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## SHORT CONTRIBUTION

# The time dependence of reference pressure in limited region available potential energy budget equations

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## 1. Introduction

In previous papers Smith (1969) and Vincent & Chang (1973) have formulated available potential energy (APE) budget equations in pressure coordinates applicable to fixed and moving systems. A subtle aspect of these formulations is that the Lagrangian derivative of the global reference pressure is zero, $dp_r/dt = 0$.

Recently, Egger (1976), in comments on a paper by Boer (1975), has pointed out that in general $dp_r/dt$ and thus, the term in the APE equation in which $dp_r/dt$ is contained are not zero. In his response, Boer (1976) confirms Egger's conclusion and further shows that the $dp_r/dt$ term is zero only when averaged over the entire globe.

The exchange of correspondence between Egger and Boer deals with the time dependence of the global reference pressure. The intent of the present paper is to extend these discussions by examining the time variability of the "local" reference pressure, as defined by Johnson (1970), and formulating mathematical expressions for its numerical evaluation in pressure coordinates. The authors' preliminary calculations indicate that the term containing temporal changes in the local reference pressure can be as large as other available potential energy budget terms. Thus, inclusion of such determinations is essential for a complete diagnostic energy budget study.

## 2. Role in Eulerian budget equations

Smith (1969) has defined the available potential energy of a limited region ($A_j$) in terms of its contribution to the global available potential energy. For a fixed region of area $\sigma_j$ extending from the earth's surface, $s$, to the top of the atmosphere, $u$, the appropriate expression is

$$A_j = \frac{c_p}{\sigma_j g} \int_{\sigma_j} \int_{p_u}^{p_s} \frac{p^\kappa - p_r^\kappa}{p^\kappa} T \, dp \, d\sigma, \tag{1}$$

where

$p, T$ = pressure, temperature,
$g = 9.8 \text{ m s}^{-2}$,
$c_p$ = specific heat capacity of air at constant pressure
$R$ = gas constant for air, $\Big\}\ \kappa = R/c_p$
$p_r$ = global reference pressure.

Note that Smith's original expression has been modified by dividing by $\sigma_j$.

Smith & Horn (1969) extend the concept of contributed available potential energy by partitioning (1) according to

$$A_j = \frac{c_p}{\sigma_j g} \int_{\sigma_j} \int_{p_u}^{p_s} \frac{p^\kappa - p_{rj}^\kappa}{p^\kappa} T \, dp \, d\sigma + \frac{c_p}{\sigma_j g} \int_{\sigma_j} \int_{p_u}^{p_s} \frac{p_{rj}^\kappa - p_r^\kappa}{p^\kappa} T \, dp \, d\sigma. \tag{2}$$

The value of the parameter $p_{rj}$, referred to as the "local" reference pressure, at a particular point is the limited area average pressure on the potential temperature surface $\theta$ passing through that point and can be mathematically defined as

$$p_{rj}(\theta, t) = \frac{1}{\sigma_j} \int_{\sigma_j} \int_{\theta_u}^{\theta} \frac{\partial p}{\partial \theta} \, d\theta \, d\sigma. \tag{3}$$

As described by Smith & Horn, the first term on the right side of (2) is the available potential energy that the limited region would possess if in passing to the local reference state the region were not allowed to interchange mass with its environment, while the second is the contribution to the global available potential energy due to the departure of the local reference state from that of the total atmosphere. Johnson (1970) provides a rigorous development of the partitioning given by (2) and refers to the first term as the "baroclinic" contribution to the global available potential energy and the second as the "barotropic" contribution. Further elaboration of the budget equations in pressure coordinates is given in Vincent & Chang (1973).

In order to focus attention on $p_{rj}$, only the baroclinic contribution (denoted by $A_j^*$) is considered here. Noting the developments by Johnson and Vincent & Chang, the formulation given by Smith (1969) can be applied to $A_j^*$ to yield for the budget equation

$$\begin{aligned}\frac{\partial A_j^*}{\partial t} &= \frac{1}{\sigma_j g}\int_{\sigma_j}\int_{p_u}^{p_s}\frac{p^\kappa - p_{rj}^\kappa}{p^\kappa}\,Q\,dp\,d\sigma \\ &+ \frac{1}{\sigma_j g}\int_{\sigma_j}\int_{p_u}^{p_s}\omega\alpha\,dp\,d\sigma \\ &- \frac{c_p}{\sigma_j g}\int_{\sigma_j}\int_{p_u}^{p_s}\nabla_p\cdot\frac{p^\kappa - p_{rj}^\kappa}{p^\kappa}\,T\mathbf{V}\,dp\,d\sigma \\ &- \frac{c_p}{\sigma_j g}\int_{\sigma_j}\int_{p_u}^{p_s}\frac{T}{p^\kappa}\frac{dp_{rj}^\kappa}{dt}\,dp\,d\sigma - \frac{c_p}{\sigma_j g} \\ &\times \int_{\sigma_j}\frac{p_s^\kappa - p_{rjs}^\kappa}{p_s^\kappa}\,T_s\frac{\partial p_s}{\partial t}\,d\sigma, \end{aligned} \tag{4}$$

where

$\alpha$ = specific volume,

$Q$ = diabatic heating rate per unit mass,

$\omega$ = vertical motion in pressure coordinates ($dp/dt$),

$\mathbf{V}$ = horizontal wind vector.

In writing (4), Smith's original equation has been simplified by imposing the boundary conditions

$$\omega_s = \omega_u = 0;\ \partial p_u/\partial t = 0.$$

Of interest in this paper is the fourth term on the right side of (4). The Lagrangian derivative of $p_{rj}$ can be expanded in the form

$$\frac{dp_{rj}}{dt} = \frac{\partial p_{rj}}{\partial t} + \dot{\theta}\frac{\partial p_{rj}}{\partial \theta}, \tag{5}$$

where $\dot{\theta} = d\theta/dt$. Note that since $p_{rj}$ is constant on an isentropic surface, $\nabla_\theta p_{rj} = 0$. From (3), the first term on the right side of (5) becomes

$$\begin{aligned}\frac{\partial p_{rj}}{\partial t} &= \frac{1}{\sigma_j}\int_{\sigma_j}\int_{\theta_u}^{\theta}\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \theta}\right)d\theta\,d\sigma \\ &= \overline{\int_{\theta_u}^{\theta}\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \theta}\right)d\theta}, \end{aligned} \tag{6}$$

where $(\bar{\ })$ = area average over $\sigma_j$. Applying the continuity equation in isentropic coordinates to (6) yields

$$\begin{aligned}\frac{\partial p_{rj}}{\partial t} &= \overline{\int_{\theta_u}^{\theta}\left[\nabla_\theta\cdot\frac{\partial p}{\partial \theta}\mathbf{V} - \frac{\partial}{\partial \theta}\left(\dot{\theta}\frac{\partial p}{\partial \theta}\right)\right]d\theta} \\ &= \overline{\int_{\theta_u}^{\theta} -\nabla_\theta\cdot\frac{\partial}{\partial \theta}\mathbf{V}\,d\theta} - \overline{\dot{\theta}\frac{\partial p}{\partial \theta}}. \end{aligned} \tag{7}$$

The second term on the right side of (5) becomes

$$\begin{aligned}\dot{\theta}\frac{\partial p_{rj}}{\partial \theta} &= \frac{\dot{\theta}}{\sigma_j}\int_{\sigma_j}\int_{\theta_u}^{\theta}\frac{\partial}{\partial \theta}\left(\frac{\partial p}{\partial \theta}\right)d\theta\,d\sigma \\ &= \dot{\theta}\,\overline{\frac{\partial p}{\partial \theta}}. \end{aligned} \tag{8}$$

In both (7) and (8) it is assumed that $(\partial p/\partial\theta)\theta_u = 0$. Combining (7) and (8) yields

$$\begin{aligned}\frac{dp_{rj}}{dt} &= \overline{\int_{\theta_u}^{\theta} -\nabla_\theta\cdot\frac{\partial p}{\partial \theta}\mathbf{V}\,d\theta} - \overline{\dot{\theta}\frac{\partial p}{\partial \theta}} \\ &+ \dot{\theta}\,\overline{\frac{\partial p}{\partial \theta}}. \end{aligned} \tag{9}$$

Returning to (4), the $dp_{rj}/dt$ term (denoted for simplicity as $DP$) can be rewritten as

$$DP = -\frac{c_p\kappa}{\sigma_j g}\int_{\sigma_j}\int_{p_u}^{p_s}\frac{T}{p_{rj}}\left(\frac{p_{rj}}{p}\right)^{\kappa}\frac{dp_{rj}}{dt}\,dp\,d\sigma$$

$$= -\frac{R}{g}\int_{\theta_u}^{\theta_s}\frac{\theta}{p_{rj}}\left(\frac{p_{rj}}{p_{00}}\right)^{\kappa}\overline{\frac{dp_{rj}}{dt}\frac{\partial p}{\partial\theta}}\,d\theta, \tag{10}$$

where $p_{00} = 1000$ mb. It is not necessary to include $\theta$ and $p_{rj}$ under the area averaging operator since they are constant on a potential temperature surface.

Inserting (9) into (10) yields

$$DP = \frac{-R}{g}\int_{\theta_u}^{\theta_s}\frac{\theta}{p_{rj}}\left(\frac{p_{rj}}{p_{00}}\right)^{\kappa}\overline{\left\{\frac{\partial p}{\partial\theta}\int_{\theta_u}^{\theta}-\nabla_\theta\cdot\frac{\partial p}{\partial\theta}\mathbf{V}\,d\theta\right.}$$

$$\overline{\left.-\frac{\partial p}{\partial\theta}\left(\dot{\theta}\frac{\partial p}{\partial\theta}\right)+\frac{\partial p}{\partial\theta}\dot{\theta}\left(\overline{\frac{\partial p}{\partial\theta}}\right)\right\}}\,d\theta. \tag{11}$$

Since the area over which $A_j^*$ is evaluated is the same as that over which $p_{rj}$ is determined, the two bar operators in (11) represent averages over the same area and the last two terms cancel. Thus,

$$DP = -\frac{R}{\sigma_j g}\int_{\sigma_j}\int_{\theta_u}^{\theta_s}\frac{\theta}{p_{rj}}\left(\frac{p_{rj}}{p_{00}}\right)^{\kappa}\overline{\left\{\int_{\theta_u}^{\theta}-\nabla_\theta\cdot\frac{\partial p}{\partial\theta}\mathbf{V}\,d\theta\right\}}\frac{\partial p}{\partial\theta}\,d\theta\,d\sigma. \tag{12}$$

Clearly, (9) and (12) show that in general $dp_{rj}/dt$ and $DP \neq 0$. In addition, (12) confirms Boer's (1976) conclusion that $DP = 0$ in global APE studies.

Final reduction of (12) is accomplished by rewriting the isentropic gradient operator in terms of equivalent isobaric quantities using the relationship

$$\nabla_\theta\cdot\frac{\partial p}{\partial\theta}\mathbf{V} = \nabla_p\cdot\frac{\partial p}{\partial\theta}\mathbf{V}+\frac{\partial}{\partial p}\left(\frac{\partial p}{\partial\theta}\mathbf{V}\right)\cdot\nabla_\theta p$$

$$= \nabla_p\cdot\frac{\partial p}{\partial\theta}\mathbf{V}-\frac{\partial}{\partial p}\left(\frac{\partial p}{\partial\theta}\mathbf{V}\right)\cdot\nabla_p\theta\left(\frac{\partial\theta}{\partial p}\right)^{-1}. \tag{13}$$

Inserting (13) in (12) and transforming both potential temperature integrations to pressure integrations yields the final form for $DP$

$$DP = -\frac{R}{\sigma_j g}\int_{\sigma_j}\int_{p_u}^{p_s}\frac{T}{p_{rj}}\left(\frac{p_{rj}}{p}\right)^{\kappa}\overline{\left\{\int_{p_u}^{p}\left[-\frac{\partial\theta}{\partial p}\nabla_p\cdot\frac{\partial p}{\partial\theta}\mathbf{V}\right.\right.}$$

$$\overline{\left.\left.+\frac{\partial}{\partial p}\left(\frac{\partial p}{\partial\theta}\mathbf{V}\right)\cdot\nabla_p\theta\right]dp\right\}}\,dp\,d\sigma. \tag{14}$$

In addition to yielding an equation applicable to calculations in isobaric coordinates, the development contained in (10)–(14) shows that $DP$ is caused by transport processes and that when treated as an area averaged quantity, $DP$ is not influenced directly by diabatic heating ($\dot{\theta}$).

Computationally, (14) contains one significant difficulty. Care is required to avoid circumstances in which $\theta$ becomes or approaches a hydrostatically neutral or hydrostatically unstable state. An alternate approach to evaluating $DP$ is to expand $dp_{rj}/dt$ in Eulerian form which, applying the continuity equation is isobaric coordinates, yields

$$\frac{dp_{rj}}{dt} = \frac{\partial p_{rj}}{\partial t} - \nabla_p\cdot p_{rj}\mathbf{V} + \frac{\partial(p_{rj}\omega)}{\partial p}, \tag{15}$$

an expression which can be substituted into the first form of (10). Although this approach does not possess the physical insight provided in the preceding discussion, it does avoid the problem noted for (14), and in addition has the strong appeal of simplicity. However, it includes the disadvantage of requiring the determination of a time-derivative quantity.

## 3. Role in quasi-Lagrangian budget equations

From Vincent & Chang (1973) the quasi-Lagrangian APE budget equation for a moving system can be written as

$$\frac{\delta A_j^*}{\delta t} = \frac{1}{\sigma_j g} \int_{\sigma_j} \int_{p_u}^{p_s} \frac{p^\kappa - p_{rj}^\kappa}{p^\kappa} Q \, dp \, d\sigma + \frac{1}{\sigma_j g} \times \int_{\sigma_j} \int_{p_u}^{p_s} \omega\alpha \, dp \, d\sigma + \frac{c_p}{\sigma_j g} \times \int_{\sigma_j} \int_{p_u}^{p_s} \nabla_p \cdot \frac{p^\kappa - p_{rj}^\kappa}{p^\kappa} T(\mathbf{W} - \mathbf{V}) \, dp \, d\sigma - \frac{c_p}{\sigma_j g} \int_{\sigma_j} \int_{p_u}^{p_s} \frac{T}{p^\kappa} \frac{dp_{rj}}{dt} dp \, d\sigma, \tag{16}$$

where

$$\mathbf{W} = \frac{\delta x}{\delta t} \mathbf{i} + \frac{\delta y}{\delta t} \mathbf{j},$$

is the motion of the lateral boundaries, and $\delta$ is the quasi-Lagrangian derivative operator. In this formulation it is assumed that the magnitude of $\sigma_j$ does not change and that the vertical boundaries of the moving system remain constant.

As in (4), the term of interest here is the fourth term on the right side of (16). Much of the development parallels that presented in the preceding section. However, care must be taken to account for the motion of the atmospheric system in the evaluation of $dp_{rj}/dt$.

The derivative of $p_{rj}$ in quasi-Lagrangian coordinates can be written as:

$$\frac{dp_{rj}}{dt} = \frac{\delta p_{rj}}{\delta t} + \left(\dot{\theta} - \frac{\delta\theta}{\delta t}\right) \frac{\partial p_{rj}}{\partial\theta}. \tag{17}$$

The first term on the right side of (17) can be written as

$$\frac{\delta p_{rj}}{\delta t} = \frac{1}{\sigma_j} \int_{\sigma_j} \int_{\theta_u}^{\theta} \frac{\partial}{\partial t}\left(\frac{\partial p}{\partial\theta}\right) d\theta \, d\sigma + \frac{1}{\sigma_j} \int_{\sigma_j} \int_{\theta_u}^{\theta} \nabla_\theta \cdot \frac{\partial p}{\partial\theta} \mathbf{W} \, d\theta \, d\sigma + \frac{1}{\sigma_j} \int_{\sigma_j} \frac{\delta\theta}{\delta t} \frac{\partial p}{\partial\theta} d\sigma. \tag{18}$$

Applying the continuity equation in isentropic coordinates as in (7), (18) becomes

$$\frac{\delta p_{rj}}{\delta t} = \overline{\int_{\theta_u}^{\theta} -\nabla_\theta \cdot \frac{\partial p}{\partial\theta}(\mathbf{V} - \mathbf{W}) \, d\theta} - \overline{\left(\dot{\theta} - \frac{\delta\theta}{\delta t}\right) \frac{\partial p}{\partial\theta}}. \tag{19}$$

Also, following (8),

$$\left(\dot{\theta} - \frac{\delta\theta}{\delta t}\right) \frac{\partial p_{rj}}{\partial\theta} = \left(\dot{\theta} - \frac{\delta\theta}{\delta t}\right) \overline{\frac{\partial p}{\partial\theta}}. \tag{20}$$

Thus, inserting (19) and (20) into (17) yields

$$\frac{dp_{rj}}{dt} = \overline{\int_{\theta_u}^{\theta} -\nabla_\theta \cdot \frac{\partial p}{\partial\theta}(\mathbf{V} - \mathbf{W}) \, d\theta} - \overline{\left(\dot{\theta} - \frac{\delta\theta}{\delta t}\right)} \overline{\frac{\partial p}{\partial\theta}} + \overline{\left(\dot{\theta} - \frac{\delta\theta}{\delta t}\right)} \overline{\frac{\partial p}{\partial\theta}}. \tag{21}$$

Finally, substituting (21) into (10) and following the same procedures described for (11) through (14), with $\mathbf{V}$ and $\dot{\theta}$ replaced by $(\mathbf{V} - \mathbf{W})$ and $(\dot{\theta} - \delta\theta/\delta t)$, yields the expression for the $dp_{rj}/dt$ term in quasi-Lagrangian coordinates ($DP'$)

$$DP' = -\frac{R}{\sigma_j g} \int_{\sigma_j} \int_{p_u}^{p_s} \frac{T}{p_{rj}} \left(\frac{p_{rj}}{p}\right)^\kappa \times \left\{ \overline{\int_{p_u}^{p} \left[ -\frac{\partial\theta}{\partial p} \nabla_p \cdot \frac{\partial p}{\partial\theta}(\mathbf{V} - \mathbf{W}) \, dp \right.} \right. \left. \overline{\left. + \frac{\partial}{\partial p} \left\{ \frac{\partial p}{\partial\theta} (\mathbf{V} - \mathbf{W}) \right\} \cdot \nabla_p \theta \right] dp} \right\} dp \, d\sigma. \tag{22}$$

As discussed for the Eulerian equations, an alternate approach would be to substitute

$$\frac{dp_{rj}}{dt} = \frac{\delta p_{rj}}{\delta t} + \nabla_p \cdot p_{rj}(\mathbf{V} - \mathbf{W}) + \frac{\partial p_{rj}}{\partial p}\left(\omega - \frac{\delta p}{\delta t}\right) \tag{23}$$

into (10).

## 4. Concluding remarks

The preceding formulations are restricted to an analysis of the time dependence of the $p_{rj}$ in the baroclinic contribution to the global available potential energy (Johnson, 1970). No discussion of the impact on the barotropic contribution has been presented. Restriction to the baroclinic contribution simplifies the development because the area over which the $p_{rj}$ is determined is the same as that over which $A_j^*$ is evaluated. The importance of identifying the same areas for both $p_{rj}$ and $A_j^*$ is seen in (11). If the two areas were not the same, as is the case for a portion of the barotropic contribution, the last two terms of (11) would not cancel and (14) and (22) would be more complex.

In addition, recall that the quasi-Lagrangian equations have been developed with the assumption that the magnitude of $\sigma_j$ does not change. Allowing variations in $\sigma_j$ would yield an additional term in both (16) and (21).

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