\omega_{1,2} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1 + 4\omega_g}$$

if the pressure gradient is such that

$$\omega_g \gtreqless -\tfrac{1}{4} \tag{2.11}$$

We will now try to find ω which makes the integral

$$I = \int\int L^2(\omega)\, dx\, dy$$

a minimum when the condition (2.11) is not satisfied and therefore $L(\omega) \neq 0$. The variational method gives that the condition for minimum absolute residual is:

$$L(\omega) \cdot (1 + 2\omega) = 0 \tag{2.12}$$

and since $L(\omega) \neq 0$ this condition is satisfied only by

$$\omega = -\tfrac{1}{2} \tag{2.13}$$

If we use the same iterative method as mentioned before to solve eq. (2.10), we will show that the result satisfies condition (2.13):

$$\begin{aligned}\omega^{n+2} - \omega^n &= -\kappa[L(\omega^{n+1}) - L(\omega^n)] \\ &= -\kappa[(\omega^{n+1})^2 - (\omega^n)^2 + \omega^{n+1} - \omega^n] \\ &= -\kappa[(\omega^{n+1} - \omega^n)(\omega^{n+1} + \omega^n + 1)] \\ &= -\kappa^2 L(\omega^n)\left(1 + 2\frac{\omega^{n+1} + \omega^n}{2}\right)\end{aligned}$$

If this scheme converges, we see that in the limit

$$\overline{\omega^n} \equiv \frac{\omega^{n+1} + \omega^n}{2} = -\frac{1}{2}$$

which satisfies the condition (2.13).

2.4. *The iterative method in practice*

The iterative method which is described in Section 2.2, converges slowly also in the elliptic areas. To get an idea about the rate of convergence we inspect the quasi-geostrophic balance equation

$$f_0 \nabla^2 \psi - \nabla^2 \varphi = 0$$

From eq. (2.4) we get the following relation for the error in the stream function after n iterations, ($\varepsilon^n \equiv \psi^n - \psi$):

$$\varepsilon^{n+1} = \varepsilon^n + \kappa f_0 \frac{m^2}{d^2} \nabla^2 \varepsilon^n$$

where m is the map scale factor, d is the grid length and ∇^2 is the five points difference equivalent to the Laplacian operator. As this is a linear equation, we may consider the convergence of each Fourier component of the error separately.

We then get:

$$\varepsilon_{k,l}^{n+1} = \varepsilon_{k,l}^n (1 - \kappa b_{k,l})$$

where

$$b_{k,l} = 2f_0 \frac{m^2}{d^2} (2 - \cos kd - \cos ld) \tag{2.14}$$

and k and l are the wave numbers in x and y direction respectively.

After n iterations the relative error on a given wave component is

$$\frac{\varepsilon_{k,l}^n}{\varepsilon_{k,l}^0} = (1 - \kappa b_{k,l})^n \quad \text{Method 1} \tag{2.15}$$

If we instead use the method of changing the sign of κ, the relative error will be

$$\frac{\varepsilon_{k,l}^n}{\varepsilon_{k,l}^0} = (1 - (\kappa b_{k,l})^2)^{n/2} \quad \text{Method 2} \tag{2.16}$$

The conditions for convergence are, from the above expressions:

$0 < \kappa < 2/b_{max}$ Method 1

$|\kappa| < \sqrt{2}/b_{max}$ Method 2

It is the shortest wave components which determine the choice of κ. The wave length $2d$ makes in this simplified case $b_{max} = 8f_0(m^2/d^2)$, and if we choose $\kappa = 1.4/b_{max}$, both methods will converge. The following table illustrates the rate of convergence of two different waves and the difference between the two methods.

The main thing is that the long waves converge slowly and that method 2 is unusable for these waves. For the shorter waves both methods converge faster and method 2 should give a reasonable accuracy after about 100 iterations.

To be more sure about the convergence of method 2 used in the non-linear balance equation, we have solved an elliptic case using the two methods. As an initial guess the solution, ψ_l, of the linear balance equation is used. Fig. 1 shows the residual at a point as a function of the number of iterations. It is well demonstrated that about 200 iterations are necessary with method 2 to reduce the residual to 3% of its initial value while method 1 gives the same accuracy after about 20 iterations. Fig. 2 shows a cross section of ψ_l along the x-axis and the difference from this guess field which is achieved after 400 iterations using the two methods. If we assume the result of method 1 to be the correct solution, we see that the error of method 2 has a large scale since the difference between the results has the same sign in the whole cross section. We also see that the convergence is slower in the high pressure area than in the low pressure area.

Table 1

Number of iterations necessary	Long wave ($20d$) 10% error	Short wave ($6d$) 1% error
Method 1	$n = 65$	$n = 11$
Method 2	$n = 3757$	$n = 70$

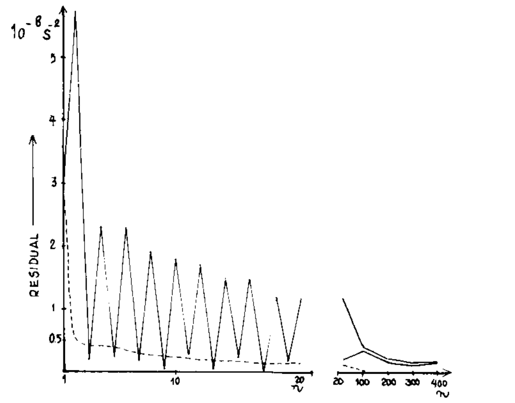

Fig. 1. The residual of the balance equation (corrected height field) at a point as a function of the number of iterations (n), using two methods: ---- method 1 (same sign of κ), ——— method 2 (changing sign of κ). The limits of the residual are drawn to the right in the figure.

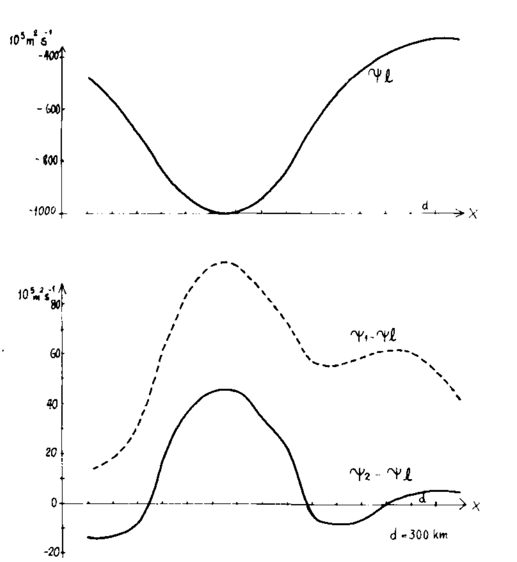

Fig. 2. Upper part. A cross section of the solution of the linear balance equation ψ_l. *Lower part.* ψ_1 and ψ_2 are the solutions of the balance equation using method 1 and method 2 respectively after 400 iterations. The differences from the guess value are plotted. The height analysis as in Fig. 1.

The error is reduced by about 50% on a cyclone scale, and so the error is much less on a scale 4–8d which is the scale of the hyperbolic areas and also the scale of the difference between the standard solution of the balance equation and the new solution (see Fig. 6). We conclude therefore that although the iterative method is slow, it is sufficient for the purpose of our experiment, provided that the initial guess is almost correct on scales greater than 10d.

With the above results in mind we therefore solve the elliptic balance equation using a standard method (Miyakoda, 1960) and find the stream function ψ_e which is in balance with the corrected height z_e. This stream function is the best possible guess, and it is used in the iteration eq. (2.4). We use a relaxation coefficient $\kappa = 10^{14}\ \mathrm{m^2\,s}$ and stop after 400 iterations. The resulting stream function ψ_h should then be accurate enough on the scales in interest. The corresponding height z_h, which also differs from the analysis, is found by reversing the balance equation.

3. Results

We tried the method on our routine analysis of the height in 300 mb, 19 November 1973, 00z. The method of analysis is an iterative method described by Haug (1959) and modified by Bjørheim (unpublished). Background fields are 12-h prognoses with a four-level balanced model (Grønås & Lystad, 1975). The area of operation is shown in Fig. 3. We focus our interest on the limited Atlantic area shown in the figure.

The analysis in this area is shown in Fig. 4. The quality is probably poor since there are very few observations. The geostrophic vorticity is computed and mapped in relation to the Coriolis

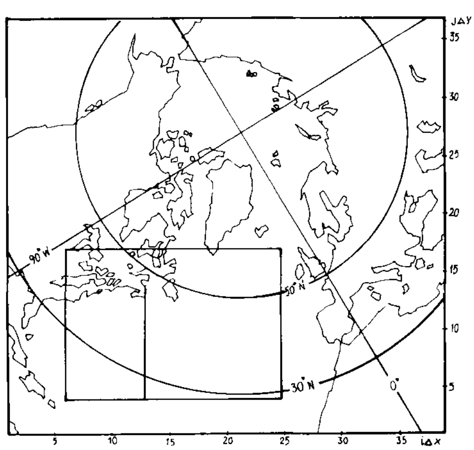

Fig. 3. The area of integration. The geographical area of Figs. 4 to 7 and Fig. 9 is the inner rectangle and the cross section used in Fig. 8 is indicated.

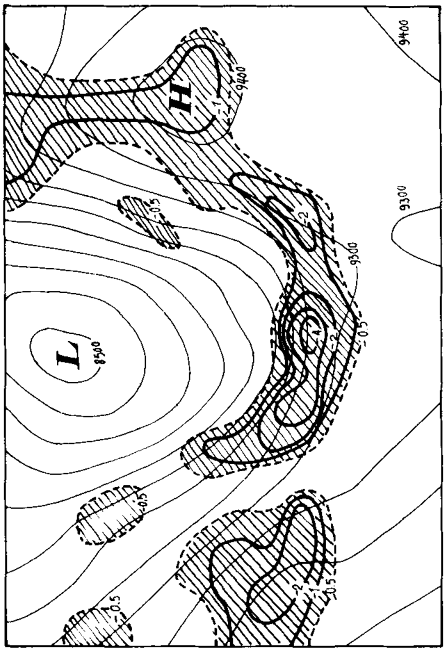

Fig. 4. The height analysis z of 300 mb, 19 November 1973, 00 GMT. Equidistance 100 m. In the dashed areas the geostrophic vorticity is less than $-\frac{1}{2}f$. The isolines inside these areas show the geostrophic vorticity divided by f.

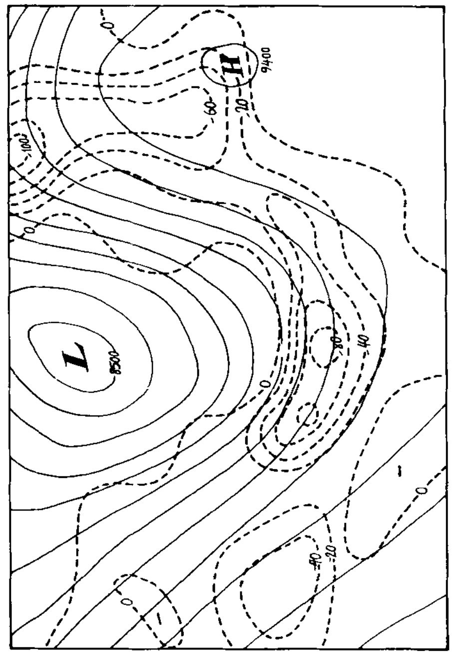

Fig. 5. The corrected height analysis z_e and the corrections $z_e - z$.

parameter. We find the hyperbolic areas where this quotient is less than minus one-half. We notice two main regions. One is in connection with the large anticyclonic shear on the southern flank of the jet. The other is in connection with the main high pressure area where the anticyclonic curvature is large. We see that in both areas the anticyclonic geostrophic vorticity is also much larger than the Coriolis parameter, which indicates dynamical instability or an analysis error.

Fig. 5 shows the analysis after it has been corrected so that the elliptic condition is satisfied, z_e. Maximum corrections are found in the regions mentioned above, but the affected areas are much wider. The method of correction used is a simplification of the method given by Shuman (1957). This is an iterative method where the height in a hyperbolic point is lowered so that the condition is satisfied. There is no compensation for this correction to the adjacent points. We have also tried another correction method where the geostrophic vorticity is increased in the hyperbolic points and decreased by the same amount in the neighbour

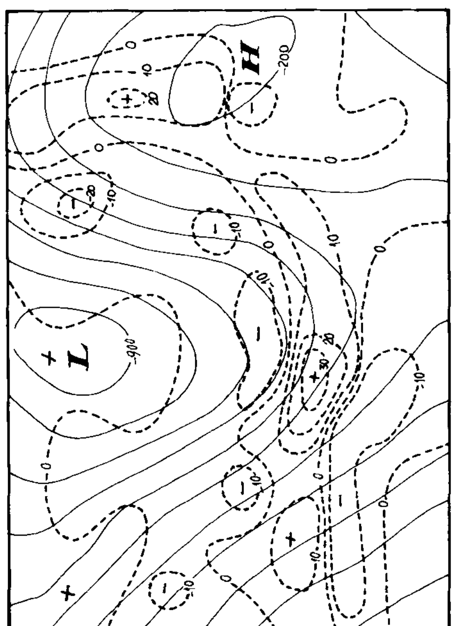

Fig. 6. The stream function ψ_h found by the presented method (equidistance $100 \cdot 10^5$ m^2 s^{-1}) and the difference from the stream function found by the conventional method, $\psi_h - \psi_e$.

points with highest cyclonic vorticity. This method conserves the mean height of the field, but the amplitude of the corrections are of the same magnitude as in the simplified method.

The stream function, ψ_h, found by our iterative method, and which gives the minimum unbalance to the original height field, is shown in Fig. 6. The difference between this field and the solution ψ_e of the elliptic balance equation is also mapped on the figure $(\psi_h - \psi_e)$.

The height field z_h in balance with ψ_h is shown in Fig. 7 together with the quotient between the geostrophic vorticity and the Coriolis parameter as in Fig. 5. As we see, the new height field has hyperbolic areas corresponding to those on the original height field. The amplitude of the computed geostrophic vorticity is smaller, but even on this field we find a small area in the jet where the absolute vorticity is negative. From this we draw a main conclusion that the presented method gives balanced fields with less constraints on the height field.

By comparing the three height fields z, z_e and z_h we also clearly see that the new field z_h is a better

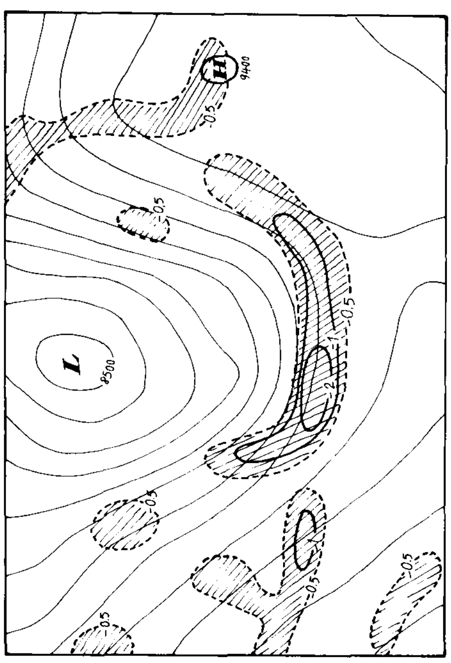

Fig. 7. The height z_h in balance with the stream function ψ_h on Fig. 6. In the dashed areas the geostrophic vorticity is less than $-\frac{1}{2}f$. Isolines inside these areas show geostrophic vorticity.

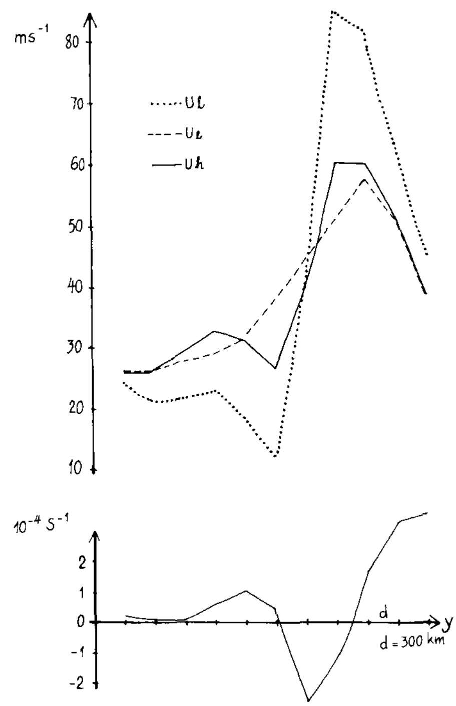

Fig. 8. Upper part: Wind profiles through the jet, U, computed from three different stream functions: ψ_l, ψ_e and ψ_h. *Lower part:* The expression $g/f\nabla^2 z + f/2$. The cross section used is shown in Fig. 3.

approximation than z_e to the original field z around the hyperbolic areas. The new field contains more small-scale structures that correspond to the original analysis. Perhaps this is most clearly seen in connection with the jet area where the anticyclonic shear is increased considerably. This is demonstrated in Fig. 8, which shows a cross section of the wind through the jet computed using central differences of the stream function ψ_h, ψ_e and the linear solution ψ_l. We see that the wind profile from ψ_h tends to follow the linear solution. It is possible that our system has not converged enough, and that the maximum speed in the jet should be even closer to the linear solution. On the other hand the jet stream is curved cyclonically and the wind speed should be below the wind speed from ψ_l.

The difference $z_h - z$ is mapped on Fig. 9. In the jet area there is a correction of 40 m which is approximately 40% of the correction made to find z_e, Fig. 5. There are, however, also corrections on

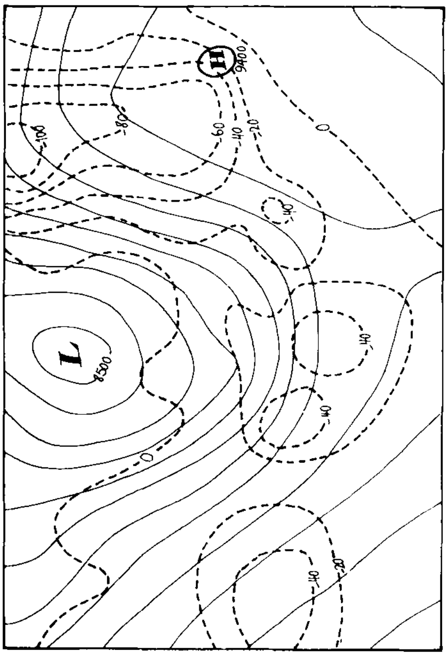

Fig. 9. The height z_h and the difference from the analysis, $z_h - z$. Compare with Fig. 5.

the cyclonic flank of the jet which are not present in Fig. 5, but they contribute to maintain higher shear. In the ridge area, the maximum corrections which result from our method are only slightly less than those resulting from the standard method, but the shape of the corrections differ a little from those mapped in Fig. 5, and is such that the anticyclonic vorticity is higher. This change seems to be present both in the curvature of the ridge and in the anticyclonic shear.

4. Concluding remarks

A method is given to find a stream function which is in minimum unbalance with the height field. From this stream function a corrected height field in full balance is found by solving the balance equation for the height field.

We have shown that the field found in this way is a better approximation to the original height field than an elliptic height field found after corrections by standard methods. The height field found using the method described indeed does not satisfy the condition of ellipticity with respect to the balance equation.

We tried our method on an analysis where the hyperbolic areas were primarily a result of inadequate data coverage and analysis techniques. It is an open question how often such areas occur in the atmosphere on the synoptic scales. In regions where little data is available, it is probably wise to perform height analysis with the elliptic criterion as a constraint. The criterion should, however, probably be avoided in connection with tropospheric jets where large anticyclonic shear seems to be common. In the same way corrections in the jet areas should be avoided when standard methods are applied on analyses. If hyperbolic areas are allowed, our method could be applied. The method, however, converges slowly and work should be done to speed up the rate of convergence, for instance by applying block relaxation.

5. Acknowledgements

We would like to thank Dr Ragnar Fjørtoft for having initiated this work and for many interesting ideas and discussions. We would also like to thank Anne Lise Knebel for typing this report.

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6. Appendix

The method of variation of parameters

The problem of minimizing the residual of eq. (2.1) can be stated in this way:

(1) $L(\psi)$ is defined in the interior of the area by:

$$L(\psi) \equiv \nabla . f\nabla\psi + 2J(\psi_x, \psi_y) - \nabla^2\varphi \tag{A1}$$

$L(\psi) \equiv 0$ on the boundary.

The indexes x, y mean partial derivatives.

(2) $\nabla^2\varphi$ given in the interior of the area.

(3) ψ given on the boundary.

(4) $I(\psi) \equiv \int\int L^2(\psi)\, dx\, dy$.

Find a stream function ψ which gives a minimum I.

To solve this problem we introduce a variational parameter ε and a function $\mu(x, y, \varepsilon)$ defined by:

$$\mu(x, y, \varepsilon = 0) = \psi(x, y) \tag{A2}$$

$\mu(x, y, \varepsilon) - \psi(x, y) = 0$ on the boundary

Here $\psi(x, y)$ is assumed to be a solution to the problem above.

A Taylor expansion of μ around $\varepsilon = 0$ gives:

$$\mu(x, y, \varepsilon) = \psi(x, y) + \left(\frac{\partial\mu}{\partial\varepsilon}\right)_{\varepsilon=0} \cdot \varepsilon + \frac{1}{2}\left(\frac{\partial^2\mu}{\partial\varepsilon^2}\right)_{\varepsilon=0} \cdot \varepsilon^2 + \cdots$$

We see that the boundary condition for μ implies that

$$\left(\frac{\partial\mu}{\partial\varepsilon}\right)_{\varepsilon=0} = \left(\frac{\partial^2\mu}{\partial\varepsilon^2}\right)_{\varepsilon=0} = \cdots = 0 \tag{A3}$$

on the boundary

We now define the variational integral

$$I(\varepsilon) = \int\int L^2(\mu)\, dx\, dy \tag{A4}$$

and since $I(0)$ is assumed to be a minimum value of the integral, the first derivative has to be zero:

$$\left(\frac{dI}{d\varepsilon}\right)_{\varepsilon=0} = 0 \tag{A5}$$

This derivative has to be calculated utilizing the definition (A1) of $L(\mu)$:

$$\frac{dI}{d\varepsilon} = 2\int\int L\frac{\partial L}{\partial\varepsilon}\, dx\, dy \tag{A6}$$

$$\begin{aligned}
\frac{\partial L}{\partial\varepsilon} &= \nabla\cdot f\nabla\frac{\partial\mu}{\partial\varepsilon} + 2\frac{\partial}{\partial\varepsilon}J(\mu_x, \mu_y) \\
&= \nabla\cdot f\nabla\frac{\partial\mu}{\partial\varepsilon} + 2\left[J\left(\frac{\partial\mu_x}{\partial\varepsilon}, \mu_y\right) + J\left(\mu_x, \frac{\partial\mu_y}{\partial\varepsilon}\right)\right] \\
&= \nabla\cdot f\nabla\frac{\partial\mu}{\partial\varepsilon} + 2\left[\mathbb{v}_x\frac{\partial\mu_y}{\partial\varepsilon} - \mathbb{v}_y\frac{\partial\mu_x}{\partial\varepsilon}\right] \\
&= \nabla\cdot f\nabla\frac{\partial\mu}{\partial\varepsilon} + 2\nabla\cdot\left[(-\mathbb{v}_y\mathbb{i} + \mathbb{v}_x\mathbb{j})\cdot\nabla\frac{\partial\mu}{\partial\varepsilon}\right] \\
&= \nabla\cdot\left[((f + 2\mu_{yy})\,\mathbb{i}\mathbb{i} - 2\mu_{xy}\,\mathbb{i}\mathbb{j} - 2\mu_{xy}\,\mathbb{j}\mathbb{i} + (f + 2\mu_{xx})\,\mathbb{j}\mathbb{j})\cdot\nabla\frac{\partial\mu}{\partial\varepsilon}\right] \\
&\equiv \nabla\cdot\left[\mathscr{P}\cdot\nabla\frac{\partial\mu}{\partial\varepsilon}\right]
\end{aligned} \tag{A7}$$

The velocity $\mathbb{v}$ is deduced from the stream function μ

$$\mathbb{v} = \mathbb{k} \times \nabla\mu$$

and $\mathscr{S}$ is the symmetric tensor

$$\mathscr{S} = \begin{pmatrix} f + 2\mu_{yy}, & -2\mu_{xy} \\ -2\mu_{xy}, & f + 2\mu_{xx} \end{pmatrix}$$

Expression (A7) is used in the calculations of Section 2.2 (eq. 2.8). We now return to the expression (A6) and insert for $\partial L/\partial\varepsilon$:

$$\frac{dI}{d\varepsilon} = 2\int\int L\nabla\cdot\left(\mathscr{S}\cdot\nabla\frac{\partial\mu}{\partial\varepsilon}\right) dx\,dy$$

$$=2\left[\int\int\nabla\cdot\left(L\mathscr{S}\cdot\nabla\frac{\partial\mu}{\partial\varepsilon}\right) dx\,dy - \int\int\nabla L\cdot\mathscr{S}\cdot\nabla\frac{\partial\mu}{\partial\varepsilon}\,dx\,dy\right]$$

$$=2\left[\oint_R L\left(\mathscr{S}\cdot\nabla\frac{\partial\mu}{\partial\varepsilon}\right)_n dl - \int\int\nabla\frac{\partial\mu}{\partial\varepsilon}\cdot\mathscr{S}\cdot\nabla L\,dx\,dy\right]$$

$$=2\left[-\int\int\nabla\cdot\left(\frac{\partial\mu}{\partial\varepsilon}\mathscr{S}\cdot\nabla L\right) dx\,dy + \int\int\frac{\partial\mu}{\partial\varepsilon}\nabla\cdot(\mathscr{S}\cdot\nabla L)\,dx\,dy\right]$$

$$=2\left[-\oint_R\frac{\partial\mu}{\partial\varepsilon}(\mathscr{S}\cdot\nabla L)_n\,dl + \int\int\frac{\partial\mu}{\partial\varepsilon}\nabla\cdot(\mathscr{S}\cdot\nabla L)\,dx\,dy\right]$$

$$=2\int\int\frac{\partial\mu}{\partial\varepsilon}\nabla\cdot(\mathscr{S}\cdot\nabla L)\,dx\,dy \qquad \text{(A8)}$$

The transformations of an area integral to a boundary integral are done using the Gauss theorem, and the boundary integral vanishes because of the boundary conditions: $L = 0$ and $\partial\mu/\partial\varepsilon = 0$. The index n means the component of the vector normal to the boundary and l is a line element along the boundary.

The condition for the integral $I(0)$ to be a minimum is then:

$$\int\int\left(\frac{\partial\mu}{\partial\varepsilon}\right)_{\varepsilon=0}\nabla\cdot(\mathscr{S}(\psi)\cdot\nabla L(\psi))\,dx\,dy = 0$$

for all values of $(\partial\mu/\partial\varepsilon)_{\varepsilon=0}$.

This is only possible when $\psi(x,y)$ satisfies the condition:

$$\nabla\cdot(\mathscr{S}(\psi)\cdot\nabla L(\psi)) = 0 \qquad \text{(A9)}$$

This is a fourth-order partial differential equation in ψ.

РЕШЕНИЕ УРАВНЕНИЯ БАЛАНСА С МИНИМУМОМ КОРРЕКЦИИ ПОЛЯ МАССЫ

Если уравнение баланса решается относительно функции тока обычными методами, то поле массы должно удовлетворять условиям эллиптичности. В этой статье мы описываем метод нахождения функции тока, когда это условие не удовлетворяется. Метод является итерационным методом, развитым Фьортофтом (1962). Показано, что если метод сходится, то он дает решение, в котором остаток уравнения баланса минимален.

Метод испытывался на анализе высот поверхности 300 мб, которая содержит довольно большие области гиперболичности. Сходимость метода очень медленная, поэтому в качестве первого приближения использовалось решение, полученное обычным методом. Из решения мы находим поле высот, находящееся в балансе с функцией тока, путем решения уравнения баланса для поля высот. Показывается, что это поле является лучшей аппроксимацией первоначального поля высот, чем эллиптическое поле, найденное после коррекции стандартным методом. Новое поле высот не удовлетворяет условию эллиптичности по отношению к уравнению баланса.