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Frictional effects on thermally forced waves

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ABSTRACT

A consistent linear theory is developed for determining the importance of boundary layer friction for thermally forced gravity and Rossby wave modes. Analytic case studies are developed on a mid-latitude β-plane in order to retain separability of the linearized equations in the presence of boundary layer friction in a slab mixed layer. It is found that frictional convergence is important in synoptic-scale Rossby waves (but not gravity waves). If the feedback between the boundary layer and the interior solution is not properly taken into account, however, it is found that the boundary layer convergence will be incorrectly evaluated and may be considerably overestimated.

1. Introduction

Wave motions in the tropics have been associated with forced internal equatorial wave modes. Both Rossby and mixed gravity–Rossby wave modes have been identified with observed synoptic-scale easterly waves in the tropics (Hayashi, 1970, 1971b; Holton, 1970; Yamasaki, 1969, 1971a; Lindzen, 1974a, 1974b) while gravity modes have been associated with cloud cluster-scale motions (Lindzen, 1974a). Inviscid internal waves have their own convergence field directly associated with their dynamic structure. The co-operative effect between the low-level moisture convergence due to the wave and the cumulus heating which drives the wave is called "wave-CISK" (Conditional Instability of the Second Kind).

Both Hayashi (1970) and Lindzen (1974a) solved a forced problem on an equatorial β-plane, where the heating was proportional to the vertical velocity at cloud base. The problem was set up as an eigenvalue problem for each of the equatorial modes by demanding that the vertical velocity at cloud base was such as to provide a heating which forced a vertical velocity equal to that necessary to provide the heating. The eigenvalue was the (complex) vertical wavelength or, equivalently, the (complex) frequency or growth rate.

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The effect of boundary layer friction on the structure and stability properties of equatorial wave modes has been investigated in several ways. Holton et al. (1971) and Yamasaki (1971b) used the pressure imposed by a free barotropic equatorial mode (Matsuno, 1966) to determine the boundary layer convergence in a neutral Ekman layer with a empirical eddy viscosity, where the pressure was taken constant throughout the layer. They found infinite convergence at the top of the boundary layer at the critical latitude, defined such that the Coriolis parameter equals the Doppler frequency of the wave. Hayashi (1971a) and Chang (1973) made similar calculations using a finite depth boundary layer and found that the singularity at the critical latitude was removed. Hayashi (1971a) also parameterized the neutral boundary layer by a linear frictional law. He found that the convergence field using the linear frictional drag law was quite similar to that using the viscous (Ekman-like) parameterization for both synoptic-scale mixed gravity–Rossby and Rossby modes. The boundary layer analyses of Holton et al. (1971), Yamasaki (1971b), Hayashi (1971a) and Chang (1973) are of limited applicability, however, to the study of internal equatorial wave modes. If the pressure imposed by an *a priori* determined inviscid mode is used to drive the boundary layer on an equatorial β-plane then the horizontal structure of the response will not be the same as that of the imposed mode since the presence of friction modifies the horizontal structure of the mode. To say it another way, the presence of a frictional boundary layer (either of the linear drag or viscous type) destroys the separability of the equations into a horizontal and vertical structure equation so that a single mode is not a proper solution to the complete system of equations. This was not a severe problem for the studies mentioned above since they were looking for the qualitative behavior of the boundary layer response and not a completely consistent solution.

A consistent internal (baroclinic) interior solution, however, does not exist except in the presence of heating. Boundary layer convergence serves as both a moisture (and therefore heating) source which drives the motion and as a lower boundary condition on the interior. Thus a completely consistent solution with the interior solution determined by the boundary layer convergence as well as vice versa is necessary for a true picture of the structure and response of a frictionally modified wave-CISK mode. First, the feedback from the boundary layer to the interior will couple the interior modes. Second, the presence of the boundary layer itself modifies the pressure "imposed" by the interior solution so that the actual pressure which drives the boundary layer is, for a given heating, in general less than that for the inviscid solution.

Hayashi (1971b) correctly determined a consistent internal frictional solution on the equatorial β-plane as an extension of his inviscid eigenvalue calculation (Hayashi, 1970) by calculating the complete solution for a given eigenvalue including boundary layer friction on a two-dimensional grid. The solutions were consistent extensions of single inviscid interior modes which properly incorporated the modification of the interior pressure by the boundary layer. Yamasaki (1969, 1971a) also numerically evaluated the structure of the most unstable modes including a linear surface drag and a vertically varying mean flow.

Holton (1975) has recently used a barotropic model to evaluate the effect of boundary layer friction on wave convergence. The pressure specified by a given inviscid barotropic mixed gravity–Rossby mode with a given equivalent depth is used to drive a frictional slab mixed layer. The ratio between the convergence in the boundary layer and the convergence in the inviscid interior is considered to measure the importance of boundary layer friction in controlling the low-level moisture convergence. However, since the interior pressure is specified in Holton's model and there is no feedback between the boundary layer and the interior, the importance of boundary layer friction may be considerably overestimated. In a consistent baroclinic formulation (which the barotropic system is supposed to model) the presence of the boundary layer itself modifies the pressure "imposed" by the interior through an effective boundary condition between vertical velocity and pressure at the top of the boundary layer. Since the feedback generally reduces the pressure, the resulting boundary layer convergence is correspondingly reduced. Holton does attempt to include some feedback between the boundary layer and interior through an effective mass sink which models the cumulus heating due to boundary layer moisture convergence. However, even in this model the only feedback is through the heating, while in a completely consistent (baroclinic) solution the feedback is through both the heating and the effective boundary condition. In any case the inclusion of the mass sink in Holton's model does not qualitatively effect the solutions, so that the layers are effectively decoupled.

2. Definition of problem

It is the purpose of this paper to use as simple a model as possible in order to isolate and quantify the importance of boundary layer friction on low-level inviscid wave convergence and contrast the results when the boundary layer and interior are not coupled. The most direct means of determining the effect of boundary layer friction on thermally driven wave modes is to consider the response to a given heating, at the top of the boundary layer. Only at this level can a direct comparison be made with a barotropic model. The remainder of this paper will consider the formulation and solution of this forced problem, with and without friction. The relationship between the solutions to the forced problem and Holton's (1975) results will be given after the forced problem is formulated. The formulation is similar to those of Hayashi and Lindzen, but contains the following modifications and additions:

(1) Instead of solving the problem on an equatorial β-plane as is done by Lindzen (1974a), Hayashi (1970, 1971b) and Holton (1975), a mid-latitude β-plane is used. This has the disadvantage that equatorially trapped modes cannot be studied and the critical latitude must be avoided, but this disadvantage is slight since the modes of particular

interest here are gravity and Rossby modes, which are common to both the tropics and mid-latitudes. Thus the mixed gravity–Rossby mode explicitly studied by Holton (1975) cannot be evaluated. However, a simple study can be made on the mid-latitude β-plane using a measure of the importance of boundary layer friction identical to Holton's model. Thus the ability of the barotropic model to evaluate the influence of boundary layer friction can be isolated and quantified. The major advantage of the mid-latitude β-plane is that the linearized equations are separable even in the presence of boundary layer friction, since the coefficients are independent of both horizontal coordinates. This is in contrast with Hayashi (1971b) who had to use a two-dimensional numerical integration, since on the equatorial β-plane the presence of friction in the boundary layer changes the meridional structure, thereby coupling different horizontal modes in the vertical and making separation impossible.

(2) The vertical structure equation is solved using a constant static stability (in a modified vertical coordinate), and a forcing magnitude consistent with the moisture budget as in Hayashi (1970), but with a vertical distribution corresponding to that provided by a single non-entraining cumulus tower. Thus, the vertical distribution of heating is the same as that used in Lindzen (1974a).

(3) The vertical structure equation is solved analytically and a radiation condition is used at the top as in Lindzen (1974a). However, the vertical velocity forced by the specified heating is not required to equal the vertical velocity which provides the moisture flux so that an eigenvalue problem is not solved as in Hayashi (1970, 1971b) or Lindzen (1974a).

The modifications due to boundary layer friction will be studied using a linear surface drag law. The importance of frictional convergence is measured by direct comparison between inviscid and frictional cases so that the frictional contribution can be directly evaluated, instead of indirectly through growth rates as in Hayashi (1971b).

3. Formulation of equations and boundary conditions

Let $z^* = -H_0 \ln(p/p_0)$ be the vertical coordinate in log-pressure coordinates where p is the pressure, p_0 the surface pressure and H_0 is a constant arbitrary scale height (= 8.73 km in these calculations). The vertical velocity in log-pressure coordinates is

$$w^* = \frac{\partial z^*}{\partial t} + \mathbf{v}\cdot\nabla z^* + w\,\frac{\partial z^*}{\partial z} \tag{1}$$

where w is the true vertical velocity, $\mathbf{v}$ is the horizontal velocity, and the gradient is taken at constant z. The equations will be derived on a mid-latitude β-plane and are linearized on a basic state at rest. The momentum, continuity and thermodynamic equations are

$$\frac{\partial u}{\partial t} - fv = -\frac{\partial \Phi}{\partial x} + F_x \tag{2}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{\partial \Phi}{\partial y} + F_y \tag{3}$$

$$\nabla\cdot\mathbf{v} + \frac{\partial w^*}{\partial z^*} - \frac{w^*}{H_0} = 0 \tag{4}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial z^*}\right) + w^* S = \frac{R}{C_p H_0}\,Q \tag{5}$$

where x and y are the zonal and meridional coordinates taken positive to the east and north, respectively. The coordinate system is centered at latitude θ_0 and $f = f_0 + \beta y$ is the Coriolis parameter where

$$f_0 = 2\Omega \sin\theta_0, \quad \beta = 2\Omega(\cos\theta_0)/R_0$$

Ω is the angular frequency of the earth's rotation and R_0 is the earth's radius. The system will be centered on latitude $\theta_0 = 15°$ in all calculations. Φ is the geopotential gz and S is the basic state static stability in log-pressure coordinates,

$$S = \frac{R}{H_0}\left(\frac{\partial T}{\partial z^*} + \frac{RT}{C_p H_0}\right)$$

where T is the basic state temperature, R is the gas constant, g is the acceleration due to gravity and C_p is the specific heat. Here S is taken to have the constant value 1.31×10^{-4} sec^{-2}. Q is the diabatic heating rate per unit mass and $\mathbf{F} = (F_x, F_y)$ is the friction.

In the well-mixed convective boundary layer in the tropics, in which the mixing of potential temperature implies a neutral lapse rate ($S = 0$) and in which the observed vertical shear of the wind is small, a vertically averaged boundary layer formulation is appropriate. This is the same formulation as is used by Holton (1975). The momentum equations can be integrated from the surface to the top of the boundary layer where the stresses are assumed to vanish. Then, using a linearized surface stress law $\boldsymbol{\tau}\ (z = 0) = \rho C_D U \mathbf{v}$ where $\boldsymbol{\tau}$ is the surface stress, ρ is the density, C_D is an empirical surface drag constant and U is a typical velocity magnitude, the appropriate frictional law in the boundary layer is found to be $\mathbf{F} = -D\mathbf{v}$ where $D = C_D U/z_B$. Taking the curl of the momentum equations and letting $\zeta = v_x - u_y$,

$$\zeta_t + f\,\nabla\cdot\mathbf{v} + \beta v = -D\zeta \tag{6}$$

Taking the divergence one obtains

$$(\nabla\cdot\mathbf{v})_t + \nabla^2\Phi - f\zeta + \beta u = -D(\nabla\cdot\mathbf{v}) \tag{7}$$

The vorticity and divergence equations have been derived so that the mid-latitude β-plane approximation can be consistently applied. If the meridional extent of the motion is not large, so that $|\beta y| \ll f_0$ then f may be replaced by f_0, thereby retaining first-order effects of the variation of f with latitude, but at the same time simplifying the analysis considerably by eliminating the y-dependence of the coefficients. The time and horizontal dependence may then be completely separated out from the equations by using a single Fourier component and assuming solutions of the form $\exp\{i(kx + ly - \sigma t)\}$ where k and l are zonal and meridional wavenumbers, respectively, and σ is the frequency. The velocity field is most conveniently decomposed into non-divergent and irrotational components so that $\mathbf{v} = \nabla\chi + \mathbf{n}\times\nabla\psi$ where ψ is the stream function and χ is the velocity potential. $\mathbf{n}$ is a vertical unit vector. Then (4), (5), (6) and (7) become

$$i\sigma(k^2 + l^2)\psi - f_0(k^2 + l^2)\chi + ik\beta\psi = D(k^2 + l^2)\psi \tag{8}$$

$$i\sigma(k^2 + l^2)\chi - (k^2 + l^2)\Phi + f_0(k^2 + l^2)\psi + ik\beta\chi = D(k^2 + l^2)\chi \tag{9}$$

$$-(k^2 + l^2)\chi + w^*_z - w^*/H_0 = 0 \tag{10}$$

$$-i\sigma\Phi_z + w^*S = Q(R/(C_pH_0)) \tag{11}$$

where the star (*) superscript on z^* is omitted for notational convenience. Consistent with the small meridional extent assumed for the motion, terms of order $\beta l/(f_0(k^2 + l^2))$ have been neglected. The approximation is valid for meridional wavelengths much less than 16,000 km (corresponding to scales ~2700 km).

The system of equations can easily be reduced to a single second-order equation in w^*. A change of variables $w^* = w'\exp(z/(2H_0))$ simplifies the result to the vertical structure equation

$$w'_{zz} + [SK/\sigma - 1/(4H_0^2)]w' = [KR/(\sigma C_p H_0)]Q\exp(-z/2H_0) \tag{12}$$

where

$$K = \frac{-(k^2 + l^2)(\sigma + iD + \beta k/(k^2 + l^2))}{f_0^2 - (\sigma + iD + \beta k/(k^2 + l^2))^2}$$

The heating is taken to be due to a constant mass flux between cloud base z_B and cloud top z_T, corresponding to a single non-entraining cumulus tower. The source term is due to adiabatic cooling which is not realized in the cloud due to the release of latent heat which balances the cooling. $Q = C_pSH_0^2\rho(z_B)w_c(z_B)/(R\rho H)$ where $H = RT/g$ is the actual basic state scale height and $\rho(z_B)w_c(z_B)$ is the mass flux into the cloud. The total heating due to the cloud is

$$\int_{z_B}^{z_T} \rho Q\,dz = C_p\rho(z_B)w_c(z_B)[T(z_T) - T(z_B) + g(z_T - z_B)/C_p] \tag{13}$$

The moisture budget is satisfied by tying the level of cloud base to the lifting condensation level, and the cloud top to the level where cloud air would lose buoyancy given a surface temperature and humidity. In order to be consistent with the moisture budget, the total heating must also equal the flux of latent heat due to the moisture flux into the clouds. Since the moisture remains highly mixed up to the trade inversion the moisture flux into the clouds is evaluated as the trade inversion (z_{TI}). Therefore the total heating is $Lq\rho(z_{TI})w_c(z_{TI})$ where q is the specific humidity and L is the latent heat of water vapor. Assuming a constant mass flux in the clouds so that $\rho(z_B)w_c(z_B) = \rho(z_{TI})w_c(z_{TI})$, consistency requires

$$Lq = C_p[T(z_T) - T(z_B)] + g(z_T - z_B)$$

which provides a restriction on cloud top height for a given humidity. For all cases considered here a surface temperature of 25 °C and a specific humidity $q = 12.3$ g/kg is assumed. The lifting condensation level $z_B \simeq 1$ km and the consistent cloud top height is $z_T = 12$ km.

The vertical structure equation will be solved in a three-layer system so that interface conditions between the regions, as well as boundary conditions at the ground and top (infinity) will be needed. In linearized form, $w = (-i\sigma/g)\Phi + (H/H_0)w^*$. Equations (8), (9) and (10) can be solved to obtain

$$w^*_z - w^*/H_0 = -iK\Phi \tag{14}$$

so that $w = w^*(H/H_0 - \sigma/(gKH_0)) + (\sigma/(gK))w^*_z$. The w^*_z term is quite small for all reasonable vertical scales, and the lower boundary condition $w = 0$ can be accurately approximated by $w^* = 0$ at $z = 0$. The upper boundary condition in the unbounded atmosphere is taken to be the radiation condition $w' \sim \exp(i\lambda z)$ where $\mathrm{Re}(\lambda) > 0$ so that the phase will propagate downward for $\sigma < 0$ (waves with phase speed easterly with respect to the basic flow). Downward phase propagation implies an upward energy flux so that there is no energy source at infinity. The interface conditions between regions are continuity of dp/dt (or w^*) and w, the first so that pressure remains continuous and the second so that material surfaces remain continuous. w and w^* continuous imply w' and Φ are continuous.

4. Formulation of case studies

Case studies of the solutions to the forced problem are made using an inviscid and frictional boundary layer formulation in order to consistently determine the effect which friction has on thermally forced Rossby and gravity wave modes.

Inviscid solution (subscript "I")

A three-layer model is used with a lower boundary layer to model the mixed layer ($0 < z < z_B$), a middle heating region to model the forcing due to cumulus towers ($z_B < z < z_T$) and an upper radiation region in which energy propagates upward ($z > z_T$). In order to directly evaluate the effect of friction in the mixed layer, the frictionless forced problem is first solved, corresponding to the solution of (12) with $D = 0$. The stability in the boundary layer is taken to be zero ($S = 0$) so as to model the mixing of potential temperature.

In the boundary layer $\partial\Phi/\partial z = 0$, so the geopotential can be set equal to Φ_I, an unknown constant, in the lower layer. Then the solution to (14) in the lower layer is

$$w'_I = -2iH_0K_0\Phi_I \sinh(z/2H_0) \tag{15}$$

where $K_0 \equiv K$ evaluated with $D = 0$. In the middle layer, $S \neq 0$ and the heating is that given in section 3. (12) then becomes

$$w'_{zz} + \lambda_0^2 w' = w_c(z_B)(H_0/H(z_B))(\lambda_0^2 + 1/4H_0^2) \exp(z/2H_0 - z_B/H_0) \tag{16}$$

where

$$\lambda_0^2 = SK_0/\sigma - 1/(4H_0^2) \tag{17}$$

In the middle layer the solution is

$$w'_I = B_1 \exp(i\lambda_0 z) + B_2 \exp(-i\lambda_0 z) + w_c(H_0/H(z_B)) \exp(z/2H_0 - z_B/H_0) \tag{18}$$

where B_1 and B_2 are constants. In the upper radiation layer there is no forcing so (16) implies $w' = A\exp(i\lambda_0 z)$ where A is a constant and $\mathrm{Re}(\lambda_0) > 0$. The interface conditions are continuity of w' and Φ at z_B and z_T, which can be used to solve for w' and then w^*, w and Φ at z_B. The solutions for $w(z_B)$, etc., will be evaluated using a vertical wavelength corresponding to a real vertical wavenumber λ_0. The value of (real) σ is chosen using (17) to solve for the root corresponding to a Rossby or gravity mode.

Frictional solution (subscript "D")

In order to determine the effects of a frictional boundary layer, a linear drag formulation of friction is evaluated. The formulation and solution are the same as in the inviscid case except that the mixed layer is modified by friction corresponding to $\mathbf{F} = -D\mathbf{v}$. Thus the solution to (14) in the lower layer is

$$w'_D = -2iH_0K_D\Phi_D \sinh(z/2H_0) \tag{19}$$

(instead of (15)), where $K_D \equiv K$ evaluated with $D \neq 0$ and Φ_D is the boundary layer geopotential ($\Phi_D \neq \Phi_I$). The solution follows as in the inviscid case, and the choice of λ_0 and σ for gravity and Rossby modes is the same.

The ratio between the boundary layer convergence forced by the specified heating when a frictional boundary layer is included to that which results when the boundary layer friction is zero is equal to the ratio $\hat{R}$ between the forced vertical velocities at z_B,

$$\hat{R} \equiv \frac{w_D'(z_B)}{w_I'(z_B)} = \frac{K_D \Phi_D}{K_0 \Phi_I} \tag{20}$$

Thus this ratio, which measures the importance of friction on boundary layer convergence for the thermally forced modes, depends on the geopotential (or pressure) response at the top of the boundary layer. The ratio R^* between boundary layer convergence with and without friction using a *specified* geopotential Φ_S at the top of the boundary layer is, however,

$$R^* = \frac{K_D \Phi_S}{K_0 \Phi_S} = \frac{K_D}{K_0} \tag{21}$$

Thus if the geopotential response is effectively reduced by the presence of the boundary layer ($\Phi_D \ll \Phi_I$), the importance of friction for boundary layer convergence may be considerably overestimated by R^*. The ratio R^* is identically the ratio between boundary layer and interior convergence δ_s/δ which would be obtained for these modes with Holton's (1975) barotropic model if a mid-latitude β-plane were used (cf. the discussion in Holton (1975), Section 3). The use of a mid-latitude β-plane simplifies the analysis considerably but does not alter the essential interior-boundary layer interaction (which is omitted in the barotropic analysis) for internal gravity or Rossby modes.

5. Discussion of results

Boundary layer frictional drag was taken to be $D = 10^{-5}$ sec^{-1} in all cases, corresponding to $C_D = 0.002$, $U = 5$ m/sec and $z_B = 1$ km. For both Rossby and gravity modes the horizontal wavelengths L_x $(= 2\pi/k)$ and L_y $(= 2\pi/l)$ were taken equal for simplicity. Tables 1, 2 and 3 show the magnitude and phase which measures the ratio of frictional to inviscid response when the effect of the boundary layer is properly taken into account ($\hat{R}$), when it is not (R^*), and the ratio between the two measures ($R^*/\hat{R}$) for representative gravity modes with $L_x = 1000$ km and Rossby modes with $L_x = 2000$ and 6000 km.

From Table 1 it can be seen that the effect of boundary layer friction on gravity modes is generally small ($\hat{R} \simeq 1$) since they are high frequency. Thus frictional effects are relatively weak, and the error in using an inconsistent formulation is also small ($R^*/\hat{R} \simeq 1$). Tables 2 and 3 indicate, however, that for Rossby modes frictional effects are generally quite large ($\hat{R} \gg 1$). The error resulting from using a specified pressure is also generally large ($R^*/\hat{R} \gg 1$), but is smallest for small vertical wavelength (L_z) and large horizontal wavelength (L_x). The reason may be easily seen by comparing the values of K_D and K_0. $K_D \simeq K_0$ only when D is much smaller than $\sigma_\beta \equiv \sigma + \beta k/(k^2 + l^2)$ so that the boundary layer is a small perturbation on the imposed inviscid pressure (so that $\Phi_D \simeq \Phi_I$ also). This condition is most easily satisfied for Rossby waves, for which $\sigma \simeq -\beta k/(k^2 + l^2 + f_0^2(\lambda_0^2 + 1/4H_0^2)/S)$, when $L_z (= 2\pi/\lambda_0)$ is small or L_x is large

Table 1. *Magnitude and phase of ratio which measures importance of boundary layer friction for gravity wave ($L_x = L_y = 1000$ km)*

L_z is vertical wavelength of mode. $\hat{R}$ and R^* are, respectively, consistent and inconsistent measures of frictional effects (discussed in text). $R^*/\hat{R}$ is the ratio between the two measures

L_z (km)	$\lvert\hat{R}\rvert$	arg ($\hat{R}$)	$\lvert R^*\rvert$	arg (R^*)	$\lvert R^*/\hat{R}\rvert$	arg ($R^*/\hat{R}$)
1.5	0.68	2°	0.39	61°	0.58	59°
3	0.81	4	0.83	29	1.03	25
6	0.90	4	0.97	10	1.07	6
10	0.96	3	0.99	5	1.04	2
15	0.99	3	1.00	3	1.02	1
20	0.99	2	1.00	3	1.01	0
25	0.99	2	1.00	2	1.01	0
30	0.99	2	1.00	2	1.01	0
40	1.00	1	1.00	1	1.00	0
50	1.00	1	1.00	1	1.00	0
60	1.00	1	1.00	1	1.00	0
90	1.00	1	1.00	1	1.00	0

so that σ_β is large. Thus the error is not reduced as the vertical wavelength increases. For synoptic-scale Rossby waves the error is large and the effect of friction on boundary layer convergence is considerably overestimated by R^* unless D is very small ($\ll 10^{-5}$ sec^{-1}) so that the boundary layer is a small perturbation on the interior.

The reduced effect of boundary layer friction in the completely consistent formulation is due to the constraint imposed by the coupling of interior and boundary layer dynamics. In the inconsistent solution, which uses a specified pressure to drive the boundary layer, frictional boundary layer convergence may induce a large vertical velocity which is incompatible with the interior dynamics. However, in a consistent solution both w and Φ are continuous at z_B so that such large vertical velocities are not allowed. The pressure response adjusts (is reduced) so as to be consistent with both the interior dynamics and the boundary layer relation (19) between w and Φ. The adjustment of w and Φ is in both magnitude and phase and depends strongly on the vertical mode (L_z) which is being considered. On an equatorial β-plane it would also be expected to depend on the latitude. Thus there is no simple adjustment in the effective forcing (heating) which can be made to account for the feedback between boundary layer and interior.

6. Conclusion

The results presented here indicate that boundary layer friction has a significant effect on synoptic-scale Rossby waves. This is not surprising since boundary layer pumping is large for these vorticity waves. The dominance of frictional con-

Table 2. *Same as Table 1 for Rossby wave with $L_x = L_y = 2000$ km*

L_z (km)	$\lvert\hat{R}\rvert$	arg ($\hat{R}$)	$\lvert R^*\rvert$	arg (R^*)	$\lvert R^*/\hat{R}\rvert$	arg ($R^*/\hat{R}$)
1.5	0.96	12°	3.0	74°	3.1	63°
3	0.98	22	3.4	76	3.5	54
6	1.18	40	5.3	81	4.5	41
10	1.64	43	10	85	6.2	32
15	2.33	61	19	86	8.3	26
20	3.02	65	31	89	10	24
25	3.71	65	46	89	12	24
30	4.47	66	66	90	14	24
40	6.11	64	118	90	19	26
50	7.03	62	156	90	22	27
60	8.50	60	221	90	26	30
90	11.4	52	395	90	35	39

Table 3. *Same as Table 1 for Rossby wave with $L_x = L_y = 6000$ km*

L_z (km)	$\lvert\hat{R}\rvert$	arg ($\hat{R}$)	$\lvert R^*\rvert$	arg (R^*)	$\lvert R^*/\hat{R}\rvert$	arg ($R^*/\hat{R}$)
1.5	0.90	7°	1.3	52°	1.4	45°
3	0.85	15	1.3	52	1.5	38
6	0.85	27	1.3	54	1.6	26
10	0.96	39	1.5	58	1.6	18
15	1.16	49	1.8	63	1.5	14
20	1.40	57	2.2	69	1.6	12
25	1.68	61	2.7	72	1.6	12
30	2.07	65	3.4	76	1.6	12
40	3.00	68	5.3	81	1.8	13
50	3.63	69	6.7	83	1.8	14
60	4.68	69	9.1	85	2.0	17
90	7.08	64	15	88	2.2	23

vergence over inviscid convergence is not nearly as strong as that estimated using a measure of boundary layer frictional effects which neglects the boundary layer-interior coupling evident in the complete baroclinic solution. A barotropic model cannot properly account for the complex interaction between a baroclinic interior and a frictional boundary layer. Thus, even though the mixed gravity–Rossby mode was not explicitly studied here, a barotropic model such as Holton's (1975) cannot be expected to correctly evaluate the frictional boundary layer contribution for such a wave, where frictional effects are order one.

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REFERENCES

Chang, C-P. 1973. On the depth of the equatorial planetary boundary layer. *J. Atmos. Sci. 30*, 436–443.

Hayashi, Y. 1970. A theory of large-scale equatorial waves generated by condensation heat and accelerating the zonal wind. *J. Met. Soc. Japan 48*, 140–160.

Hayashi, Y. 1971a. Frictional convergence due to large-scale equatorial waves in a finite-depth Ekman layer. *J. Met. Soc. Japan 49*, 450–456.

Hayashi, Y. 1971b. Large-scale equatorial waves destabilized by convective heating in the presence of surface friction. *J. Met. Soc. Japan 49*, 458–466.

Holton, J. 1970. A note on forced equatorial waves. *Mon. Wea. Rev. 98*, 614–615.

Holton, J. 1975. On the influence of boundary layer friction on mixed Rossby–gravity waves. *Tellus 27*, 107–115.

Holton, J., Wallace, J. & Young, J. 1971. On boundary layer dynamics and the ITCZ. *J. Atmos. Sci. 28*, 275–280.

Lindzen, R. 1974a. Wave-CISK in the tropics. *J. Atmos. Sci. 31*, 156–179.

Lindzen, R. 1974b. Wave-CISK and tropical spectra. *J. Atmos. Sci. 31*, 1447–1449.

Matsuno, T. 1966. Quasi-geostrophic motions in the equatorial area. *J. Met. Soc. Japan 44*, 25–42.

Yamasaki, M. 1969. Large-scale disturbances in the conditionally unstable atmosphere in low latitudes. *Papers in Met. and Geophys. 20*, 289–336.

Yamasaki, M. 1971a. A further study of wave disturbances in the conditionally unstable model tropics. *J. Met. Soc. Japan 49*, 391–415.

Yamasaki, M. 1971b. Frictional convergence in Rossby waves in low latitudes. *J. Met. Soc. Japan 49, Syono Memorial Volume.*

ВЛИЯНИЕ ТРЕНИЯ НА ТЕРМИЧЕСКИ ВОЗБУЖДАЕМЫЕ ВОЛНЫ

Развивается линейная теория с целью выяснения важности трения в пограничном слое для возбуждаемых термически гравитационных волн и волн Россби. Проводится аналитическое исследоване на β-плоскости в средних широтах, что позволяет разделить переменные в линейных уравнениях при наличии трения в пограничном слое. Найдено, что конвергенция за счёт трения существенна для волн Россби синоптического масштаба (но не для гравитационных волн). Если обратная связь между пограничным слоем и внутренним решением не принимается в расчёт, то оценка конвергенции будет неправильной и может оказаться значительно завышенной.