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Optimization of atmospheric models

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ABSTRACT

An entirely new type of equation for forecasting atmospheric parameters is derived by applying variational methods to a mathematical model of the atmosphere. The method also defines vertical eigenfunctions to the model. In a simplified case some of the eigenfunctions are compared with empirically obtained data, and conclusions are drawn regarding the validity of some of the approximations in the mathematical model.

1. Introduction

In the terminology of numerical weather prediction a forecasting parameter is a function of x, y and t only (or λ, ϕ and t) substituting the dependent variable we wish to forecast by means of non-linear predictive equations. A compact and very general form for the transformation from a time and space dependent variable to a parameter may be obtained by a series expansion. Taking for instance the stream function $\psi(x, y, t, p)$ as dependent variable, the expansion

$$\psi(x, y, t, p) = \Sigma\alpha_n(x, y, t)F_n(p) \tag{1.1}$$

will determine the forecasting parameters $\alpha_n(x, y, t)$ as soon as the functions $F_n(p)$ are given. In integrated models these will be suitably chosen continuous weighting functions, while, taking them as delta functions, we obtain instead parameters representing conditions at prescribed levels.

The form of the series expansion (1) shows that what we in fact are doing, when carrying out the transformation, is to utilize the method of variable separation, so common and efficient in solving linear partial differential equations. However, when application is made to non-linear partial differential equations, the purpose of the method is not very clear, and it is therefore of interest to make a comparison between these two very different situations.

In the case of a linear equation, the expansion (1) is not arbitrary. Instead, it is defined by the requirement that each separate term in the series should *exactly* satisfy the equation, and furthermore that the functions $F_n(p)$, in this case the eigenfunctions of the problem, should satisfy certain boundary conditions, The rate of convergence of the series is not of particular interest, it is determined partly by initial conditions, i.e. the extent to which various modes are excited initially, and partly by damping or diffusing terms in the equation which generally result in high frequency and high wavenumber modes vanishing rapidly.

In the non-linear case particular solutions that *exactly* satisfy the equation do not exist, except in a few special cases which are of no interest. Therefore, in our usual approach to a non-linear problem, the functions $F_n(p)$ in (1) remain undetermined and arbitrary, and one may choose any complete orthogonal set provided boundary conditions can be satisfied. For reasons of formal simplicity Fourier functions are often selected, but again it should be noted that these are not in any specific way related to the equation. Due to the non-linearity, energy transfer will take place between possible modes, and the rate of convergence in the series—or in other terms the energy spectrum—will depend on the choice of the orthogonal set. Obviously one may by misfortune make a poor choice and end up with a very slow convergence. In the application discussed here this is equivalent to a need for many parameters before a satisfactory approximation of the dependent variable is obtained.

In order to economize on the number of parameters and, by this, also on computer time and capacity we obviously need a method by which a best choice or a consistent derivation of the expanding functions $F_n(p)$ can be made. With this purpose in mind it is natural to return to the principle of eigenfunctions, but in view of the non-linearity, to generalize and extend their definition. Retaining the condition that they should satisfy proper boundary conditions we may now define them by the relaxed requirement that they should satisfy the equation not exactly, but instead *as well as possible*. This means that the determination of eigenfunctions transforms into a well-defined variational problem. The definition of eigenfunctions to linear equations will remain unchanged as a special case, since here the minimum will be exactly zero.

2. Some properties of generalized eigenfunctions

In order to clarify the ideas discussed above we shall consider a dependent variable $u(x, y, t)$ the evolution of which is governed by the non-linear equation

$$M(u) = 0 \tag{2.1}$$

M being a non-linear operator. Equation (2.1) is supposed to be valid in the area S, and u should satisfy given boundary conditions along the boundary L. Furthermore u is supposed to satisfy some initial conditions.

For $u(x, y, t)$ we now introduce an approximation denoted by $v(x, y, t)$. We may require v to be of the form

$$v(x, y, t) = \alpha(t) f(x, y) \tag{2.2}$$

but other restrictions are also possible. With the imposed limitation, v can no longer satisfy eq. (2.1) and we therefore have

$$M(v) = R(x, y, t) \tag{2.3}$$

where the values of the residual R depend on the approximation or restriction made in v. Within these limitations we now wish to make the best possible choice of v and we therefore introduce the condition that R should be minimized in the least square sense when integrated over S and over the time interval $(0, T)$.

The variational problem is thus

$$\int_S \int_T R^2 \, ds \, dt = \text{minimum}$$

or in view of (2.3)

$$\int_S \int_T R \, \delta M(v) \, ds \, dt = 0 \tag{2.4}$$

Since variations by definition are small, this equation will be linearized in terms of the variation δv. If $M(u)$, for instance, contains the term $u \, \partial u/\partial x$ we shall have

$$\int_S \int_T R \, \delta \left(v \frac{\partial v}{\partial x} \right) ds \, dt = \int_S \int_T \left(R \frac{\partial v}{\partial x} \, \delta v + Rv \, \delta \frac{\partial v}{\partial x} \right) ds \, dt$$

By partial integration the second term may be transformed so that the final result becomes

$$\int_S \int_T R \, \delta \left(v \frac{\partial v}{\partial x} \right) ds \, dt = \int_S \int_T \left(R \frac{\partial v}{\partial x} - \frac{\partial Rv}{\partial x} \right) \times \delta v \, ds \, dt + \int_Y \int_T [Rv \, \delta v]_{x_1}^{x_2} \, dy \, dt$$

Here, if v is given on the boundary we have $\delta v = 0$ at x_1 and x_2 and the second integral vanishes. If this condition is not imposed on v the second integral, possibly together with other partially integrated terms will provide so-called natural boundary conditions on v. In any case the first integral above will separate out, and together with other terms multiplying the variation of δv in S and T give an equation of the form

$$\int_S \int_T M'(R, v) \, \delta v \, ds \, dt = 0 \tag{2.5}$$

where M' is a new operator on v and R, involving non-linear partial derivatives. In particular, if M includes partial time derivatives M′ will also do so and is thus a prognostic operator.

In (2.5) we may now consider the case when the restriction on $v(x, y, t)$ is one of variable separation

according to (2.2). We thus have

$$\delta v = \alpha(t)\,\delta f(x,y) + f(x,y)\,\delta\alpha(t) \tag{2.6}$$

where δf and $\delta\alpha$ naturally are independent. Equation (2.5) therefore separates into the following two

$$\int_T M'(R,v)\,\alpha(t)\,dt = 0 \tag{2.7}$$

and

$$\int_S M'(R,v)\,f(x,y)\,ds = 0 \tag{2.8}$$

Going back now to (2.5) it is seen that if the variation δv is unrestricted everywhere, then

$$M'(R,v) = 0 \tag{2.9}$$

will be a correct prognostic equation for v. If on the other hand the variation δv is restricted according to (2.6) then the operatore $M'(R, v)$ will be orthogonal to $\alpha(t)$ over the interval $(0, T)$ and to $f(x, y)$ over S. The predictive equation (2.9) is no longer exactly valid but will, if applied to an initial field v produce an error field which is orthogonal to $\alpha(t)$ and $f(x, y)$.

In practical application (2.9) will not be used for forecasting purposes. Instead eq. (2.7), being integrated over t, will provide a non-linear partial differential equation which together with given and natural boundary conditions will determine $f(x, y)$ as a solution. The coefficients in this equation will depend on statistics of $\alpha(t)$ and its time derivatives. In a similar way eq. (2.8), being integrated over S, will provide the necessary predictive equation for $\alpha(t)$, in which the coefficients will depend on surface statistics of $f(x, y)$ and its derivatives.

If we multiply eq. (2.8) by $\alpha(t)$ and integrate over time we obtain a relation between these two types of statistics. This corresponds to the usual relation between eigenvalues that one finds in linear problems. It follows that by this method different modes can be determined where, in each separate mode, an interdependence exists between time scale and horizontal scale.

Considering the predictive equation (2.8) it is easily shown by partial integration that if the operator M of (2.1) includes the time derivative $\partial/\partial t$, then the operator M′ will have $\partial^2/\partial t^2$ as highest order time derivative. This means that the forecast equation for α is very different from the initial forecast equation (2.1). In a similar way we find that the order of space derivatives will be increased.

We therefore arrive at the somewhat surprising conclusion that in order for an approximation to satisfy a prognostic equation as well as possible *its evolution in time should not be calculated by introduction in this same equation,* but instead in a different equation of higher order. It should be noticed that this also holds true if in (2.2) we prescribe the function $f(x, y)$ and thus only deal with a variation of α. The resulting forecast equation for $\alpha(t)$ will still be equation (2.8).

The procedure given above may now be extended to include more terms in the approximation of $u(x, y, t)$. Taking, for instance,

$$v = \alpha_1(t)\,f_1(x,y) + \alpha_2(t) f_2(x,y) \tag{2.10}$$

we have

$$\delta v = \alpha_1\delta f_1 + f_1\delta\alpha_1 + \alpha_2\delta f_2 + f_2\delta\alpha_2 \tag{2.11}$$

and since these variations are independent, eqs. (2.7) and (2.8) transform into

$$\text{(a)}\quad \int_T M'(R,v)\,\alpha_1\,dt = 0$$

$$\text{(b)}\quad \int_S M'(R,v)\,f_1\,ds = 0$$

$$\text{(c)}\quad \int_T M'(R,v)\,\alpha_2\,dt = 0$$

$$\text{(d)}\quad \int_S M'(R,v)\,f_2\,ds = 0$$

which shows that the orthogonality considerations made before still hold. Since it is not possible in (2.10) and (2.11) to discriminate between the first and the second term, an extra condition has to be prescribed. Various possibilities may exist but the simplest is probably to consider $f_1(x, y)$ as prescribed, determined instead from (2.7) and thus excluding eq. (a) from (2.11). Equation (c) in (2.11) will then determine f_2 while forecast equations for α_1 and α_2 are obtained from eqs. (b) and (d). This procedure may then be extended to any required number of terms.

3. The model atmosphere

In order now to apply the method to the construction of an atmospheric model we shall here consider the simplest possible case, starting from the vorticity and adiabatic equations in the following, approximate form

$$\frac{\partial \nabla^2 \psi}{\partial t} + \beta \frac{\partial \psi}{\partial x} + J(\psi, \nabla^2 \psi) - f_0 \frac{\partial \omega}{\partial p} = 0 \tag{3.1}$$

and

$$\frac{\partial^2 \phi}{\partial t \, \partial p} + J\left(\psi, \frac{\partial \phi}{\partial p}\right) + \omega \sigma_0 = 0 \tag{3.2}$$

Elimination of ω and introduction of the geostrophic approximation $f_0 \psi = \phi$ gives

$$\frac{\partial \nabla^2 \psi}{\partial t} + \beta \frac{\partial \psi}{\partial x} + J(\psi, \nabla^2 \psi) + f_0^2 \frac{\partial}{\partial p} \frac{1}{\sigma_0} \left[\frac{\partial^2 \psi}{\partial t \, \partial p} + J\left(\psi, \frac{\partial \psi}{\partial p}\right) \right] = 0 \tag{3.3}$$

We now introduce the following approximation for the stream-functions

$$\psi(x, y, t, p) \simeq \alpha(x, y, t) F(p) \tag{3.4}$$

and obtain from (3.3) the relation

$$\left(\frac{\partial \nabla^2 \alpha}{\partial t} + \beta \frac{\partial \alpha}{\partial x} \right) F + J(\alpha, \nabla^2 \alpha) F^2 + f_0^2 \frac{\partial \alpha}{\partial t} \frac{d}{dp} \left(\frac{1}{\sigma_0} \frac{dF}{dp} \right) = R(x, y, t, p) \tag{3.5}$$

where R is the residual due to the restricted form of the approximation. For the same reason, and due to the geostrophic approximation, the Jacobian from the adiabatic equation will take the form

$$f_0^2 J(\alpha, \alpha) \frac{d}{dp} \left(\frac{F}{\sigma} \frac{dF}{dp} \right)$$

and thus vanish. It should, however, be pointed out that friction would invalidate the geostrophic approximation and make the Jacobian non-vanishing. The term may therefore be of importance. We shall return to this question later on.

In order to have a boundary condition at the surface of the earth we introduce for the geopotential ϕ an approximation consistent with (3.4) or

$$\phi(x, y, t, p) = F_0(p) + f_0 \alpha(x, y, t) F(p)$$

where $F_0(p)$ represents a standard atmosphere.

Taking the surface of the earth to be flat with $\phi = 0$ and differentiating the expression on the right-hand side with respect to time, we obtain

$$\frac{dF_0}{dp} \frac{dp}{dt} + f_0 \alpha \frac{dF}{dp} \frac{dp}{dt} + f_0 \frac{d\alpha}{dt} F = 0$$

or

$$f_0 \frac{d\alpha}{dt} F + \omega \left(\frac{dF_0}{dp} + f_0 \alpha \frac{dF}{dp} \right) = 0 \tag{3.6}$$

Introducing the same approximation into the adiabatic equation, it may be written

$$f_0 \frac{d\alpha}{dt} \frac{dF}{dp} + \omega \sigma_0 = 0 \tag{3.7}$$

noting that here the operator d/dt only includes horizontal derivatives. In (3.6) we now neglect the second term in the parentheses multiplying ω and obtain from a combination of (3.6) and (3.7) the boundary condition at $p = p_s$

$$S_0 \frac{dF}{dp} - \sigma_0 F = 0$$

with $S_0 = dF_0/dp$, the negative value of standard specific volume at surface pressure.

With regard to the upper boundary at $p = 0$ the only condition required is that F should remain finite since this point is singular due to $1/\sigma_0$ behaving like p^2 for small p if we assume an isothermal upper stratosphere.

4. Variation of α

With the form given in (3.4) for the approximation of ψ, the variational problem directly separates into the two equations

$$\int_S \int_T \int_P R \, \delta_\alpha R \, ds \, dt \, dp = 0$$

and

$$\int_S \int_T \int_P R\, \delta_F R\, ds\, dt\, dp = 0 \tag{4.1}$$

with R given by (3.5).

Here the variational symbol δ_α indicates that only α and not F is to be varied and vice versa for δ_F. S is the horizontal area of integration with $ds = dx\, dy$. The boundary of S will be denoted L with dL a line element on this boundary. Furthermore T indicates the interval of time over which the forecast equation is to be valid and P the vertical pressure interval, here taken to be $(0, p_s)$ with p_s equal to surface pressure.

In order to have a more compact form of the equation we also introduce the notation

$$F^* = f_0^2 \frac{d}{dp}\left(\frac{1}{\sigma_0}\frac{dF}{dp}\right) \tag{4.2}$$

The variational equation to be considered here is first

$$\int_S \int_T \int_P R\, \delta_\alpha \left[\left(\frac{\partial \nabla^2 \alpha}{\partial t} + \beta \frac{\partial \alpha}{\partial x}\right) F + J(\alpha, \nabla^2 \alpha) F^2 + \frac{\partial \alpha}{\partial t} F^*\right] ds\, dt\, dp = 0 \tag{4.3}$$

where partial integrations have to be carried out in order to obtain integrands which explicitly contain $\delta\alpha$ or its lowest possible derivatives as a factor.

The manipulations are rather long and details of them have therefore been collected in the appendix. For the discussion it is sufficient to give the resulting full equations. After separation into different and independent variations of α or its derivatives we obtain the following integrals, which all must vanish separately.

$$\int_S \int_T \int_P \left[F \frac{\partial \nabla^2 R}{\partial t} + F^* \frac{\partial R}{\partial t} + F^2 J(R, \nabla^2 \alpha) + F^2 \nabla^2 J(\alpha, R) + F \beta \frac{\partial R}{\partial x}\right] \delta\alpha\, ds\, dt\, dp = 0 \tag{4.4}$$

$$\int_S \int_P [(F \nabla^2 R + F^* R)\delta\alpha]_0^T\, ds\, dp = 0 \tag{4.5}$$

$$\oint_L \int_P \left[FR\, \delta \frac{\partial \alpha}{\partial n}\right]_0^T dL\, dp = 0 \tag{4.6}$$

$$\oint_L \int_P \left[F \frac{\partial R}{\partial n} \delta\alpha\right]_0^T dL\, dp = 0 \tag{4.7}$$

$$\oint_L \int_T \int_P \left\{\left[F \frac{\partial^2 R}{\partial t\, \partial n} - F^2 \frac{\partial J(R, \alpha)}{\partial n}\right] dL + \left. + [F^2 R \nabla(\nabla^2 \alpha) + FR \nabla f]\, d\mathbb{L}\right\} \delta\alpha\, dt\, dp = 0 \tag{4.8}$$

$$\oint_L \int_T \int_P \left[-F \frac{\partial R}{\partial t} + F^2 J(R, \alpha)\right] \delta \frac{\partial \alpha}{\partial n} dL\, dt\, dp = 0 \tag{4.9}$$

$$\oint_L \int_T \int_P F^2 R \nabla\alpha\, \delta \nabla^2 \alpha\, d\mathbb{L}\, dt\, dp = 0 \tag{4.10}$$

The main problem with regard to eqs. (4.4)–(4.10) is to decide if they are exactly satisfied by restrictions on α or its derivatives or if they give so-called natural boundary or initial conditions, that have to be satisfied in order for R to be a minimum.

Since $\delta\alpha$ is arbitrary in S and T the necessary condition for eq. (4.4) to be satisfied, directly gives the following forecast equation for R

$$\int_P \left\{F\left(\frac{\partial \nabla^2 R}{\partial t} + \beta \frac{\partial R}{\partial x}\right) + F^2 [J(R, \nabla^2 \alpha) + \nabla^2 J(\alpha, R)] + F^* \frac{\partial R}{\partial t}\right\} dp = 0 \tag{4.11}$$

Taking (3.5) into account, we see that this in reality is a predictive equation for α of second order in time and of fourth order in S. Thus, in order to carry out a time integration using (4.10) we have to know α and $\partial\alpha/\partial t$ initially. On the other hand, since we wish to apply the equation to all kinds of initial fields, we do not wish to fix α at $T = 0$ once and for all, and this means that $\delta\alpha$ may at this instant be arbitrary. In order for (4.5) to hold we must then have

$$\int_P [F \nabla^2 R + F^* R]\, dp = 0 \quad t = 0 \tag{4.12}$$

which again, taking (3.5) into account, determines the initial $\partial\alpha/\partial t$ as soon as initial α is given. It should be noted that the coefficients in eqs. (4.11) and (4.12) are obtained by integration over p of second or third order products of functions of F. The construction of a model on the basis of these equations will therefore require a knowledge of F.

Turning now to the remaining equations we shall choose the simplest possible case and assume the area S to be so large that we can have $\alpha = 0$ and furthermore $\partial\alpha/\partial n$ specified along L at all times. This makes $\delta\alpha = \delta\,\partial\alpha/\partial n = 0$ in eqs. (4.6)–(4.9) which thereby are satisfied. Furthermore the condition (4.10) is also satisfied since $\nabla\alpha \cdot d\mathbb{L} = d\alpha$ taken along L which vanishes with $\alpha = 0$. The variation $\delta\nabla^2\alpha$ along L is thus free.

5. Variation of F

In order to simplify notations when carrying out the second variation of (4.1) we introduce

$$A = \frac{\partial\nabla^2\alpha}{\partial t} + \beta\frac{\partial\alpha}{\partial x};\quad B = J(\alpha,\ \nabla^2\alpha);\quad C = f_0^2\frac{\partial\alpha}{\partial t} \tag{5.1}$$

and have thus to consider

$$\int_S\int_T\int_P R\,\delta\left[AF + BF^2 + C\frac{d}{dp}\left(\frac{1}{\sigma_0}\frac{dF}{dp}\right)\right] \times ds\,dt\,dp = 0 \tag{5.2}$$

For the third term inside the brackets we obtain after partial integration

$$\int_S\int_T\int_P RC\,\delta\frac{d}{dp}\left(\frac{1}{\sigma_0}\frac{dF}{dp}\right)ds\,dt\,dp =$$

$$\int_S\int_T C\left[\frac{R}{\sigma_0}\delta\frac{dF}{dp} - \frac{1}{\sigma_0}\frac{\partial R}{\partial p}\delta F\right]_0^{p_s} ds\,dt$$

$$+ \int_S\int_T\int_P C\frac{\partial}{\partial p}\left(\frac{1}{\sigma_0}\frac{\partial R}{\partial p}\right)\delta F\,ds\,dt\,dp \tag{5.3}$$

where, in the first term on the right-hand side the integrand will vanish at $p = 0$ provided R and $\partial R/\partial p$ are finite. At the boundary $p = p_s$ we have from (3.8) the relation

$$S_0\,\delta\frac{dF}{dp} = \sigma_0\,\delta F$$

so that the natural boundary condition on R, taking F unspecified and $\delta F = 0$ at $p = p_s$ becomes

$$\int_S\int_T C\left[\frac{R\sigma_0}{S_0} - \frac{\partial R}{\partial p}\right]ds\,dt = 0 \tag{5.3}$$

Carrying out remaining variations in (5.2) and taking (5.3) into account we obtain

$$\int_S\int_T\left[R(A + 2BF) + C\frac{\partial}{\partial p}\left(\frac{1}{\sigma_0}\frac{\partial R}{\partial p}\right)\right] \times ds\,dt = 0 \tag{5.4}$$

which in view of the definition of R is a fourth order non-linear differential equation for F. Denoting

$$\int_S\int_T a(x, y, t)\,b(x, y, t)\,ds\,dt = \overline{ab}$$

and introducing R from (3.5) eqs. (5.4) and (5.3) become

$$\frac{d}{dp}\frac{1}{\sigma_0}\frac{d}{dp}\left[\overline{AC}F + \overline{BC}F^2 + \overline{C^2}\frac{d}{dp}\left(\frac{1}{\sigma_0}\frac{dF}{dp}\right)\right] + \overline{AC}\frac{d}{dp}\left(\frac{1}{\sigma_0}\frac{dF}{dp}\right) + 2\overline{BC}F\frac{d}{dp}\left(\frac{1}{\sigma_0}\frac{dF}{dp}\right) + \overline{A^2}F + 3\overline{ABF^2} + 2\overline{B^2F^3} = 0 \tag{5.5}$$

and at $p = p_s$

$$\frac{\sigma_0}{S_0}\left[\overline{AC}F + \overline{BC}F^2 + \overline{C^2}\frac{d}{dp}\left(\frac{1}{\sigma_0}\frac{dF}{dp}\right)\right] = \frac{d}{dp}\left[\overline{AC}F + \overline{BC}F^2 + \overline{C^2}\frac{d}{dp}\left(\frac{1}{\sigma_0}\frac{dF}{dp}\right)\right] \tag{5.6}$$

If we divide by $\overline{C^2}$ and furthermore apply a normalizing condition on F, only four statistical quantities involving α are needed in order to solve

eq. (5.5) with the boundary conditions (5.6) and (3.8). However, these are not known since, as already mentioned, the evolution equation for α and thus also statistical quantities on α will depend on F.

We find here an interdependence of the same kind as in the case of linear partial differential equations where eigensolutions for separated variables are related through an equation between the eigenvalues. In the non-linear case discussed here similar equations relating statistics of α and F may be obtained by multiplying eq. (4.11) by α and integrating over s and t and by multiplying (5.5) by F and integrating over p.

It is, however, obvious that even with relations of this kind being satisfied an infinite number of solutions will exist and that in order to find those that are relevant to and important in our atmospheric model it will be necessary to take forcing and dissipation into account in the variational problem. This is, however, way beyond the purpose of this paper and we shall, therefore, in the following paragraph utilize empirical data in order to determine approximate solutions to (5.5).

Before we do this it is, however, of interest to discuss in some detail the meaning of the residual $R(x, y, t, p)$ in eq. (3.5).

Instead of the approximation (3.4) we shall in equation (3.3) introduce

$$\psi(x, y, p, t) = \alpha(x, y, t)F(p) + \psi'(x, y, t, p)$$

and at the same time presuppose that at time $t = 0$ the initial field is exactly represented by αF, or, in other words that

$$\psi'(x, y, 0, p) = 0$$

At $t = 0$ eq. (3.5) may be written

$$M(\alpha F) = \frac{\partial}{\partial t}\left[\nabla^2 \psi' + f_0^2 \frac{\partial}{\partial p}\left(\frac{1}{\sigma_0}\frac{\partial \psi'}{\partial p}\right)\right]$$

where $M(\alpha F)$ is the expression on the left-hand side of (3.5). The residual R is thus a measure of the instantaneous creation of potential vorticity in ψ', i.e. in modes that are not present in the field. By minimizing R^2 we find vertical modes with the property that the transfer of potential entropy from one excited mode to other non-excited modes is as small as possible.

6. Observational data

In the previous sections we have determined approximations to the stream function $\psi(x,y,p,t)$ of the form $\alpha(x,y,t)F(p)$ with an optimized persistance in the mathematical model of the atmosphere. It is then natural to assume that *if the model is sufficiently realistic* similar vertical modes should be dominant in the real atmosphere. If $\hat{\psi}(x,y,t,p)$ represents measured values of the stream function we therefore formulate the following variational problem

$$\iiint [\hat{\psi}(x, y, p, t) - \tau(x, y, t)f(p)]^2 \times ds\, dt\, dp = \text{minimum} \qquad (6.1)$$

and expect that the function $f(p)$, determined from empirical data should be similar to one of the solutions $F(p)$ to (5.5).

Furthermore, if differences between $F(p)$ and $f(p)$ are found, it also seems natural to assume that these will depend on deficiencies in the mathematical model, for instance a neglect of important terms in its derivation.

Turning now to the variational problem (6.1) it is not surprising that we recognize the basic principle in the derivation of empirical orthogonal functions —in too many applications thought of only as eigenvectors to a covariance matrix. Carrying out the variation we obtain a system of two integral equations from which either τ or f can be eliminated. Choosing to eliminate τ we find

$$\mu \int K(p,p')F(p')dp' = f(p) \qquad (6.2)$$

or in matrix notation

$$Kf = \lambda f \quad \lambda = 1/\mu$$

where the kernel $K(p,p')$ is the covariance function (matrix)

$$K(p, p') = \int_S \int_T \hat{\psi}(x, y, t, p)\, \hat{\psi}(x, y, t, p')\, ds\, dt$$

and μ, the eigenvalue of the Fredholm integral equation is defined by

$$\frac{1}{\mu} = \int_S \int_T \tau^2(x, y, t)\, ds\, dt \cdot \int_P f^2(p)\, dp$$

The eigenfunctions of (6.2) form an orthogonal set and give in a series expansion of observed stream

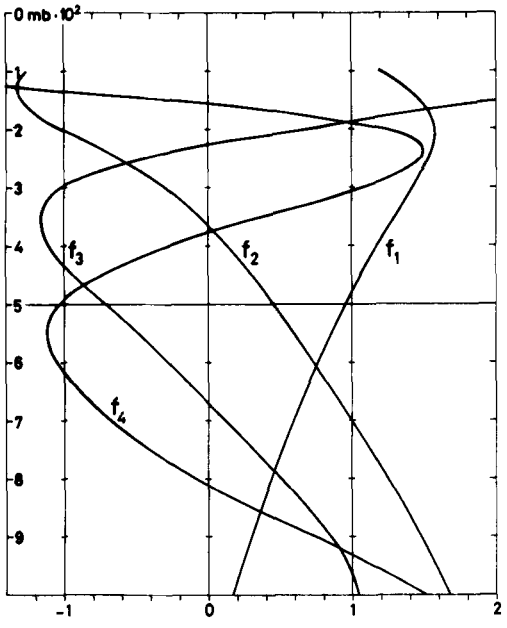

Fig. 1. First four functions $f(p)$ in expansion of geopotential data into empirical orthogonal functions.

function data the fastest possible convergence with an optimized variance reduction for each term.

We have in (6.1) and (6.2) used the stream function as atmospheric variable. Since no expansion into empirical orthogonal functions exists on such data we shall make use of results from expansions of observed geopotential heights, taking advantage of the geostrophic approximation. A certain caution is necessary since the approximation is not always valid.

Expansions of geopotential data have been made by Obukhov (1960), Holmström (1963) and others and we shall, for reference, utilize results obtained by Holmström. The first four empirical functions $f(p)$ are reproduced in Fig. 1. They are normalized to 1 over the interval 1000–100 mb. The dominance of the first mode in these hemispheric data from 7 days in October 1959 is very pronounced with r.m.s. values of corresponding τ-functions being 236.2, 42.5, 20.1 and 10.2 m respectively.

7. Linearized model

For the comparison between theoretical and empirical results we shall not in this paper use the full equation (5.5) but instead consider a simplified version which is more easily discussed. Since the circulation pattern of the atmosphere to a major part is characterized by waves of rather sinusoidal form and relatively small amplitude, it seems reasonable to assume that eq. (5.5) would give realistic vertical profile functions even if the corresponding α-field is linearized. We shall therefore only consider small perturbations on a zonal flow and for α introduce

$$\alpha = -U_0 y + \hat{\alpha} e^{ik(x-ct)} \tag{7.1}$$

where U_0 and $\hat{\alpha}$ are constants and $\hat{\alpha}$ is small.

Instead of introducing this expression in (5.5) we may proceed in a different way. From (3.5) and (6.1) we obtain

$$R = [(k^2 c + \beta)F - k^2 U_0 F^2 - cF^*]ik\,\hat{\alpha} e^{ik(x-ct)}$$

and from (4.10)

$$\int [(k^2 c + \beta)F - k^2 U_0 F^2 - cF^*]^2 dp = 0$$

where second order quantities in $\hat{\alpha}$ have been neglected. If we look for stable solutions with the phase velocity c being real, this relation cannot hold unless the integrand vanishes at all p. We therefore have

$$\frac{d}{dp}\left(\frac{1}{\sigma_0}\frac{dF}{dp}\right) = \frac{k^2 c + \beta}{f_0^2 c} F - \frac{k^2 U_0}{f_0^2 c} F, \tag{7.2}$$

where the expression for F^* in (4.2) has been reintroduced.

A few characteristic properties of solutions to eq. (7.2) were discussed by Holmström (1964) but the equation was not properly derived and the presentation of computational results not adequate. An extended treatment is therefore given here.

Equation (7.2) which for σ_0 = constant has elliptic functions as solutions, has with σ_0 variable also certain interesting properties. First of all it is directly seen that if $F = 1$ we obtain the Rossby phase velocity relation. A slightly modified form is also obtained if the equation is integrated over p with the boundary conditions $dF/dp = 0$ at 1000 mb and $dF/\sigma_0 dp = 0$ at $p = 0$ the modification depending on the integrated value of the normalized function F.

By taking $F = \mu G$ where μ is a constant it is also easily shown that the coefficient of F^2 has no influ-

ence on the form of the solution. It determines only its amplitude and sign and may therefore be arbitrarily chosen so as to make the solution normalized and to be positive at $p = 1000$ mb. As will be seen the properties of the solution are only determined by its absolute value at 1000 mb and by the coefficient in (7.2) multiplying F. This coefficient has here the character of an eigenvalue but the similarity with linear equations is not complete. It is convenient for the demonstration to introduce the notation

$$a^2 = \frac{k^2 c + \beta}{f_0^2 c}; \qquad b^2 = \frac{k^2 U_0}{f_0^2 c} \tag{7.3}$$

so that the equation becomes

$$\frac{d}{dp}\left(\frac{1}{\sigma_0}\frac{dF}{dp}\right) = a^2 F - b^2 F^2 \tag{7.4}$$

If we introduce

$$F = \mu G + \nu \tag{7.5}$$

where μ and ν are constants, we find from (7.4)

$$\mu \frac{d}{dp}\left(\frac{1}{\sigma_0}\frac{dG}{dp}\right) = (a^2 - 2b^2\nu)\mu G + \nu(a^2 - \nu b^2) - b^2\mu^2 G^2 \tag{7.6}$$

Taking ν to satisfy the relation

$$a^2 - \nu b^2 = 0 \tag{7.7}$$

eq. (7.6) transforms into

$$\frac{d}{dp}\left(\frac{1}{\sigma_0}\frac{dG}{dp}\right) = -a^2 G - \mu b^2 G^2 \tag{7.8}$$

which only differs from (7.4) by the sign of the eigenvalue and possibly by a different normalization of G depending on the arbitrary constant μ.

Multiplying now (7.8) by F, (7.4) by G and taking the difference we obtain

$$\frac{d}{dp}\left[\frac{G}{\sigma_0}\frac{dF}{dp} - \frac{F}{\sigma_0}\frac{dG}{dp}\right] = 2a^2 FG - b^2 FG(F - \mu G) = a^2 FG$$

where the relations (7.5) and (7.7) have been taken into account. It is seen that F and G are orthogonal functions provided they satisfy a boundary condition of the type given in (3.8). On this condition we have two (7.4) pairs of mutually orthogonal solutions corresponding to opposite signs of the eigenvalue and related linearly through eq. (7.5).

With this result in mind it is interesting to compare the two first empirical functions given in Fig. 1. Determining the equivalents of μ and ν in (7.5) so as to give the best possible fit in the intervals 1000–100 mb and 1000–250 mb we obtain from the values of f_1 the curves f_2' and f_2'' in Fig. 2, compared with f_2. In spite of the difference above 250 mb the similarity between f_2 and f_2'' is striking and seems to imply the validity of this characteristic property of (7.4) also in empirical data.

A further point of interest in eq. (7.4) is the role of the non-linear term. Taking a^2 positive the effect of the linear term is to make the solution diverge from the p-axis. For positive F the non-linear term with its negative sign is therefore necessary in order to recurve the solutions towards zero and to satisfy realistic boundary conditions. For negative F the solution is seen to diverge and consequently no boundary condition at $p = 0$ can be satisfied.

Turning now to numerical integrations of eq. (7.4) a sample of calculated profiles for F is shown in Fig. 3. Integrations have been carried out from prescribed initial values of F at 1000 mb in steps of 1 mb and in the first step satisfying the boundary condition (3.8). Values for σ_0 have been taken from Holmström (1963), interpolated in the interval 1000–200 mb and from there on extrapolated assuming a smooth transition to an isothermal stratosphere. The solutions have not been renormalized.

Calculations in Fig. 3 have been made with $a^2 = 5 \cdot 10^{-4}$ and $b^2 = 6.25 \cdot 10^{-4}$. Taking $f_0 = 10^{-4}$ and $\beta = 10^{-11}$ various values of k (or wavelength L) correspond to the values of c and U_0 given in Fig. 4. Different initial values of F at $p = 1000$ mb have been prescribed and we shall use these for reference.

From Fig. 3 it is seen that all curves become negative before reaching 5 mb. They will then go rapidly towards infinite negative values and the curves in this calculation do therefore not represent solutions that satisfy the upper boundary condition. In some cases this may be due to the integration step 1 mb, being too large close to the singularity, but this point is not of special interest. More interesting is the fact that vertical gradients as well as curvatures become very large at tropopause level

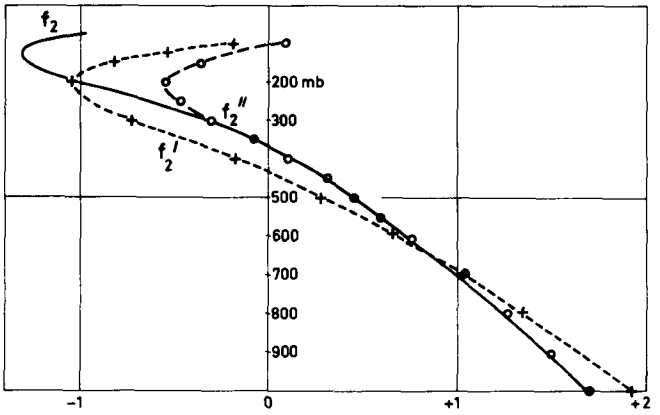

Fig. 2. Comparison between the function $f_2(p)$ of Fig. 1 with corresponding functions $f_2'(p)$ and $f_2''(p)$ calculated from $f_1(p)$.

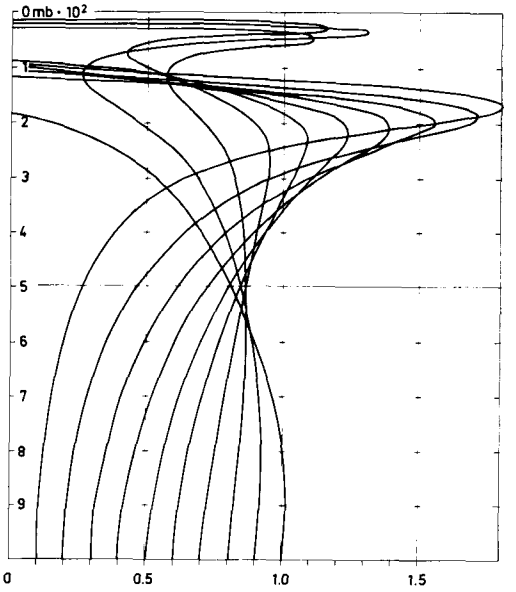

Fig. 3. Profiles $F(p)$ calculated from eq. (6.4) with $a^2 = 5 \cdot 10^{-4}$, $b^2 = 6.25 \cdot 10^{-4}$ and varying values of F at $p =$ 1000 mb.

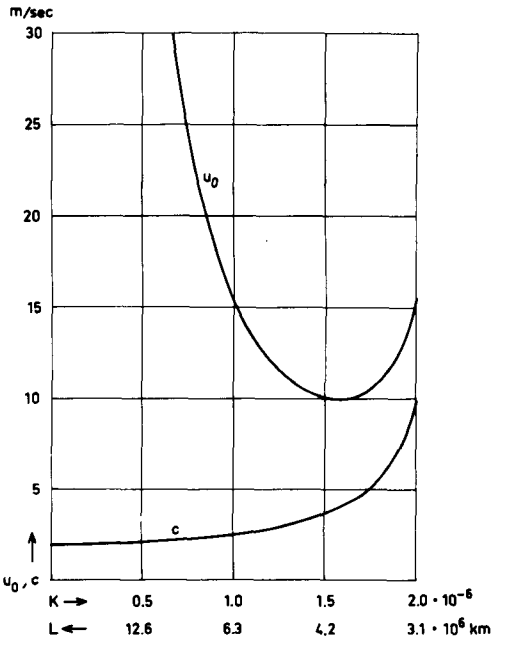

Fig. 4. Relation between wave number k, phase velocity c and zonal wind U_0 for values of a^2 and b^2 used in Fig. 3.

and one would therefore expect turbulence to be generated in this region. The neglect of friction in the initial set of equations is therefore likely to be of crucial importance.

The effect of neglect of friction is also evident in the curves 0.6–1.0, giving unrealistic values of the wind at the ground, taking the geostrophic approximation into account. The drag should lower considerably the value of the function s at 1000 mb.

Naturally a rederivation of the equations should therefore be made, including an expression for the friction from the beginning. This would also affect the geostrophic approximation and make for instance the Jacobian in the thermodynamic equation non-vanishing. It would, however, also necessitate inclusion of an external forcing to

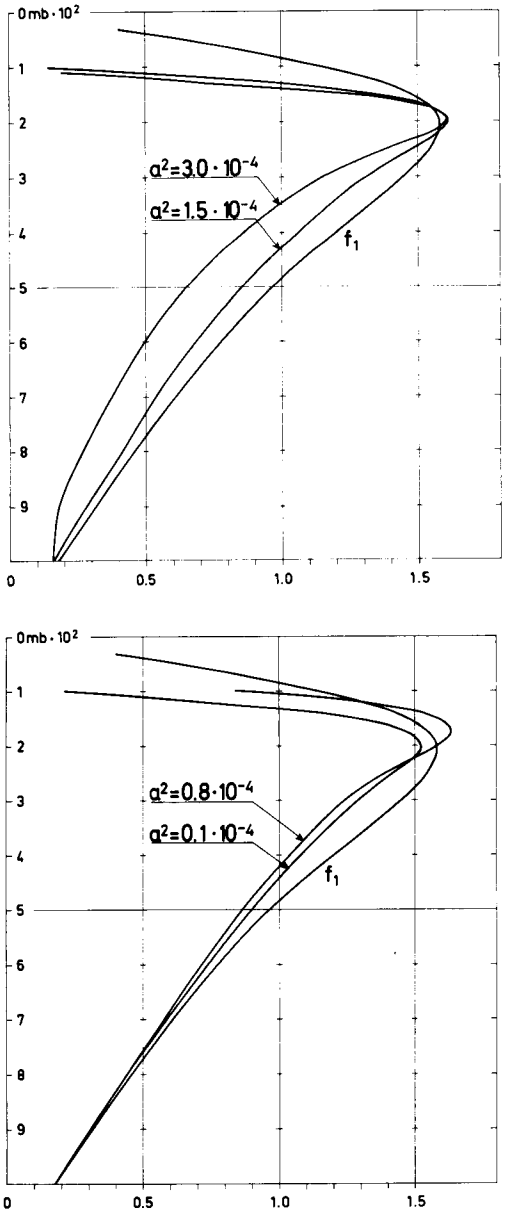

Fig. 5a, b. Profiles $F(p)$ calculated from eq. (7.4) with relaxed boundary condition at $p = 1000$ mb and compared with empirical $f_1(p)$. Values of a^2 are given beside the curves.

balance dissipation and this would not only complicate the treatment but also make the results dependent on rather arbitrary assumptions. We shall therefore take surface friction into account in this paper simply by relaxing the lower boundary condition (3.8) and carry out the calculations and the comparison assuming that at 1000 mb the theoretical solution $F(p)$ should have the same value and the same slope as the empirical function $f(p)$.

Recomputing F with this boundary condition we obtain the results shown in Fig. 5 for varying values of a^2 and b^2. Up to levels around 250 mb some of the curves show a striking similarity with the empirical one but above this level we again notice a much larger curvature and much steeper gradients. This may be due to lack of friction but other approximations may also be involved.

We also notice that solutions remain rather similar for a wide range of values of a^2. This implies that the vertical structure of the model atmosphere and most probably also the real atmosphere does not to a large extent depend on the circulation pattern itself but much more on the average static stability and on surface friction. This is corroborated by the dominance of the first empirical orthogonal function in expansions of the geopotential. One may here also see a reason why one-parameter models indeed give comparatively good results in short-term forecasting.

8. Conclusion

The purpose in this paper has been to draw attention to the fact that in parameterizing a non-linear predictive equation—as we always do in meteorology—the parameters should not be predicted with the same equation as the variable they replace. Instead, a higher order equation may be derived which minimizes the approximation made when introducing the parameters. The variational method employed also defines vertical eigensolutions to the mathematical model of the atmosphere. Some of these should be similar to those found from empirical data. If this condition is not satisfied the conclusion should be that the mathematical model does not sufficiently well-simulate the real atmosphere. In such a case one cannot expect the predictive equations to give useful results.

Starting from a set of approximative equations and drastically simplifying computations we have shown that the eigenfunctions to the mathematical model have certain characteristic properties common with the real atmosphere. However, certain

obvious differences exist. These may be due to neglect of friction in the mathematical model but other sources of error are also possible, for instance the geostrophic approximation. A more comprehensive treatment of the problem is therefore required in order to determine the simplest possible mathematical model of the atmosphere which still has "eigenfunctions" that sufficiently well-resemble those found in the atmosphere.

9. Appendix

Starting with the first term in (4.2) we first integrate partially with respect to time, obtaining

$$\int_S\int_T\int_P R\,F\,\delta\,\frac{\partial\nabla^2\alpha}{\partial t}\,ds\,dt\,dp = \int_S\int_P \times [R\,F\,\delta\nabla^2\alpha]_0^T\,ds\,dp - \int_S\int_T\int_P F\,\frac{\partial R}{\partial t}\,\delta\nabla^2\alpha\,ds\,dt\,dp \tag{A.1}$$

since F is a function of p only. On both terms on the right hand side we now utilize Green's theorem

$$\int_S a\,\nabla^2 b\,ds = \int_S b\,\nabla^2 a + \oint_L \left(a\,\frac{\partial b}{\partial n} - b\,\frac{\partial a}{\partial n}\right) dL \tag{A.2}$$

This gives directly

$$\left.\begin{aligned}&\int_S\int_T\int_P R\,F\,\delta\,\frac{\partial\nabla^2\alpha}{\partial t}\,ds\,dt\,dp\\ &= \int_S\int_P [F\,\nabla^2 R\,\delta\alpha]_0^T\\ &\quad + \oint_L\int_P F\left[R\delta\frac{\partial\alpha}{\partial n} - \frac{\partial R}{\partial n}\delta\alpha\right]_0^T dL\,dp\\ &- \int_S\int_T\int_P F\,\frac{\partial\nabla^2 R}{\partial t}\,\delta\alpha\,ds\,dt\,dp\\ &\quad - \oint_L\int_T\int_P F\left(\frac{\partial R}{\partial t}\delta\frac{\partial\alpha}{\partial n} - \frac{\partial^2 R}{\partial t\,\partial n}\delta\alpha\right)\\ &\quad \times\ dL\,dt\,dp\end{aligned}\right\} \tag{A.3}$$

The next term to be dealt with is the Jacobian, where we first notice that, since δ is a linearizing operator, we have

$$\int_S\int_T\int_P R\,F^2\,\delta J(\alpha,\nabla^2\alpha)\,ds\,dt\,dp = \int_S\int_T\int_P R\,F^2[J(\delta\alpha,\nabla^2\alpha) - J(\delta\nabla^2\alpha,\alpha)]\,ds\,dt\,dp \tag{A.4}$$

We shall also utilize the following two relations, which are easily verified

$$J(a,b) = \mathbb{k}\cdot(\nabla a + \nabla b) = \mathbb{k}\cdot[\nabla\times(a\nabla b)] \tag{A.5}$$

and, if $c = c(x,y)$

$$cJ(a,b) = J(ac,b) - aJ(c,b) \tag{A.6}$$

Combining (A.5) and (A.6) we have

$$\int_S cJ(a,b)\,ds = \int_S J(ac,b)\,ds - \int_S aJ(c,b)\,ds = \int_S [\nabla\times(ac\,\nabla b)]\,d\mathbf{s} - \int aJ(c,b)\,ds$$

or, using Stokes theorem

$$\int_S cJ(a,b)\,ds = \int_L ac\,\nabla b\cdot d\mathbb{L} - \int_S aJ(c,b)\,ds \tag{A.7}$$

In these equations $d\mathbf{s}$ is a vector surface element of S and $d\mathbb{L}$ a vector line element along L.

Combining now (A.4) and (A.7) and taking $c = R$, $a = \delta a$ and $b = \nabla^2\alpha$ in the first case and $a = \delta\nabla^2\alpha$ and $b = \nabla^2\alpha$ in the second, we find

$$\begin{aligned}&\int_S\int_T\int_P RF^2\,\delta J(\alpha,\nabla^2\alpha)\,ds\,dt\,dp\\ &= \oint_L\int_T\int_P F^2 R\,\nabla(\nabla^2\alpha)\,\delta\alpha\cdot d\mathbb{L}\,dt\,dp\\ &\quad - \int_S\int_T\int_P F^2 J(R,\nabla^2\alpha)\,\delta\alpha\,ds\,dt\,dp\\ &\quad - \oint_L\int_T\int_P F^2 R\,\nabla\alpha\cdot\delta\nabla^2\alpha\,d\mathbb{L}\,dt\,dp\\ &\quad + \int_S\int_T\int_P F^2 J(R,\alpha)\,\delta\nabla^2\alpha\,ds\,dt\,dp\end{aligned}$$

or, using (A.2) on the last integral

$$\left.\begin{aligned}
&\int_S \int_T \int_P RF^2\,\delta J(\alpha, \nabla^2\alpha)\,ds\,dt\,dp \\
&= -\int_S \int_T \int_P F^2\,[J(R, \nabla^2\alpha) \\
&\quad + \nabla^2 J(\alpha, R)]\,\delta\alpha\,ds\,dt\,dp \\
&\quad + \oint_L \int_T \int_P F^2 R[\nabla(\nabla^2\alpha)\,\delta\alpha \\
&\quad - \nabla\alpha\delta\nabla^2\alpha]\cdot d\mathbb{L}\,dt\,dp \\
&\quad + \oint_L \int_T \int_P F^2 \left[J(R, \alpha)\,\frac{\partial\delta\alpha}{\partial n} \right. \\
&\quad \left. - \frac{\partial J(R, \alpha)}{\partial n}\,\delta\alpha \right] dL\,dt\,dp
\end{aligned}\right\} \quad [A.8]$$

The variation of the β-term is easiest to carry out if we write it under the form of a Jacobian. Repeating previous procedures we find

$$\left.\begin{aligned}
&\int_S \int_T \int_P \beta F R\,\delta\frac{\partial\alpha}{\partial x}\,ds\,dt\,dp \\
&= \int_S \int_T \int_P F[J(R\delta\alpha, f) - \\
&\quad - J(R, f)\,\delta\alpha]\,ds\,dt\,dp \\
&= \oint_L \int_T \int_P FR\,\nabla f\,\delta\alpha\cdot d\mathbb{L}\,dt\,dp \\
&\quad - \int_S \int_T \int_P F\beta\,\frac{\partial R}{\partial x}\,\delta\alpha\,ds\,dt\,dp
\end{aligned}\right\} \quad (A.9)$$

Collecting present results and including also the transformation of the time-derivative from the adiabatic equation we find

$$\left.\begin{aligned}
&-\int_S \int_T \int_P \left[F\,\frac{\partial\nabla^2 R}{\partial t} + F^*\,\frac{\partial R}{\partial t} \right. \\
&\quad + F^2 J(R, \nabla^2\alpha) + F^2\nabla^2 J(\alpha, R) \\
&\quad \left. + F\beta\,\frac{\partial R}{\partial x} \right] \delta\alpha\,ds\,dt\,dp + \int_S \int_P \\
&\quad \times[(F\nabla^2 R + F^*R)\,\delta\alpha]_0^T\,ds\,dp \\
&\quad + \oint_L \int_P F \left[R\,\delta\frac{\partial\alpha}{\partial n} \right. \\
&\quad \left. - \frac{\partial R}{\partial n}\,\delta\alpha \right]_0^T dL\,dp \\
&\quad + \oint_L \int_T \int_P \left\{ F \left(\frac{\partial^2 R}{\partial t\,\partial n}\,\delta\alpha \right. \right. \\
&\quad \left. - \frac{\partial R}{\partial t}\,\delta\frac{\partial\alpha}{\partial n} \right) \\
&\quad + F^2 \left[J(R, \alpha)\,\delta\frac{\partial\alpha}{\partial n} \right. \\
&\quad \left. \left. - \frac{\partial J(R, \alpha)}{\partial n}\,\delta\alpha \right] \right\} \\
&\quad \times dL\,dt\,dp + \oint_L \int_T \int_P \{F^2 R\,[\nabla(\nabla^2\alpha)\,\delta\alpha \\
&\quad - \nabla\alpha\,\delta\nabla^2\alpha] + FR\nabla f\,\delta\alpha\}\cdot d\mathbb{L}\,dt\,dp = 0
\end{aligned}\right\} \quad (A.10)$$

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ОПТИМИЗАЦИЯ АТМОСФЕРНЫХ МОДЕЛЕЙ.

Выводится совершенно новый тип уравнений для прогноза атмосферных параметров путём приложения вариационных методов к математической модели атмосферы. Метод позволяет также определить вертикальные собственные функции модели. В простейшем случае некоторые собственные функции сравниваются с эмпирическими данными и делаются выводы о справедливости некоторых аппроксимаций в математической модели.