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Forced plume characteristics

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ABSTRACT

This paper describes an investigation of turbulent forced plumes generated by the steady release of mass, momentum and buoyancy from a finite source in a uniform environment. The entrainment coefficient assumption has been made; however a variable coefficient has been used so that the characteristics of the transition from jet to plume behaviour can be presented more clearly. A modified Boussinesq approximation has been made which corrects for the effect of density variations in the mass and momentum fluxes. The treatment reveals that the characteristics of forced plumes can best be discussed by reference to the Froude number. For all cases of forced plumes (except a pure jet) the Froude number approaches a particular constant value as the height increases.

1. Introduction

Flows referred to as plumes, forced plumes, buoyant jets and jets may all be considered part of a more general category of free turbulent flows. There are many important geophysical examples of these flows in such fields as meteorology, oceanography, air and water pollution, etc. The term "plume" has been used to describe flows arising when buoyancy is supplied continuously and "jet" has been used when momentum is supplied continuously. Flows which have the characteristics of both jets and plumes have been called "forced plumes" or "buoyant jets".

It has been observed by many experimenters that jets and plumes have a sharp boundary between the turbulent fluid and their surroundings. The flows increase their mass flux by the process of entrainment of surrounding fluid across this boundary. The interface between the turbulent flow and the surroundings is continuously moving, however the time-averaged profiles of velocity and density deficiency are quite smooth and can be accurately fitted by Gaussian profiles. The mean radius for both jets and plumes has been found to increase linearly with distance from the source.

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Batchelor (1954) pointed out that the linear growth of radius with height implies that the mean entrainment velocity is proportional to a characteristic upward velocity. This statement, now regarded as a basic assumption and called the entrainment coefficient assumption, was first used by Morton, Taylor and Turner (1956) in the analysis of plumes. They made two additional major assumptions:

(i) That the profiles of mean vertical velocity and mean density deficiency are similar at all heights

(ii) That the density difference between the ambient fluid and the buoyant fluid is negligible except in the buoyancy term in the conservation of momentum equation, where it provides the driving force and in the conservation of density deficiency equation. This assumption is commonly called the Boussinesq approximation and is discussed in considerable detail in Turner (1973).

Taking cylindrical polar coordinates (x, r) with the x axis vertical and the source at the origin and supposing that p, U and V are the time-averaged density, axial and radial velocities, Morton, Taylor and Turner then formulated, for a uniform density ambient fluid, the following conservation equations:

(i) volume

$$\frac{d}{dx}\int_0^{\infty} 2\pi r U\,dr = -\lim_{r\to\infty}(2\pi r V) = 2\pi\alpha u b \qquad (1)$$

(ii) vertical momentum

$$\frac{d}{dx}\int_0^\infty 2\pi r U^2\,dr = \int_0^\infty 2\pi r g\left(\frac{\rho_\infty - p}{\rho_\infty}\right)dr \quad (2)$$

(iii) density deficiency

$$\frac{d}{dx}\int_0^\infty 2\pi r U\left(\frac{\rho_\infty - p}{\rho_\infty}\right)dr = 0 \quad (3)$$

where ρ_∞ is the ambient density, u and b are characteristic axial velocity and radius scales and α is the entrainment coefficient.

Morton (1959) has extended the work of Morton, et al. (1956) to include plumes generated from sources of finite size which deliver fluxes of mass, momentum and buoyancy. These plumes have been called "forced plumes" by Morton. One limitation of Morton's approach is that the values of α for a jet ($\alpha_J = 0.051$) and a plume ($\alpha_p = 0.085$) differ significantly and so a forced plume cannot be characterized by a single value. Initially the forced plume will exhibit jet-like characteristics ($\alpha = 0.051$) and ultimately it must exhibit plume-like characteristics ($\alpha = 0.085$) and so a constant entrainment coefficient can give only approximate answers. A further limitation of Morton's work is the Boussinesq approximation, which excludes many industrial applications.

The problem of determining an appropriate entrainment coefficient has been examined by List & Imberger (1973). They used dimensional arguments and experimental results to show that in a uniform density environment the entrainment function and the buoyancy function are functions of the local Froude number and the spreading angle. Further, dimensional analysis revealed that the spreading angle is constant for both jets and plumes and the experimental results showed that the angles are almost identical. List and Imberger assumed that the spreading angles are equal and that the angle for a forced plume is constant and equal to the jet and plume value. In this paper List and Imberger's formulation for a variable entrainment coefficient has been used because it is more accurate than the assumption of constant α with the inherent problem of selecting an appropriate α value. Their results can be expressed as follows:

$$\frac{d}{dx}\int_0^\infty 2\pi r U\,dr = \sqrt{\frac{\pi}{8}}\,P^2\,C_p\left(1 + \frac{2R}{3R_p}\right) \quad (4)$$

$$\frac{d}{dx}\int_0^\infty 2\pi r U^2\,dr = \frac{8}{3}\pi g\,\frac{QF}{P^2}\,\frac{C_p}{R_p} \quad (5)$$

where

$$\left.\begin{aligned} Q &= \int_0^\infty 2rU\,dr \\ P^2 &= \int_0^\infty 4rU^2\,dr \\ F &= \int_0^\infty 2rU\left(\frac{\rho_\infty - p}{\rho_\infty}\right)dr \\ R &= 4\sqrt{2\pi}\,g\,\frac{Q^2F}{P^5} \end{aligned}\right\} \quad (6)$$

and C_p and R_p are constants. Before the terms in equation (6) can be evaluated, it is necessary to make some assumption about the variation of U and p with x and r. It is normal to assume that time averaged profiles are similar at all heights and can be represented by Gaussian distributions. There is considerable experimental support for these assumptions for both jets (see Albertson, Dai, Jensen & Rouse, 1950) and plumes (see Rouse, Yih & Humphrey, 1952). Thus assuming

$$\left.\begin{aligned} U &= u e^{-r^2/b^2} \\ \frac{\rho_\infty - p}{\rho_\infty} &= \varepsilon e^{-r^2/\lambda^2 b^2} \end{aligned}\right\} \quad (7)$$

it can be shown that

$$\alpha = 0.051 + \frac{0.650}{Fr^2} \quad (8)$$

where

$$Fr = \frac{u}{\sqrt{g\,\varepsilon\,b}}\,.$$

In the following work a pure plume is considered to be one arising from a virtual point source of buoyancy alone. For such a plume the Froude number has a particular constant value Fr_p. The value depends upon the definitions, scaling and value of α_p (0.085) used and in the present context $Fr_p = 4.37$. In the case of a pure jet $Fr_J = \infty$. If fluid is ejected from a finite source with the properties U, b and $\varepsilon \neq 0$ such that $Fr \neq Fr_p$ then Fr can be greater or less than Fr_p and the properties will change in a way such that $Fr \to Fr_p$. A plume produced by accelerating fluid initially at rest will have $Fr = 0$ at the source. For cases where $Fr < Fr_p$ then eq. (8) leads to the result that $\alpha > 0.085$; as there is no experimental or theoretical support for this it seems more logical to assume in this case that $\alpha = 0.085$. For cases in which $Fr \ll Fr_p$ some of the underlying assumptions of this method would not be valid e.g. the flow may not be fully turbulent. However as $Fr \to Fr_p$ comparatively quickly, the caution that the assumptions may not be valid need be borne in mind only very close to the source.

2. Formulation of equations

In the following a variable entrainment coefficient α, as defined by equations (4) and (8) will be used and a modified Boussinesq approximation will be made. If equations (6) and (7) are substituted into (3), (4) and (5) it can be shown that

$$\frac{d}{dx}(\pi Q) = C_p \sqrt{\frac{\pi}{2} P^2} + \frac{2}{3}\frac{C_p}{R_p} g \frac{\pi F(\pi Q)^2}{\left(\frac{\pi}{2} P^2\right)^2} \tag{9}$$

$$\frac{d}{dx}\left(\frac{\pi}{2} P^2\right) = g\pi B = \frac{4}{3}\frac{C_p}{R_p} g \frac{\pi F \pi Q}{\frac{\pi}{2} P^2} \tag{10}$$

$$\frac{d}{dx}(g\pi F) = 0 \tag{11}$$

where

$$B = 2\int_0^\infty r\left(\frac{\rho_\infty - \rho}{\rho_\infty}\right) dr.$$

These equations can be further simplified to

$$\frac{d}{dx}(Q) = 2\alpha_J P + g\frac{BQ}{P^2} \tag{12}$$

$$\frac{d}{dx}(P) = \frac{gB}{P} \tag{13}$$

$$\frac{d}{dx}(F) = 0 \tag{14}$$

where

$$\alpha_J = \frac{C_p}{2\sqrt{2\pi}}.$$

In the formulation of eqs. (9) and (10) the usual Boussinesq approximation was made. This approximation can be improved considerably if density variations are allowed for in the terms for mass and momentum flux. The terms Q and P^2 can be redefined as follows:

$$\left.\begin{aligned} \tilde{Q} &= \int_0^\infty 2rU\left(1 - \frac{\rho_\infty - \rho}{\rho_\infty}\right) dr \\ \tilde{P}^2 &= \int_0^\infty 4rU^2\left(1 - \frac{\rho_\infty - \rho}{\rho_\infty}\right) dr. \end{aligned}\right\} \tag{15}$$

Now substituting eq. (7) into eqs. (6) and (15) gives

$$B = \lambda^2 b^2 \varepsilon$$

$$F = \frac{\lambda^2 b^2 u \varepsilon}{1 + \lambda^2}$$

$$\tilde{Q} = b^2 u - F$$

$$\tilde{P}^2 = b^2 u^2\left(1 - \frac{2\varepsilon\lambda^2}{1 + 2\lambda^2}\right)$$

These terms can now be non-dimensionalized with respect to $(1 + \lambda^2)F$ and g to give

$$\hat{Q} = \frac{\tilde{Q}}{(1 + \lambda^2)F}$$

$$\hat{P} = \frac{\tilde{P}}{((1 + \lambda^2)F)^{3/5} g^{1/5}}$$

$$s = \frac{2\alpha_J x g^{1/5}}{((1 + \lambda^2)F)^{2/5}}$$

$$\hat{b} = \frac{b g^{1/5}}{((1 + \lambda^2)F)^{2/5}} = \frac{1}{\hat{P}}\left(\hat{Q} - \frac{1}{(1 + \lambda^2)(1 + 2\lambda^2)}\right)^{1/2} \times \left(\hat{Q} + \frac{1}{1 + \lambda^2)}\right)^{1/2}$$

$$\hat{u} = \frac{u}{((1+\lambda^2)F)^{1/5} g^{2/5}} = \frac{\hat{P}^2}{\hat{Q} - \dfrac{1}{(1+\lambda^2)(1+2\lambda^2)}}$$

Equations (12), (13) and (14) can now be reduced to 3 eqs. in 3 unknowns $\hat{Q}$, $\hat{P}$ and ε.

$$\frac{d\hat{Q}}{ds} = \hat{P} + \frac{\hat{Q}}{2\alpha_J \hat{P}^4}\left(\hat{Q} - \frac{1}{(1+\lambda^2)(1+2\lambda^2)}\right) \quad (16)$$

$$\frac{d\hat{P}}{ds} = \frac{1}{2\alpha_J \hat{P}^3}\left(\hat{Q} - \frac{1}{(1+\lambda^2)(1+2\lambda^2)}\right) \quad (17)$$

$$\hat{Q} = \frac{1}{\lambda^2 \varepsilon} - \frac{1}{1+\lambda^2} \quad (18)$$

Equations (16) and (17) have a first integral

$$\frac{\hat{Q}}{\hat{P}} = s + \left(\frac{\hat{Q}}{\hat{P}}\right)_0.$$

This integral provides a very convenient method for checking the accuracy of solutions.

3. Solution of equations

Let us define

$$\hat{F}r^2 = \left(\frac{\hat{P}^5}{\hat{Q}}\right) \Big/ \left(Q - \frac{1}{(1+\lambda^2)(1+2\lambda^2)}\right)$$

then eq. (16) can be written as

$$\frac{d\hat{Q}}{ds} = \hat{P}\left(1 + \frac{1}{2\alpha_J \hat{F}r^2}\right) \quad (19)$$

A constraint, which must now be remembered, is the limitation on the range of α. By comparing eq. (19) with eqs. (4) and (8) it can be seen that the maximum value of $1/2\alpha_J \hat{F}r^2$ is $\frac{2}{3}$ and $\hat{F}r_p^2 = 14.71$. When $\hat{F}r^2 = \hat{F}r_p^2$ the equations reduce to the pure plume problem and when $\hat{F}r^2 = \infty$ the equations reduce to the pure jet problem. Initial conditions can be conceived which give $\hat{F}r^2 < \hat{F}r_p^2$ and for these cases $1/2\alpha_J \hat{F}r^2$ was set equal to $\frac{2}{3}$. For all cases, except the pure jet solution $\hat{F}r^2 \to \hat{F}r_p^2$ as $s \to \infty$. This is shown clearly in Fig. 3.

Equations (16), (17) and (18) have been numerically integrated using a Runge-Kutta technique for a range of initial conditions $\hat{P}_0$ and ε_0. However before discussing the results it is worthwhile reviewing very briefly the results of pure plume and jet theories.

(i) Pure plume

$b = \frac{6}{5}\alpha_p x = 0.102\, x$

$u \propto x^{-1/3}$

(ii) Pure jet

$b = 2\alpha_J x = 0.102\, x$

$u \propto x^{-1}$

As discussed earlier, the rate of spreading for plumes and jets is the same.

In Fig. 1 the normalized height $s/\hat{P}_0$ is plotted against the normalized radius $\hat{b}/\hat{P}_0$ for $\varepsilon_0 = 1.0$ and $\hat{P}_0 = 0.1$, 1, 10, 100 and 1000. The theories for both pure plumes and jets leads us to expect $db/dx = 0.102$ and after making the appropriate substitutions this is equivalent to $d\hat{b}/ds = 1$, as observed. For the larger values of $\hat{P}_0$, $d\hat{b}/ds = 1$ throughout the range of s, however for $\hat{P}_0 = 0.1$ some initial necking was observed. The variation of velocity with height is shown in Fig. 2 for the same initial conditions. The pure theories suggest logarithmic gradients of -1 and $-\frac{1}{3}$ for jets and plumes respectively, and for the axes used these lines have slopes as shown. The transition from jet-like behaviour to plume-like behaviour is shown very clearly for $\hat{P}_0 \geqslant 10$. For small s the buoyancy

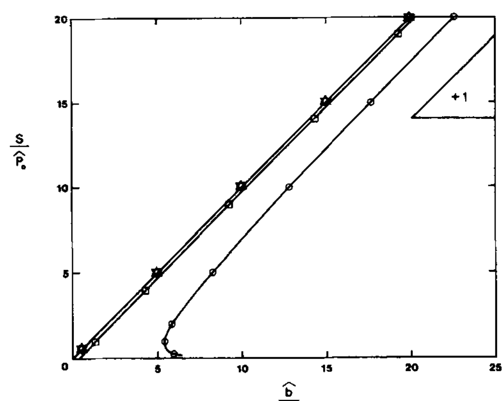

Fig. 1.

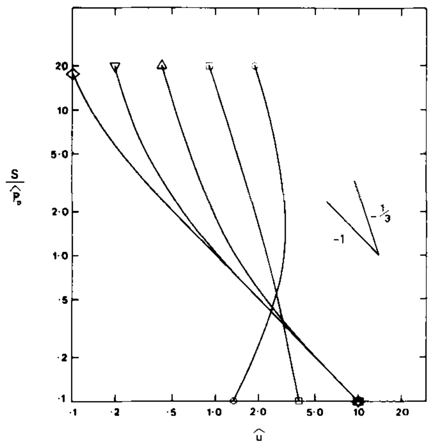

Fig. 2

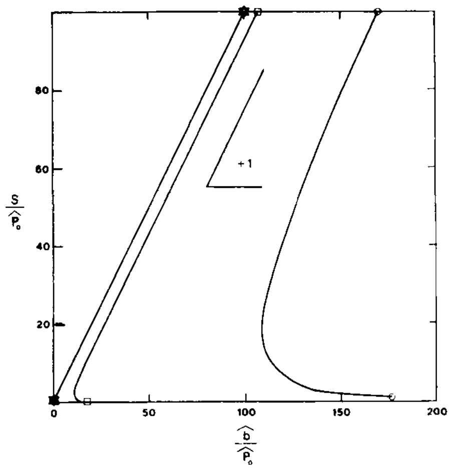

Fig. 4

force is relatively unimportant, in terms of modification of the momentum flux, but as s increases the buoyancy force continually increases the momentum flux so that at some height the original momentum flux becomes unimportant and the flow now behaves as a pure plume. The larger $\hat{P}_0$ the later the transition from jet-like to plume-like behaviour occurs. For $\hat{P}_0 = 1$ the behaviour is described fairly well by the pure plume theory. For $\hat{P}_0 = 0.1$ the buoyancy force dominates and the flow initially accelerates before settling down to the pure plume behaviour. It is this acceleration that leads to an initial reduction in radius $\hat{b}$ as observed in Fig. 1. The variation of $\hat{F}r^2$ with height is shown in Fig. 3. For $\hat{P}_0 \geqslant 10$ the $\hat{F}r^2$ is initially greater than $\hat{F}r_p^2$ and $\hat{F}r^2 \rightarrow \hat{F}r_p^2$ as $s \rightarrow \infty$. When $\hat{P}_0 = 1$ then $\hat{F}r^2 \simeq \hat{F}r_p^2$ thus it is to be expected that the velocity variation could be explained fairly accurately by the pure plume theory. For $\hat{P}_0 = 0.1$, $\hat{F}r^2 < \hat{F}r_p^2$ and thus for small s some of the underlying assumptions are probably invalid. However $\hat{F}r^2 \rightarrow \hat{F}r_p^2$ as s increases and so the assumptions would be valid over most of the range.

The variation of $\hat{b}$ with s for the same range of $\hat{P}_0$ but $\varepsilon_0 = 0.01$ is shown in Fig. 4. The features observed in Fig. 1 are evident except that this time it takes longer for the small $\hat{P}_0$ cases to approach the pure plume solution. Fig. 5 shows the variation of $\hat{u}$ with s and this figure reveals that for $\hat{P}_0 \leqslant 1$ the

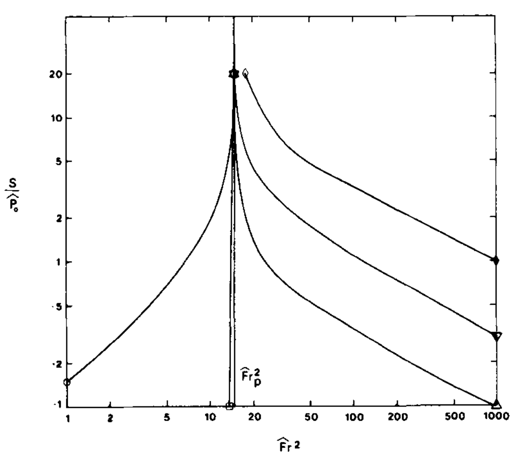

Fig. 3

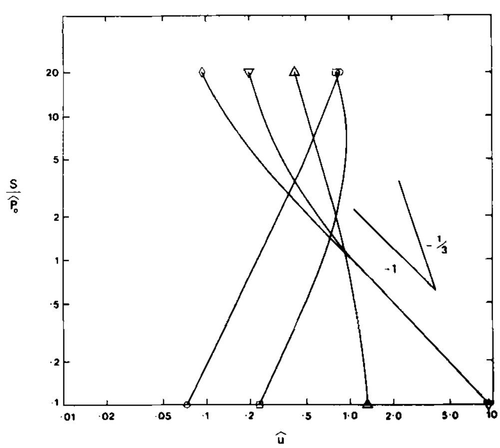

Fig. 5

height of maximum velocity increases as ε_0 decreases. Thus it is to be expected that the small $\hat{P}_0$ cases will take longer to approach the pure plume solution. This time $\hat{P}_0 = 10$ is the closest approximation to the simple plume solution and $\hat{P}_0 \geqslant 100$ exhibit the same characteristics as for $\varepsilon_0 = 1$ except that the transition to plume-like behaviour occurs at a greater height.

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ХАРАКТЕРИСТИКИ ВЫНУЖДЕННЫХ ЯЗЫКОВ

В статье описаны результаты исследования турбулентных вынужденных языков, генерируемых в однородной окружающей среде, постоянным конечным источником массы, импульса и плавучести. Сделано предположение о коэффициенте проникновения, однако, этот коэффициент считался переменным, так что можно было более ясно представить характеристики перехода возбуждаемого движения от струйного типа к языковому. Было использовано модифицированное приближение Буссинеска, учитывающее влияние изменений плотности в потоках массы и импульса. Анализ показывает, что характеристики вынужденных языков лучше всего обсуждать, используя число Фруда. Для всех случаев вынужденных языков (за исключением чистой струи) число Фруда приближается к некоторой постоянной величине по мере увеличения высоты.