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# On generation of zonal flows by interacting Rossby waves

By ARTHUR Z. LOESCH *Atmospheric Science Department, State University of New York at Albany, Albany, New York 12222, U.S.A.*

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ABSTRACT

Generation of zonal flows by finite amplitude resonantly interacting discrete Rossby waves is investigated within the context of the quasi-geostrophic barotropic model on the beta plane. Under the "channel" assumption, in which the north–south extent of the domain is taken as finite, analysis shows that a triad of Rossby waves is capable of generating a zonal flow on the time scale of the resonant interaction. Zonal currents thus produced exhibit a spatial structure similar to that obtained from the baroclinic instability mechanism: a jet in the central portion and weaker reversed flow in the northern and southern portions of the channel. If the north–south extent of the domain is taken as infinite, zonal flows of sinusoidal latitudinal structure can be generated by discrete Rossby waves via the quartet resonance mechanism. The time scale required for their generation is an order of magnitude longer than that associated with the "channel" case.

## 1. Introduction

Observational studies of the atmospheric energy cycle, reviewed by Saltzman (1970), show a redistribution of kinetic energy by mean barotropic processes among the large-scale eddies and between the eddies and the mean zonal flow. Saltzman's energy flow diagrams illustrate that all eddies in the zonal wavenumber band 0–15 supply kinetic energy to the zonal flow in the summer, and all the wavenumbers in the same band, except 3, supply kinetic energy to the zonal flow in the winter, to maintain it against frictional dissipation.

The aim of this study is to investigate in detail the manner in which large scale finite-amplitude interacting planetary waves are capable of enforcing zonal flows via the resonant interaction energy transfer mechanism. Since the interest here lies in the atmospheric domain, whose extent is finite with respect to large-scale disturbances, the wave spectrum shall be taken as discrete. Mathematically, interactions of finite-amplitude discrete waves may be modelled by using an asymptotic expansion for the wave field and treating the amplitudes of the interacting waves as slowly varying functions of time.

Resonant interactions between discrete Rossby waves in triad configuration have been shown to exist and are extensively discussed for an infinite $\beta$-plane geometry by Longuet-Higgins & Gill (1967). In particular, if one of the triad members is taken to be the zonal current, then due to vanishing of its nonlinear coupling coefficient, it can neither gain nor lose energy via an interaction with the other members of the triad. At most, in the case of nonzero initial amplitude, it acts as a catalyst, enabling energy to be exchanged between the remaining two waves.

Newell (1969) demonstrated that if the discrete spectrum assumption is removed, allowing the Rossby wave amplitudes to be slowly varying functions of space as well as time, it is possible for the waves closely neighboring the resonant waves to excite zonal flows on a time scale an order of magnitude longer than that associated with triad interactions. Newell called this the sideband resonance mechanism for generation of zonal flows. Further, he noted that quartet resonances are first felt on this longer time scale and through kinematic arguments, in which one of the members of the quartet was taken as the zonal current, pointed out that the quartet resonance mechanism is also capable of exciting zonal currents.

In the oceans where the Rossby radius of deformation is an order of magnitude smaller than in the atmosphere, planetary wave spectrum may essentially be taken as continuous and the idea of side-band resonance mechanism can apply. In the atmosphere, spectrum quantization precludes this possibility, restricting the generation of zonal flows by interacting waves, in the infinite $\beta$-plane approximation, to the quartet resonance mechanism. Investigation of the dynamics of this mechanism will constitute a part of this paper.

Recently, Loesch (1974) using the inviscid two-layer model on a $\beta$-plane of *finite* north–south extent investigated the temporal and spatial changes in the zonal flow which result from the presence of a resonant triad composed of a marginally unstable and two neutral baroclinic waves. The changes in the zonal flow were found at the second closure and shown to arise in part from the self-interaction of the marginal wave and *in part from the triad interaction* (in contrast to discrete Rossby wave interactions on an infinite $\beta$-plane). Furthermore, the time evolution of the zonal flow occurs on the time scale of the interaction. Based on these results we can anticipate that simple neutral planetary waves may, via the triad interaction mechanism, be capable of generating a zonal flow provided that in the $\beta$-plane representation, the physical domain be taken of finite north–south extent. This, the channel restriction is felt to be a more appropriate modelling approximation to the mid-latitude atmosphere than an infinite $\beta$-plane because (i) it provides a meridionally quantized wave spectrum, (ii) it restricts propagation of disturbances to the east–west direction and (iii) it isolates the mid-latitude domain, preventing any interference with the tropics.

An investigation of zonal flow generation on a mid-latitude "channel" by a triad of interacting discrete Rossby waves will constitute the first part of the paper. The generation of zonal flow on an infinite $\beta$-plane by the quartet resonance mechanism for a discrete Rossby wave spectrum will constitute the second part of the paper.

## 2. Dynamical equations

Consider a shallow layer of incompressible inviscid homogeneous fluid with a free surface, of mean depth $D$, rotating with angular velocity $\Omega$. Horizontal extent of the fluid shall either be taken as infinite or confined to a channel of infinite east–west extent and north–south width $L$. Following Rossby et al. (1939) effects of the earth's sphericity will be accounted for by allowing $\Omega$ to vary linearly with latitude coordinate $y'$, i.e.

$$2\Omega = f_0 + \beta' y'$$

If the scales ($L$, $D$, $L/U$, $U$, $DU/L$) are taken for the horizontal coordinates, the vertical coordinate, time, horizontal velocity and vertical velocity, respectively, the equation which governs the motion for small Rossby number $R_0 = Uf_0L$ is the quasi-geostrophic potential vorticity equation

$$\frac{\partial}{\partial t}(\nabla^2\psi - F\psi) + J(\psi, \nabla^2\psi - F\psi + \beta y) = 0 \quad (2.1)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$J(f, g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$

In (2.1) $x$ and $y$ are the nondimensional longitude and latitude coordinates (scaled with $L$) and $\psi$ is the nondimensional geostrophic pressure (scaled with ($UL$)). The nondimensional parameters which appear in (2.1) are

$$\beta = \frac{\beta' L^2}{U} \quad \text{(the planetary vorticity factor)}$$

$$F = \frac{f_0^2 L^2}{gD} \quad \text{(the rotational Froude number)}$$

In the case of channel flow, normal velocity of the fluid must vanish at the sidewalls, i.e.

$$\frac{\partial\psi}{\partial x} = 0 \quad \text{at} \quad y = 0, 1 \quad (2.2a)$$

$$\lim_{2X\to\infty} \int_{-X}^{X} \frac{\partial^2\psi}{\partial t\,\partial y}\,dx = 0 \quad \text{at} \quad y = 0, 1 \quad (2.2b)$$

The last boundary condition is obtained by integrating the $x$-direction momentum equation and serves to uniquely determine the zonal ftow (i.e. the $x$-independent part of $\psi$).

On an infinite $\beta$-plane (2.1) permits as an exact solution a westward propagating wave

$$\psi = Ae^{i(kx+ly-\sigma t)} + * \quad (2.3)$$

provided its frequency $\sigma$ is related to its wavenumber $\mathbf{K} = (k, l)$ by the dispersion relation

$$\sigma = -\beta k/(k^2 + l^2 + F) \tag{2.4}$$

In the channel geometry (2.1) permits as an exact solution which satisfies (2.2a) a westward propagating wave whose structure is

$$\psi = A \sin m\pi y e^{i(kx-\sigma t)} + * \tag{2.5}$$

provided

$$\sigma = -\beta k/(k^2 + m^2\pi^2 + F)$$
$$m = \pm 1, 2, 3 \ldots \tag{2.6}$$

To investigate resonant interaction among discrete Rossby waves the basic spectrum must be taken to consist of waves as in (2.3) or (2.5) whose wavenumbers and frequencies satisfy the kinematic resonance conditions. In contrast to a single wave, such a superposition of waves is an exact solution only to the linearized version of (2.1), which requires that the amplitudes of the waves be infinitesimal. Following Benny (1962), the finite-amplitude interaction problem may be handled by considering a perturbation expansion of the wave field which to the lowest order is a superposition of solutions as in (2.3) or (2.5) whose amplitudes (taken finite but small) are slowly varying functions of time. We write

$$\psi = \varepsilon\psi^{(1)} + \varepsilon^2\psi^{(2)} + \varepsilon^3\psi^{(3)} + \ldots \tag{2.7}$$

where $\varepsilon \ll 1$ is the measure of the amplitude. If we introduce the additional time scales to handle the slow amplitude variations

$$T_1 = \varepsilon t; \quad T_2 = \varepsilon^2 t; \ldots \tag{2.8}$$

the differential time operator in (2.1) transforms as

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2} + \cdots \tag{2.9}$$

Substitution of (2.7) and (2.9) into (2.1) yields a sequence of linear problems for $\psi^{(n)}$. We proceed now to investigate these in detail starting with the triad interaction in the channel geometry.

## 3. Finite-amplitude analysis

### 3.1. *A mid-latitude channel*

The $0(\varepsilon)$ problem associated with (2.1), (2.2), (2.7) and (2.9) is

$$\frac{\partial}{\partial t}(\nabla^2\psi^{(1)} - F\psi^{(1)}) + \beta\frac{\partial\psi^{(1)}}{\partial x} = 0$$
$$\frac{\partial\psi^{(1)}}{\partial x} = 0 \quad \text{at} \quad y = 0, 1 \tag{3.1}$$

A solution of (3.1) may be written in the form of three waves

$$\psi^{(1)} = \sum_{j=1}^{3} A_j(T_1, T_2, \ldots)\sin m_j\pi y e^{i(k_j x - \sigma_j t)} \tag{3.2}$$

provided

$$\sigma_j = \frac{-\beta k_j}{k_j^2 + m_j^2\pi^2 + F} \tag{3.3}$$

The triplet (3.2) will form a resonant triad if the following kinematic side conditions are satisfied

$$k_1 + k_2 + k_3 = 0$$
$$m_1 + m_2 + m_3 = 0 \tag{3.4}$$
$$\sigma_1 + \sigma_2 + \sigma_3 = 0$$

with each $\sigma_j$ given by (3.3). Existence of such triads for Rossby waves is established and thoroughly discussed by Longuet-Higgins & Gill (1967).

The $0(\varepsilon^2)$ problem associated with (2.1), (2.2), (2.7) and (2.9) is

$$\frac{\partial}{\partial t}(\nabla^2\psi^{(2)} - F\psi^{(2)}) + \beta\frac{\partial\psi^{(2)}}{\partial x} = -\frac{\partial}{\partial T_1}$$
$$\times (\nabla^{(2)}\psi^{(1)} - F\psi^{(1)}) - J(\psi^{(1)}, \nabla^2\psi^{(1)} - F\psi^{(1)})$$
$$\frac{\partial\psi^{(2)}}{\partial x} = 0 \quad \text{at} \quad y = 0, 1 \tag{3.5}$$

The inhomogeneities in (3.5) may be evaluated using (3.2). We have then

$$\frac{\partial}{\partial t}(\nabla^2 - F)\psi^{(2)} + \beta\frac{\partial\psi^{(2)}}{\partial x} = \sum_{j=1}^{3}(K_j^2 + F)\frac{dA_j}{dT_1}$$
$$\sin m_j\pi y e^{i\theta_j} + * + \sum_{1,2,3} i[N_1 \sin(m_2 + m_3)\pi y$$
$$+ M_1 \sin(m_2 - m_3)\pi y](A_2^* A_3^* e^{-i(\theta_2+\theta_3)} + *)$$
$$- \sum_{1,2,3} i[N_1 \sin(m_2 - m_3)\pi y + M_1 \sin(m_2$$
$$+ m_3)\pi y](A_2 A_3^{*(\theta_2-\theta_3)} + *) \tag{3.6}$$

where

$$\theta_j = k_j x - \sigma_j t$$

$$K_j^2 = k_j^2 + m_j^2 \pi^2$$

$$N_1 = \frac{\pi}{2}(k_2 m_3 - k_3 m_2)(K_2^2 - K_3^2)$$

$$M_1 = \frac{\pi}{2}(k_2 m_3 + k_3 m_2)(K_2^2 - K_3^2)$$

and

$$\sum_{1,2,3} a_1 b_2 c_3 = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2$$

Among the inhomogeneities in (3.6), those which are resonant with the linear operator on the left hand side, are (i) all three terms arising from the long time inhomogeneity and (ii), when conditions (3.4) are satisfied, the first three of the twelve terms arising from nonlinear inhomogeneity. Their removal, necessary to maintain the validity of expansion (2.7), yields

$$\begin{aligned}
\frac{dA_1}{dT_1} &= \frac{iN_1}{K_1^2 + F} A_2^* A_3^* \\
\frac{dA_2}{dT_1} &= \frac{iN_2}{K_2^2 + F} A_3^* A_1^* \\
\frac{dA_3}{dT} &= \frac{iN_3}{K_3^2 + F} A_1^* A_2^*
\end{aligned} \tag{3.7}$$

Eqs. (3.7) are the standard triad resonance amplitude equations whose discussion and solution for Rossby waves may be found in Longuet-Higgins & Gill (1967).

The remaining inhomogeneities in (3.6) give rise to nine forced $0(\varepsilon^2)$ solutions

$$\psi_f^{(2)} = \sum_{1,2,3} \frac{M_1}{\beta \alpha_1 4\pi^2 m_2 m_3} A_2^* A_3^* \sin(m_2 - m_3) \pi y e^{i\theta_1} + *$$

$$- \sum_{1,2,3} \frac{N_1}{\left\{ \begin{array}{l} (\sigma_2 - \sigma_3)[(k_2 - k_3)^2 + \\ \qquad \pi^2(m_2 - m_3)^2 + F] + (k_2 - k_3)\beta \end{array} \right\}} \times A_2 A_3^* \sin(m_2 - m_3)\pi y e^{i(\theta_2 - \theta_3)} + *$$

$$+ \sum_{1,2,3} \frac{M_1}{\left\{ \begin{array}{l} (\sigma_2 - \sigma_3)[(k_2 - k_3)^2 + \\ \qquad \pi^2 m_1^2 + F] + (k_2 - k_3)\beta \end{array} \right\}} \times A_2 A_3^* \sin m_1 \pi y e^{i(\theta_2 - \theta_3)} + * \tag{3.8}$$

To the above solutions we can add an homogeneous $0(\varepsilon^2)$ solution $\Phi^{(2)}(y, T)$ which satisfies (3.5) trivially and which represents the *zonal flow*. Its structure will be determined by the $0(\varepsilon^3)$ problem. Thus

$$\psi^{(2)} = \psi_f^{(2)} + \Phi^{(2)}(y, T) \tag{3.9}$$

and the $0(\varepsilon^3)$ problem associated with (2.1), (2.7), (2.9) and (3.9) is

$$\begin{aligned}
&\frac{\partial}{\partial t}(\nabla^2 \psi^{(3)} - F\psi^{(3)}) + \beta \frac{\partial \psi^{(3)}}{\partial x} = -\frac{\partial}{\partial T_2} \\
&(\nabla^2 \psi^{(1)} - F\psi^{(1)}) - \frac{\partial}{\partial T_1}\left(\frac{\partial^2}{\partial y^2} - F\right)\Phi^{(2)} \\
&-J(\psi^{(1)}, \nabla^2 \psi_f^{(2)} - F\psi_f^{(2)}) - J(\psi_f^{(2)}, \\
&\text{a } \nabla^2 \psi^{(1)} - F\psi^{(1)}) \\
&-J\left(\psi^{(1)}, \frac{\partial^2 \Phi^{(2)}}{\partial y^2} - F\Phi^{(2)}\right) - J(\Phi^{(2)}, \\
&\text{a } \nabla^2 \psi^{(1)} - F\psi^{(1)})
\end{aligned} \tag{3.10}$$

Among the inhomogeneities in (3.10) are those which are independent of $(x, t)$. They are contained in the following terms:

$$\frac{\partial}{\partial T_1}\left(\frac{\partial^2}{\partial y^2} - F\right)\Phi^{(2)}$$

$$J(\psi^{(1)}, \nabla^2 \psi_f^{(2)} - F\psi_f^{(2)})$$

$$J(\psi_f^{(2)}, \nabla^2 \psi^{(1)} - F\psi^{(1)})$$

Considering the form of the linear operator on the left-hand side of (3.10) these must vanish identically. Their removal yields the equation which governs the evolution of the mean field $\Phi^{(2)}(y, T)$.

Sparing the reader from algebraic details we have

$$\frac{\partial}{\partial T_1}\left(\frac{\partial^2}{\partial y^2} - F\right)\Phi^{(2)} = \frac{K_1^2 + F}{2\beta N_1}\frac{d|A_1|^2}{dT_1}$$
$$\times \sum_{1,2,3} [M_2(K_2^2 + F) - M_3(K_3^2 + F)]$$
$$\times\ 2m_1\pi \sin 2m_1\pi y \qquad (3.11)$$

Inspection of the mean field producing inhomogeneities shows that it is generated by interactions between the free modes $\psi^{(1)}$ given by (3.2) and the first three forced modes $\psi_f^{(2)}$ in (3.8).

Solution of (3.11), subject to boundary condition (2.2b) (which to $0(\varepsilon^3)$ has the form $\partial^2\Phi^{(2)}/\partial T_1\partial y = 0$ at $y = 0, 1$) and such that the mean streamfunction is zero initially, is

$$\Phi^{(2)}(y, T_1) = \frac{K_1^2 + F}{2\beta N_1}(|A_1|^2 - |A_1(0)|^2)$$
$$\times \sum_{1,2,3} \frac{4\pi^2 m_1^2}{4\pi^2 m_1^2 + F}[M_2(K_2^2 + F) - M_3(K_3^2 + F)]$$
$$\left(\frac{\sinh\sqrt{F}(y - \frac{1}{2})}{\sqrt{F}\cosh\sqrt{F/2}} - \frac{\sin 2m_1\pi y}{2m_1\pi}\right) \qquad (3.12)$$

It follows that the mean zonal current generated by the interaction mechanism is

$$U^{(2)}(y, T_1) = -\frac{\partial\Phi_2}{\partial y} = \frac{K_1^2 + F}{2\beta N_1}$$
$$\times\ (|A_1|^2 - |A_1(0)|^2)\sum_{1,2,3}\frac{4\pi^2 m_1^2}{4m_1^2\pi^2 + F}$$
$$\times\ [M_2(K_2^2 + F) - M_3(K_3^2 + F)]$$
$$\left(\cos 2m_1\pi y - \frac{\cosh\sqrt{F}(y - \frac{1}{2})}{\cosh\sqrt{F/2}}\right) \qquad (3.13)$$

A typical latitudinal structure of the zonal current corresponding to

$$\frac{|A_1|^2 - |A_1(0)|^2}{N_1} > 0$$

is given in Fig. 1. The figure illustrates that the strongest zonal current is generated in the central third of the channel where the current is nearly twice the magnitude and opposite in direction to that generated in either the northern or southern thirds of the channel. This spatial structure, resulting solely from the triad interaction mechanism, is qualitatively similar to that obtained by Phillips (1954) from the rectified self interaction of a finite-amplitude baroclinic wave. Quantitatively, by comparing Fig. 1 with Phillips (1954) Fig. 5, we note that in Phillip's context the strong central westerly jet is 50% wider whereas the weaker easterly north and south jets are 30% narrower. It appears, therefore, that the triad interaction mechanism is capable of generating zonal flows of the same order and similar spatial distribution as the baroclinic instability mechanism.

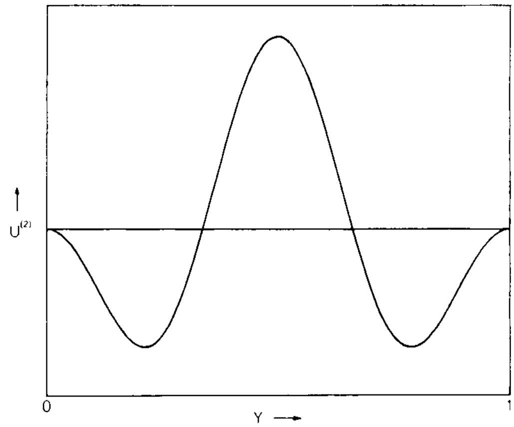


*Fig. 1.* Typical latitudinal distribution of the zonal current corresponding to $m_1 = m_3 = 1$ and $m_2 = -2$.

Following Loesch (1974) it can be shown, by zonally averaging the order Rossby number $x$-direction momentum equation, that the time change of the zonal current (3.13) is due to the mean Coriolis force (which arises from a weak meridional circulation) and due to divergence of zonally averaged Reynolds stresses, i.e.

$$\frac{\partial}{\partial T_1}U^{(2)} = \bar{v}_1 + \frac{\partial}{\partial y}\overline{\left(\frac{\partial\psi^{(1)}}{\partial x}\frac{\partial\psi_f^{(2)}}{\partial y} + \frac{\partial\psi^{(1)}}{\partial y}\frac{\partial\psi_f^{(2)}}{\partial x}\right)} \qquad (3.14)$$

In (3.14) bar represents $x$-averaged quantities and $\bar{v}_1$ is the order Rossby number nondimensional horizontal component of the mean meridional circulation. With the help of (3.2) and (3.8) we find that

$$\frac{\partial}{\partial y}\left(\overline{\frac{\partial \psi^{(1)}}{\partial x}\frac{\partial \psi_f^{(2)}}{\partial y}+\frac{\partial \psi^{(1)}}{\partial y}\frac{\partial \psi_f^{(2)}}{\partial x}}\right)=\frac{K_1^2+F}{2\beta N_1}\frac{d}{dT_1}$$

$$\times\ |A_1|^2 \sum_{1,2,3}\ [M_2(K_2^2+F)-M_3(K_3^2+F)] \times \cos 2m_1\pi y \tag{3.15}$$

and it follows with the help of (3.13) and (3.14) that

$$\bar{\sigma}_1=\frac{-(K_1^2+F)}{2\beta N_1}\frac{d}{dT_1}|A_1|^2\sum_{1,2,3} \times \frac{[M_2(K_2^2+F)-M_3(K_3^2+F)]}{4\pi^2m_1^2+F} \times \left(4\pi^2m_1^2\frac{\cosh\sqrt{F}\,(y-\frac{1}{2})}{\cosh\sqrt{F/2}}+F\cos 2m_1\pi y\right) \tag{3.16}$$

The vertical component of the mean meridional circulation follows from the zonally averaged continuity equation for an incompressible fluid

$$\left(\frac{\partial \bar{v}_1}{\partial y}+\frac{\partial \bar{w}_1}{\partial z}=0\right):$$

$$\bar{w}_1=-z\frac{\partial \bar{v}_1}{\partial y}=z\frac{K_1^2+F}{2\beta N_1}\frac{d}{dT_1}|A_1|^2\sum_{1,2,3} \times \frac{[M_2(K_2^2+F)-M_3(K_3^2+F)]}{4\pi m_1^2+F}$$

$$\left[4\pi^2m_1^2\sqrt{F}\frac{\sinh\sqrt{F}(y-\frac{1}{2})}{\cosh\sqrt{F/2}}-2m_1\pi F\sin 2m_1\pi y\right] \tag{3.17}$$

The weak meridional circulation arises because of the presence of free surface whose shape can adjust to the motions within the fluid. If the free surface is replaced by a rigid horizontal upper surface the term $F\psi$ in the governing equation (2.1) vanishes. In this case, as can easily be shown by setting $F=0$ in (3.11),

$$U^{(2)}=-\frac{K_1^2}{2\beta N_1}(|A_1|^2-|A_1(0)|^2)\sum_{1,2,3} \times (M_2K_2^2-M_3K_3^2)\cos 2m_1\pi y \tag{3.18}$$

$$\frac{\partial}{\partial T_1}U^{(2)}=\frac{\partial}{\partial y}\left(\overline{\frac{\partial \psi^{(1)}}{\partial x}\frac{\partial \psi_f^{(2)}}{\partial y}+\frac{\partial \psi^{(1)}}{\partial y}\frac{\partial \psi_f^{(2)}}{\partial x}}\right) \tag{3.19}$$

and

$$\bar{v}_1=\bar{w}_1=0 \tag{3.20}$$

Thus, as physically required by the rigid upper surface, there is no mean meridional circulation produced. The zonal current is now generated solely by the mean Reynolds stresses which in this, as well as the free surface case, arise from the interaction between the $O(\varepsilon)$ free modes and the $O(\varepsilon^2)$ forced modes.

### 3.2 *Infinite β-plane*

In this configuration, in contrast to the channel configuration, the structure of the $O(\varepsilon^2)$ forced solutions is such that there are no mean terms produced at $O(\varepsilon^3)$. Based on this and the results of Longuet-Higgins & Gill (1967), we conclude that, on an infinite $\beta$-plane, interacting Rossby waves are incapable of generating zonal flows on the time scale $T_1$. As pointed out by Newell (1969), generation of the mean flow occurs first on the time scale $T_2=\varepsilon^2 t$, and for the discrete Rossby wave spectrum, solely via the quartet resonance mechanism. The details of this mechanism are investigated below.

Suppose we have a triplet of Rossby waves

$$\sum_{j=1}^{3} A_j e^{i\phi_j}+*;\quad \phi_j=k_jx+l_jy-\sigma_jt \tag{3.21}$$

whose wavenumbers $\mathbf{K}_j=(k_j,l_j)$ and frequencies $\sigma j(\mathbf{K}_j)$ satisfy the triad kinematic resonance conditions

$$k_1+k_2+k_3=0$$
$$l_1+l_2+l_3=0 \tag{3.22a,b,c}$$
$$\sigma_1+\sigma_2+\sigma_3=0$$

or equivalently

$$\phi_1 + \phi_2 + \phi_3 = 0$$

with each $\sigma_j$ given by the dispersion relation

$$\sigma_j = \frac{-\beta k_j}{k_j^2 + l_j^2 + F} \tag{3.23}$$

Following Newell (1969) we note that if a quadruplet of Rossby waves

$$A_1 e^{i\phi_1} + A_2 e^{i\phi_2} + A_4 e^{i\phi_4} + A_5 e^{i\phi_5} + * \tag{3.24}$$

is chosen such that $\mathbf{K}_1$ and $\mathbf{K}_2$ are as in (3.22) and

$$\mathbf{K}_4 = (k_3, -l_3)$$
$$\mathbf{K}_5 = (0, 2l_3) \tag{3.25}$$

its wavenumbers and frequencies will also satisfy resonance conditions (3.22) [such wavenumber choice for modes 4 and 5 satisfies (3.22a,b) automatically and, since by (3.23) $\sigma_4 = \sigma_3$ and $\sigma_5 = 0$, (3.22c) is also satisfied]. The above choice of mode 5 yields a zonal flow of a simple sinusoidal north-south structure as a member of a resonant quartet. Its time structure may be obtained by considering cubic interactions in (2.1).

The finite-amplitude analysis may be carried out by choosing for a solution of the $0(\varepsilon)$ problem associated with (2.1), (2.7) and (2.9) a superposition of five Rossby waves, i.e.,

$$\psi^{(1)} = \sum_{j=1}^{5} A_j(T_1, T_2, \ldots) e^{i\phi_j} + * \tag{3.26}$$

with wavenumbers $\mathbf{K}_4$ and $\mathbf{K}_5$ as given by (3.25). The resonance conditions (3.22), as satisfied by modes 1, 2, 4 and 5 may be written

$$\phi_1 + \phi_2 + \phi_4 + \phi_5 = 0 \tag{3.27}$$

Mode 3 is included in the $0(\varepsilon)$ spectrum because it is *secularly* produced at $0(\varepsilon^2)$ by interactions between modes 1 and 2 and modes 4 and 5, i.e.,

$$\phi_1 + \phi_2 + \phi_3 = 0 \tag{3.28}$$

$$\phi_3 = \phi_4 + \phi_5 \tag{3.29}$$

The $0(\varepsilon^2)$ problem is given by the inhomogeneous equation (3.5). Evaluation of the inhomogeneities with the help of (3.26) yields

$$\frac{\partial}{\partial t}(\nabla^2 - F)\psi^{(2)} + \beta \frac{\partial \psi^{(2)}}{\partial x} = \sum_{j=1}^{5} (K_j^2 + F) \times \frac{dA_j}{dT_1} e^{i\phi_j} + *$$
$$+ \sum_{\substack{m,n=1 \\ m>n}}^{5} (K_m^2 - K_n^2)(k_m l_n - k_n l_m) \times [A_m A_n e^{i(\phi_m + \phi_n)} - A_m A_n^* e^{i(\phi_m - \phi_n)}] + * \tag{3.30}$$

Application of the triad resonance conditions (3.28) and (3.29) and subsequent removal of all inhomogeneities in (3.30) which are resonant with the linear operator on the left-hand side gives five coupled nonlinear equations for the evolution of $A_j$ on the time scale $T_1$:

$$\begin{aligned}
\frac{dA_1}{dT_1} &= \frac{(k_2 l_3 - k_3 l_2)(K_3^2 - K_2^2)}{K_1^2 + F} A_2^* A_3^* \\
\frac{dA_2}{dT_1} &= \frac{(k_3 l_1 - k_1 l_3)(K_1^2 - K_3^2)}{K_2^2 + F} A_1^* A_3^* \\
\frac{dA_3}{dT_1} &= \frac{(k_1 l_2 - k_2 l_1)(K_2^2 - K_1^2)}{K_3^2 + F} A_1^* A_2^* \\
&\quad - \frac{2k_3 l_3 (K_3^2 - 4l_3^2)}{K_3^2 + F} A_4 A_5 \\
\frac{dA_4}{dT_1} &= \frac{2k_3 l_3 (K_3^2 - 4l_3^2)}{K_3^2 + F} A_3 A_5^* \\
\frac{dA_5}{dT_1} &= 0
\end{aligned} \tag{3.31}$$

We note that the amplitude of the zonal flow $A_5$ does not vary on the time scale $T_1$. But the zonal flow is, to $0(\varepsilon^2)$, a member of the resonant triad composed of modes 3, 4, and 5 (because this triplet satisfies conditions (3.29)). Thus (3.31) shows, as first noted by Longuet-Higgins & Gill (1967), that in a triad configuration, the zonal flow acts as a catalyst enabling energy to be exchanged between the other two members of the triad, i.e. modes 4 and 5. Its variations will be determined at the next order, i.e. $0(\varepsilon^3)$, where cubic interactions are taken into account.

The nonresonant inhomogeneities in (3.30) give rise to eleven $0(\varepsilon^2)$ solutions

$$\psi^{(2)} = \sum_{j=2}^{5} \frac{i(k_1 l_j - k_j l_1)(K_1^2 - K_j^2)}{\left\{\begin{array}{l}(\sigma_1 - \sigma_j)[(k_1 - k_j)^2 + \\ \qquad (l_1 - l_j)^2 + F] + \beta(k_1 - k_j)\end{array}\right\}} \times A_1 A_j^* e^{i(\phi_1 - \phi_j)}$$

$$+ \sum_{j=3}^{5} \frac{i(k_2 l_j - k_j l_2)(K_2^2 - K_j^2)}{\left\{\begin{array}{l}(\sigma_2 - \sigma_j)[(k_2 - k_j)^2 + \\ \qquad (l_2 - l_j)^2 + F) + \beta(k_2 - k_j)\end{array}\right\}} \times A_2 A_j^* e^{i(\phi_2 - \phi_j)}$$

$$+ \frac{2ik_3 l_3 (K_3^2 - 4l_3^2)}{\sigma_3(k_3^2 + 9l_3^2 + F) + \beta k_3} [A_4 A_5^* e^{i(\phi_4 - \phi_5)} - A_3 A_5 e^{i(\phi_3 + \phi_5)}]$$

$$+ \left[\frac{i(k_2 l_3 + k_3 l_2)(K_2^2 - K_3^2)}{\sigma_1[k_1^2 + (l_1 + 2l_3)^2 + F] + \beta k_1} A_2^* A_4^* - \frac{2ik_1 l_3 (K_1^2 - 4l_3^2)}{\sigma_1[k_1^2 + (l_1 + 2l_3^2) + F] + \beta k_1} A_1 A_5\right] e^{i(\phi_1 + \phi_5)}$$

$$+ \left[\frac{i(k_1 l_3 + k_3 l_1)(K_1^2 - K_3^2)}{\sigma_2[k_2^2 + (l_2 + 2l_3)^2 + F] + \beta k_2} A_1^* A_4^* - \frac{2ik_2 l_3 (K_2^2 - 4l_3^2)}{\sigma_2[k_2^2 + (l_2 + 2l_3)^2 + F] + \beta k_2} A_2 A_5\right] \times e^{i(\phi_2 + \phi_5)} + * \tag{3.32}$$

The $0(\varepsilon^3)$ problem associated with (2.1), (2.7) and (2.9) is

$$\frac{\partial}{\partial t}(\nabla^2 \psi^{(3)} - F\psi^{(3)}) + \beta \frac{\partial \psi^{(3)}}{\partial x} = -\frac{\partial}{\partial T_2} \times (\nabla^2 \psi^{(1)} - F\psi^{(1)}) - J(\psi^{(1)}, \nabla^2 \psi^{(2)} - F\psi^{(2)}) - J(\psi^{(2)}, \nabla^2 \psi^{(1)} - F\psi^{(1)}) \tag{3.33}$$

Since $\psi^{(1)}$ and $\psi^{(2)}$ are known from (3.26) and (3.32), respectively, the inhomogeneity in (3.33) can be easily evaluated. Contributions which are resonant with the linear operator either as a result of the structure of $\psi^{(1)}$ and $\psi^{(2)}$ or as a result of the quartet resonance conditions (3.27) must be removed. Their removal yields

$$\begin{aligned}
\frac{dA_1}{dT_2} &= iA_1(C_{12}|A_2|^2 + C_{13}|A_3|^2 + C_{14}|A_4|^2 + C_{15}|A_5|^2) + in_1 A_2^* A_4^* A_5^* \\
\frac{dA_2}{dT_2} &= iA_2(C_{21}|A_1|^2 + C_{23}|A_3|^2 + C_{24}|A_4|^2 + C_{25}|A_5|^2) + in_2 A_1^* A_4^* A_5^* \\
\frac{dA_3}{dT_2} &= iA_3(C_{31}|A_1|^2 + C_{32}|A_2|^2 + C_{35}|A_5|^2) \\
\frac{dA_4}{dT_2} &= iA_4(C_{41}|A_1|^2 + C_{42}|A_2|^2 + C_{45}|A_5|^2) + in_4 A_1^* A_2^* A_5^* \\
\frac{dA_5}{dT_2} &= iA_5(C_{51}|A_1|^2 + C_{52}|A_2|^2 + C_{53}|A_3|^2 + C_{54}|A_4|^2) + in_5 A_1^* A_2^* A_4^*
\end{aligned} \tag{3.34}$$

The set (3.34) is a tenth order system of coupled nonlinear equations governing the evolution of amplitudes and phases of the five modes concerned on time scale $T_2$. Equations of this type have been first derived by Benny (1962) for a quartet of resonantly interacting gravity waves. The coefficients $C_{ij}$ and $n_j$ in (3.34) are all real functions of the wavenumbers $\mathbf{K}_j$. Their detailed structure is given in the Appendix. The terms on the right-hand side of (3.34) are of two types. Those inside the brackets, associated with coefficients $C_{ij}$, are all real and represent the turning of the complex wave amplitudes $A_j$ without affecting their magnitude. Physically this represents the modification of the wave frequencies $\sigma_j$ resulting from the interactions. The remaining terms, associated with the coefficients $n_j$, do affect the magnitude of $A_j$ and therefore describe the energy transfers among the modes. We note that mode $j = 3$, which is not a member of the resonant quartet (3.27), is excluded from the energy exchange on time scale $T_2$. The energy exchange among the remaining modes $j$ = 1, 2, 4 and 5 can best be illustrated by the integrals of (3.34), i.e.,

$$\frac{|A_1|^2 - |A_1(0)|^2}{n_1} = \frac{|A_2|^2 - |A_2(0)|^2}{n_2} = \frac{|A_4|^2 - |A_4(0)|^2}{n_3} = \frac{|A_5|^2 - |A_5(0)|^2}{n_4} \tag{3.35}$$

which show that these four modes form a group in which growth of one mode must be compensated for by a decay of another. It follows, if three of the interacting modes $j = 1$, 2 and 4 are present initially, they can generate the fourth mode $j = 5$, which here represents the zonal flow, from rest, to become an order-one mode on the time scale $T_2$.

## 4. Concluding remarks

The preceding results illustrate the capability of simple discrete Rossby waves to generate zonal flows through nonlinear resonant interactions. On an infinite beta plane this capability is weaker than in the physically more realistic channel configuration. Zonal currents produced in the channel geometry bear a close resemblance in spatial structure to those obtained by the baroclinic stability theory in this geometry, and contain the gross features of the observed latitudinal mean wind distribution in the lower atmosphere. Since the interaction mechanism yields a narrower westerly jet in the central portion of the channel accompanied by wider easterly flow to the north and south than the baroclinic instability mechanism, the combined picture leads to a stronger and narrower mid-latitude westerly jet than that produced by baroclinic instability alone. A similar effect was observed in the combined interaction-instability baroclinic problem by Loesch (1974). It is felt, therefore, that the resonant interaction mechanism may be one of the important mechanisms in strengthening and maintaining the mid-latitude jet against frictional dissipation. Furthermore, by altering the structure of the mean flow, the interaction mechanism affects the stability properties of the atmosphere, thereby influencing cyclogenesis and shorter term climatic changes.

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## 6. Appendix

Below are given the coefficients $n_j$ and $C_{ij}$ in eq. (3.34):

$$n_1 = \frac{\left\{ \begin{array}{l} k_2(k_2 l_3 + k_3 l_2 + 2k_3 l_3) \\ \qquad (K_2^2 - K_3^2 - 4l_1 l_3)(4l_3^2 - K_2^2) \end{array} \right\}}{2l_1\sigma_2(K_1^2 + F)} - \frac{k_1(k_2 l_3 + k_3 l_2)(K_1^2 + 4l_1 l_3)(K_2^2 - K_3^2)}{2l_2\sigma_1(K_1^2 + F)}$$

$$n_2 = \frac{\left\{ \begin{array}{l} k_1(k_1 l_3 + k_3 l_1 + 2k_3 l_3) \\ \qquad (K_1^2 - K_3^2 - 4l_2 l_3)(4l_3^2 - K_1^2) \end{array} \right\}}{2l_2\sigma_1(K_2^2 + F)} - \frac{k_2(k_1 l_3 + k_3 l_1)(K_2^2 + 4l_2 l_3)(K_1^2 + K_3^2)}{2l_1\sigma_2(K_2^2 + F)}$$

$$n_4 = \frac{\left\{ \begin{array}{l} k_2(k_1 l_2 - k_2 l_1 + 2k_1 l_3) \\ \qquad (K_2^2 - K_1^2 - 4l_1 l_3)(4l_3^2 - K_2^2) \end{array} \right\}}{2l_1\sigma_2(K_3^2 + F)} + \frac{\left\{ \begin{array}{l} k_1(k_2 l_1 - k_1 l_2 + 2k_3 l_3) \\ \qquad (K_1^2 - K_2^2 - 4l_2 l_3)(4l_3^2 - K_1^2) \end{array} \right\}}{2l_2\sigma_1(K_3^2 + F)}$$

$$n_5 = \frac{2k_1 l_3(k_2 l_3 + k_3 l_2)(K_3^2 - K_2^2)}{\sigma_1(4l_3^2 + F)} + \frac{2k_2 l_3(k_1 l_3 + k_3 l_1)(K_3^2 - K_1^2)}{\sigma_2(4l_3^2 + F)}$$

$$C_{12} = \frac{(k_1 l_2 - k_2 l_1)^2(K_2^2 - K_1^2)(K_1^2 - 2k_1 k_2 - 2l_1 l_2)}{\left\{ \begin{array}{l} (K_1^2 + F)[(\sigma_1 - \sigma_2)(K_1^2 + K_2^2 \\ \qquad + F - 2k_1 k_2 - 2l_1 l_2) + \beta(k_1 - k_2)] \end{array} \right\}}$$

$$C_{13} = \frac{(k_1 l_3 - k_3 l_1)^2(K_3^2 - K_1^2)(K_1^2 - 2k_1 k_3 - 2l_1 l_3)}{\left\{ \begin{array}{l} (K_1^2 + F)[(\sigma_1 - \sigma_3)(K_1^2 + K_3^2 \\ \qquad + F - 2k_1 k_3 - 2l_1 l_3) + \beta(k_1 - k_3)] \end{array} \right\}}$$

$$C_{14} = \frac{(k_1 l_3 + k_3 l_1)^2(K_3^2 - K_1^2)(K_1^2 - 2k_1 k_3 - 2l_1 l_3)}{\left\{ \begin{array}{l} (K_1^2 + F)[(\sigma_1 - \sigma_3)(K_1^2 + K_3^2 \\ \qquad + F - 2k_1 k_3 + 2l_1 l_3) + \beta(k_1 - k_3)] \end{array} \right\}} + \frac{\left\{ \begin{array}{l} (k_3 l_2 + k_2 l_3 + 2k_3 l_3)(k_1 l_3 + k_3 l_1) \\ \qquad (K_1^2 - K_3^2)(K_2^2 - K_3^2 - 4l_1 l_3) \end{array} \right\}}{(K_1^2 + F)4l_1 l_3 \sigma_2}$$

$$C_{15} = \frac{2k_1^2 l_3^2 (K_1^2 - 4l_3^2)(4l_1^2 - K_1^2)}{(K_1^2 + F)\sigma_1 l_2 (l_1 - l_3)}$$

$$C_{21} = \frac{(k_1 l_2 - k_2 l_1)^2 (K_2^2 - K_1^2)(K_2^2 - 2k_1 k_2 - 2l_1 l_2)}{\left\{ (K_2^2 + F)[(\sigma_1 - \sigma_2) K_1^2 + K_2^2 + F - 2k_1 k_2 - 2l_1 l_2) + \beta(k_1 - k_2)] \right\}}$$

$$C_{23} = \frac{(k_2 l_3 - k_3 l_2)^2 (K_3^2 - K_2^2)(K_2^2 - 2k_2 k_3 - 2l_2 l_3)}{\left\{ (K_2^2 + F)[(\sigma_2 - \sigma_3)(K_2^2 + K_3^2 + F - 2k_2 k_3 - 2l_2 l_3) + \beta(k_2 - k_3)] \right\}}$$

$$C_{24} = \frac{(k_2 l_3 + k_3 l_2)^2 (K_3^2 - K_2^2)(K_2^2 - 2k_2 k_3 - 2l_2 l_3)}{\left\{ (k_2^2 + F)[(\sigma_2 - \sigma_3)(K_2^2 + K_3^2 + F - 2k_2 k_3 + 2l_2 l_3) + \beta(k_2 - k_3)] \right\}} + \frac{\left\{ (k_3 l_1 + k_1 l_3 + 2k_3 l_3)(k_2 l_3 + k_3 l_2)(K_2^2 - K_3^2)(K_1^2 - K_3^2 - 4l_2 l_3) \right\}}{(K_2^2 + F) 4 l_2 l_3 \sigma_1}$$

$$C_{25} = \frac{2k_2^2 l_3^2 (K_2^2 - 4l_3^2)(4l_2^2 - K_2^2)}{(K_2^2 + F)\sigma_2 l_1 (l_2 - l_3)}$$

$$C_{31} = \frac{(k_1 l_3 - k_3 l_1)^2 (K_3^2 - K_1^2)(K_3^2 - 2k_1 k_3 - 2l_1 l_3)}{\left\{ (K_3^2 + F)[(\sigma_1 - \sigma_3)(K_1^2 + K_3^2 + F - 2k_1 k_3 - 2l_1 l_3) + \beta(k_1 - k_3)] \right\}}$$

$$C_{32} = \frac{(k_2 l_3 - k_3 l_2)^2 (K_3^2 - K_2^2)(K_3^2 - 2k_2 k_3 - 2l_2 l_3)}{\left\{ (K_3^2 + F)[(\sigma_2 - \sigma_3)(K_2^2 + K_3^2 + F - 2k_2 k_3 - 2l_2 l_3) + \beta(k_2 - k_3)] \right\}}$$

$$C_{35} = \frac{-k_3^2 (K_3^2 + 5l_3^2)(K_3^2 - 4l_3^2)}{2(K_3^2 + F)\sigma_3}$$

$$C_{41} = \frac{\left\{ (k_1 l_3 + k_3 l_1)^2 (K_3^2 - K_1^2)(K_3^2 - 2k_1 k_3 + 2l_1 l_3) \right\}}{\left\{ (K_3^2 + F)[(\sigma_1 - \sigma_3)(K_1^2 + K_3^2 + F - 2k_1 k_3 + 2l_1 l_3) + \beta(k_1 - k_3)] \right\}} + \frac{\left\{ (2k_1 l_3 + k_1 l_2 - k_2 l_1)(k_1 l_3 + k_3 l_1)(K_1^2 - K_3^2)(K_2^2 - K_1^2 + 4l_2 l_3) \right\}}{(K_3^2 + F) 4 l_1 l_3 \sigma_2}$$

$$C_{42} = \frac{(k_2 l_3 + k_3 l_2)^2 (K_3^2 - K_2^2)(K_3^2 - 2k_2 k_3 + 2l_2 l_3)}{\left\{ (K_3^2 + F)[(\sigma_2 - \sigma_3)(K_2^2 + K_3^2 + F - 2k_2 k_3 + 2l_2 l_3) + \beta(k_2 - k_3)] \right\}} + \frac{\left\{ (2k_2 l_3 + k_2 l_1 - k_1 l_2)(k_2 l_3 + k_3 l_2)(K_2^2 - K_3^2)(K_1^2 - K_2^2 + 4l_1 l_3) \right\}}{(K_3^2 + F) 4 l_2 l_3 \sigma_1}$$

$$C_{45} = C_{35}$$

$$C_{51} = C_{52} = 0$$

$$C_{53} = \frac{4k_3^2 l_3^2 (K_3^2 - 4l_3^2)}{(4l_3^2 + F)\sigma_3}$$

$$C_{54} = C_{53}$$

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## О ГЕНЕРАЦИИ ЗОНАЛЬНЫХ ПОТОКОВ ВЗАИМОДЕЙСТВУ[ЮЩИМИ ВОЛНАМИ РОССБИ

С помощью квазигеострофической баротропной модели на $\beta$-плоскости исследуется генерация зональных потоков резонансно взаимодействующими дискретными волнами Россби конечной амплитуды. В условиях "канала", когда конечно протяжение области с севера на юг, анализ показывает, что триада волн Россби способна генерировать зональное течение за время, характерное для резонансного взаимодействия. Про-изводимые таким образом зональные потоки выявляют пространственную структуру, аналогичную той, которая получается при механизме бароклинной неустойчивости: струя в центре канала и более слабые обратные течения в северной и южной частях канала. Если меридиональная протяжённость области берётся бесконечной, то дискретные волны Россби через четырёхволновый резонансный механизм могут генерировать зональные течения синусоидальной широтной структуры. Масштаб времени, требуемый для их генерации, на порядок величины длиннее, чем время, связанное с условиями "канала".