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Heat and salt transfer associated with formation of sea-ice

By J. E. WEBER,[1] *Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, England*

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ABSTRACT

The transient state of sea-ice formation due to a given constant drop in surface temperature, and the associated change in sea-water salinity due to salt rejection, has been investigated theoretically by the heat balance integral method. The water temperature is taken to be constant. Ice, water and salt were assumed to coexist in thermodynamic equilibrium at the ice–water interface, and the analysis is valid for small temperature perturbations. The ice growth rate is found to be considerably reduced due to the depression of the freezing point caused by salt rejection. Utilizing Foster's (1968) results, haline convection with typical wavelength 1–2 mm is found to occur after a characteristic time of order 1 min. Results are also presented for the ice growth problem when the ice–water interface can be regarded as a constant temperature boundary. Some effects of variable thermal properties due to brine trapping in the ice are taken into account. The case of a time-dependent surface temperature is also considered.

1. Introduction

The increase in salinity of surface waters associated with the formation of sea-ice may have a primary influence upon the formation of oceanic bottom water (see Munk, 1966). It is not yet completely understood how this process takes place. It seems, however, as if haline convection may contribute significantly to the bottom water formation (Foster, 1972).

Irrespective of how the mixing process actually occurs, it is essential to study the growth of sea-ice in order to predict the total amount of salt supplied to the water during the freezing process. For this, and various other reasons, particularly associated with environmental changes, numerous studies of the ice growth have been performed since the classical work of Stefan (1890); see Mayakut & Untersteiner (1969) where appropriate references can be found. These studies, however, commonly deal exclusively with the heat transfer problem, and neglect the coupling with the salinity field in the water. At the ice–water interface a constant temperature or heat flux is prescribed, which implicitly assumes that the salinity increase due to salt rejection is removed by some sort of turbulence in the water.

The present paper investigates the formation of sea-ice when the body of sea-water initially is at rest. Ice, water and salt are assumed to coexist in thermodynamic equilibrium at the ice–water interface, and it is shown that the coupling between the heat and salt transfer problem can not be neglected at this stage.

The salinity boundary layer produced below the ice, will ultimately become gravitationally unstable, leading to the onset of haline convection (Foster, 1968, 1969). By utilizing Foster's (1968) results, the critical time for onset of convection, as well as the critical wavelength, is predicted.

Finally, some results are presented for later stages of ice growth. It is then assumed that turbulent convection, or turbulence in general, is sufficiently effective to remove the salt excess at the ice–water interface. This, then, may be regarded as a constant temperature boundary, which is along the lines of traditional ice growth studies. Explicitly

[1] Present address: Institute of Mathematics, Mechanics Department, University of Oslo.

we have considered some effects upon the ice growth due to variable thermal properties. These variations arise from brine being trapped in the ice during the freezing process. Also the case of a variable surface temperature has been analyzed.

2. Model and basic equations

We consider a semi-infinite model where initially sea-water of uniform salt concentration S_0 is at rest. The temperature T_0 is also uniform and equals the freezing temperature. Especially for sea-water with a salinity greater than 24.7‰, this is not an unrealistic assumption. In that case, the cooling process towards freezing is accompanied by thermal convection, which ultimately leaves us with a uniform upper layer of considerable depth compared to the final ice thickness. Wind mixing will of course also contribute to this result. Measurements below natural growing ice-sheets clearly show such uniform regions (Brennecke, 1921; Tabata, 1958; Lewis, 1972).

Let the z-axis be directed vertically downward, and let t denote time. At $t = 0$, the origin is situated at the surface which is covered by an infinitely thin layer of ice. For $t > 0$, we apply a small negative temperature perturbation $-Q(t)$ at the surface. Assuming that no supercooling occurs, freezing begins, and a horizontal layer of ice starts to form. In the model, we disregard any microstructure, and take the ice–water interface to be sharp and smooth. Further we assume that the dissolved salts behave as a single solute. At the ice–water interface, ice, water and salt are assumed to coexist in thermodynamic equilibrium governed by a relation of the form

$$f(T^*, S^*) = 0 \tag{2.1}$$

where T^* is the temperature and S^* is the salinity at the phase boundary. The pressure is assumed constant during the process. Hence, at $t = 0, f(T_0^*, S_0^*) = 0$ where $T_0^* = T_0$ and $S_0^* = S_0$. Denoting the perturbated position of the phase boundary by $\zeta(t)$, thermodynamic equilibrium in general at $t > 0$ requires to a first approximation that

$$\left(\frac{dT_0^*}{dz}\right)_{z=0} \zeta = -\theta - a_0 S \tag{2.2}$$

where we have used the fact that $(dS_0^*/dz)_{z=0} = 0$. Here θ and S are the temperature and salinity perturbations at the phase boundary, and a_0 a positive constant given by

$$a_0 \equiv \left(\frac{\partial f/\partial S_0^*}{\partial f/\partial T_0^*}\right)_{z=0} = -\left(\frac{dT_0^*}{dS_0^*}\right)_{z=0} \tag{2.3}$$

where the last relation follows from (2.1). From measurements we obtain $a_0 = 54.11 \times 10^{-3}\,°C$ for salinities up to about 140‰. (Assur, 1958).

From (2.2) we observe that, if the initial temperature increases downward, a negative temperature perturbation at the surface causes the ice–water interface to move downward, while the associated positive increase of salinity at the interface, as well as liberated heat, tend to oppose the motion. Finally a new stationary position may be reached; see Gjevik (1972) for a similar problem.

In the present case there are no initial gradients, and hence

$$\theta = -a_0 S, \quad z = \zeta(t) \tag{2.4}$$

Accordingly, the ice will continue to grow to this approximation, provided the surface temperature is below the initial freezing point.

Since water expands by freezing, the ice surface will move upward during the freezing process. This small expansion is not essential for the present problem, however, and can be neglected.

When sea-water freezes, nearly pure ice is formed, and salt is rejected back into the sea. Since the advancing ice front tends to develop a microstructure in which the ice crystals and their connections form a matrix, however, highly saline brine may be trapped in the ice during the freezing process. The amount of brine trapped depends largely on the freezing rate, and rapidly formed sea-ice may have salinities up to 20‰ (Pounder, 1962).

Due to the salt content, the properties of sea-ice may differ significantly from those of pure ice. In the present context we are particularly interested in its thermal properties, first measured by Malmgren (1927). The most striking variation occurs in the specific heat at constant pressure, c_p, which may vary by several orders of magnitude in a temperature interval near the freezing point for sea-water. The latent heat of fusion, L, also varies considerably. Ono (1967) has derived analytical expressions for c_p and L, which are in agreement with experimental results. For the present purpose, it is sufficient to take into account only the leading terms in Ono's expressions; the other terms are quite small. The resulting formulae are

$$c_p = c_p^0(1 + G_1 S_i/T^2) \tag{2.5}$$

$$L = L^0(1 + G_2 S_i/T) \tag{2.6}$$

where the superscript 0 denotes value for pure ice, which we assume to be constant, and S_i is the ice salinity in parts per thousand, T is the temperature in deg. C, and G_1, G_2 are constants, having values $8537.6 \times 10^{-3}(°\mathrm{C})^2$ and $54.11 \times 10^{-3}\,°\mathrm{C}$, respectively.

The trapping process is not very well understood and it is therefore difficult to relate the ice salinity S_i to the variables of the problem. We shall therefore have to be content by assuming that S_i is constant. This merely means an adjustment of the physical constants of the problem, except that c_p and L still vary with temperature.

Since pure ice conducts heat nearly four times better than sea-water at the freezing point, we shall neglect the heat flux into the water, and assume that all heat released by the freezing process flows to the ice. This assumption first introduced by Stefan (1890), greatly simplifies the mathematical problem. This is especially the case when working with time-dependent surface temperature and variable thermal properties. Accordingly, heat balance across the ice–water interface implies

$$k_T \frac{\partial \theta}{\partial z} = \rho L \dot{\zeta}, \quad z = \zeta \tag{2.7}$$

where a dot denotes the time derivative. Further ρ is the density and k_T is the thermal conductivity of ice, which is assumed constant (k_T varies less than 10% according to Mayakut & Untersteiner (1969)). It should be noted, however, that under natural growing conditions, the air bubble content in the ice has a marked influence on k_T (Pounder, 1962).

Salt balance across the interface further requires

$$-\kappa_S \frac{\partial S}{\partial z} = (S_0 + S - S_i)\dot{\zeta}, \quad z = \zeta \tag{2.8}$$

where κ_S is the diffusivity of salt in sea-water.

Introducing an arbitrary length h, non-dimensional variables may conveniently be defined by

$$z = z' h, \quad t = t' h^2/\kappa_T,$$

$$\theta = \theta' L^0/c_p^0,$$

$$S = S'(S_0 - S_i) \tag{2.9}$$

where

$$\kappa_T = k_T/\rho c_p^0$$

The non-dimensional equations governing the unsteady growth of sea-ice and the associated change of salinity in the water, may then be written, dropping the primes, as

$$\varphi \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial z^2} \qquad 0 < z < \zeta(t) \tag{2.10}$$

$$\frac{\partial S}{\partial t} = \tau \frac{\partial^2 S}{\partial z^2}, \quad \zeta(t) < z < \infty \tag{2.11}$$

where

$$\varphi = c_p/c_p^0 \quad \text{and} \quad \tau = \kappa_S/\kappa_T \tag{2.12}$$

The boundary conditions are

$$\theta = -q(t), \quad z = 0 \tag{2.13}$$

$$\left.\begin{aligned} &\frac{\partial \theta}{\partial z} = l\dot{\zeta} && (2.14)\\ &\theta = -\alpha S && (2.15)\\ &\tau \frac{\partial S}{\partial z} = -(1 + S)\dot{\zeta} && (2.16) \end{aligned}\right\} z = \zeta(t)$$

$$S = 0, \quad z = \infty \tag{2.17}$$

where

$$q = \frac{Q(t)}{L^0/c_p^0}, \quad l = L/L^0$$

and

$$\alpha = a_0 c_p^0 (S_0 - S_i)/L^0 \tag{2.18}$$

A useful boundary condition, not involving $\dot{\zeta}$, may be obtained by differentiating (2.15) with respect to time and applying (2.10), (2.11) and (2.14). This finally yields

$$\frac{1}{l_m}\left(\frac{\partial \theta}{\partial z}\right)^2 + \frac{1}{\varphi_m}\frac{\partial^2 \theta}{\partial z^2} + \alpha\left[\frac{1}{l_m}\frac{\partial S}{\partial z}\frac{\partial \theta}{\partial z} + \tau \frac{\partial^2 S}{\partial z^2}\right] = 0, \quad z = \zeta \tag{2.19}$$

where subscript m denotes value at the ice–water interface.

3. Method of solution

The system of equations (2.10–2.17) will be solved by the heat balance integral method introduced by Goodman (1958). In principle, the heat equation is integrated over a thermal boundary-layer depth δ_T. The exact solution is approximated by a polynomial in z. By integration, this results in a first order differential equation for $\delta_T(t)$. This procedure is analogous to the momentum integral method developed by von Karman (1921) and Pohlhausen (1921) for viscous boundary-layer theory. For application of the heat balance integral method to freezing and melting problems, reference is given to the review article by Muehlbauer & Sunderland (1965).

In this particular problem, the heat equation is integrated over the ice thickness, while the salt equation is integrated over a suitably defined salinity boundary layer. Integration of (2.10) then yields

$$\frac{d}{dt}\int_0^\zeta V\,dz = \{l_m + V(\zeta)\}\dot{\zeta} - \left(\frac{\partial\theta}{\partial z}\right)_{z=0} \quad (3.1)$$

where

$$V = \int \varphi d\theta$$

Defining a salinity boundary layer δ by

$$S(\zeta + \delta) = 0 \quad (3.2)$$

$$\left(\frac{\partial S}{\partial z}\right)_{z=\zeta+\delta} = 0 \quad (3.3)$$

the salt equation (2.11) can be integrated directly in z and t, by the aid of (2.16), giving

$$\int_\zeta^{\zeta+\delta} S\,dz = \zeta \quad (3.4)$$

where we have used the fact that $\zeta = \delta = 0$ at $t = 0$. In this part of the analysis we shall, for simplicity, assume that the non-dimensional surface perturbation temperature $-q$ is constant. An analysis for variable q will be left for section 5. By differentiating (2.13) with respect to time and assuming that (2.10) is valid at the boundary, we obtain

$$\frac{\partial^2\theta}{\partial z^2} = 0, \quad z = 0 \quad (3.5)$$

and similarity for the salinity

$$\frac{\partial^2 S}{\partial z^2} = 0, \quad z = \zeta + \delta \quad (3.6)$$

Equations (3.5) and (3.6) are usually termed as "smoothing" conditions, and are often encountered in boundary layer analysis. It should be noted, however, that the inclusion of such extra constraints does not always improve the accuracy of the solution (Goodman, 1961).

According to the adopted approach, we assume that

$$\theta = C_0 + C_1 z + C_2 z^2 + C_3 z^3 \quad (3.7)$$

and

$$S = D_0 + D_1 z + D_2 z^2 + D_3 z^3 \quad (3.8)$$

where the coefficients are functions of time. By applying (2.13), (2.15) and (3.2–3.6), we obtain

$$\theta = \frac{q}{\zeta}(z - \zeta) - \frac{4\alpha z}{\delta} + C\left(\frac{z^3}{\zeta^3} - \frac{z}{\zeta}\right) \quad (3.9)$$

$$S = -\frac{4\zeta}{\delta^4}(z - \zeta - \delta)^3 \quad (3.10)$$

Equations (2.16), (2.19) and (3.1) finally provide the necessary relations between the unknown ζ, δ and C. They are a bit lengthy, however, and we have therefore placed them in an appendix. Inspection of (A.1–A.3) reveals that we have a solution of the form

$$C = \text{const.}$$

$$\zeta = Kt^{1/2}$$

$$\delta = Dt^{1/2} \quad (3.11)$$

where K and D are constants. This is of course a reflection of the fact that we apply a constant temperature perturbation at the upper ice surface. Substitution of (3.11) into (A.1–A.3) leads to a coupled system of algebraic equations in C, K and D. Defining $\epsilon = D/K$, (A.1) may be expressed as $K = K(\epsilon)$, while (A.2) give $C = C(\epsilon)$. Insertion into (A.3), then yields an implicit equation in ϵ, which is solved numerically by the method of bisection. The integral in (A.3) was solved numerically at each step by Simpson's rule.

Computations were performed with sea-ice salinities ranging from zero to appropriate values for the bulk of sea-ice (4‰). No significant change in the solution was found. It is also clear, that for such small temperature changes as considered here, significant trapping is not likely to occur. Hence S_i can be taken as zero in this part of the analysis.

4. Results and discussion

Fig. 1 exhibits the main features of the solution. The non-dimensional growth rate coefficients K for the ice thickness ζ (solid lines) and D for the salinity boundary layer δ (broken lines) are plotted as functions of the initial sea salinity S_0. Various values of the (constant) temperature drop Q at the upper ice surface have been considered. In the computations we have chosen $L^0 = 80$ cal/g, $c_p^0 =$ 0.5 cal/g°C, $\kappa_S = 7 \times 10^{-6}$ cm²/s and $\kappa_T =$ 0.01 cm²/s. We observe that the growth rate coefficient for ice increases with Q, which is obvious, and decreases with the sea salinity S_0. This latter effect is due to the depression of the freezing point caused by salt rejection during the freezing process. For a drop in surface temperature of 1 °C, the initial growth rate of ice on sea-water of 35‰, is nearly ten times less than the corresponding growth rate on fresh water.

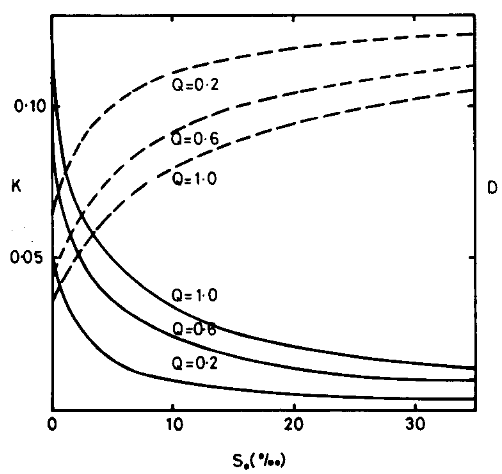

Fig. 1. Non-dimensional growth rate coefficients K for ice (solid lines) and the salinity boundary layer D (broken lines) vs. sea salinity S_0. The values of Q represent constant drops in surface temperature (in °C).

It is also observed that the salinity boundary layer thickness increases with increasing S_0, and also increases with decreasing Q. The figure shows that the salinity boundary layer in the sea is considerably thicker than the ice-sheet. The values of D for $S_0 = 0$, i.e. fresh water, has of course no physical relevance.

The analysis presented so far has assumed a purely conductive state in the fluid below the growing ice-sheet. Both theoretical analysis and laboratory experiments, however (Foster, 1968, 1969), give evidence that, after a certain period of time, the relatively saltier layer just beneath the ice becomes gravitationally unstable, and haline convection commences. Assuming the ice–water interface to be fixed in space, Foster (1968) performed a linear analysis to determine the onset of convection. The Schmidt number ν/κ_S, ν being the kinematic viscosity, was taken to be infinite, which is a good approximation for molecular values of ν and κ_S. His results for a given constant salinity step ΔS at the upper rigid boundary are relevant for the present purpose. Convection then began after a critical dimensional time

$$t_c = \frac{59}{\kappa_S} \cdot \left(\frac{\nu\kappa_S}{g\gamma\Delta S}\right)^{2/3} \tag{4.1}$$

The horizontal wavelength of the fastest growing disturbance was

$$\lambda_c = 5.1 \cdot \left(\frac{\nu\kappa_S}{g\gamma\Delta S}\right)^{1/3} \tag{4.2}$$

Here γ is the "expansion" coefficient for salt and g the acceleration of gravity. The exact erfc-profile used by Foster in his calculations, is very well approximated by a cubic profile of the form (3.10) (Goodman, 1961). Hence the present analysis enables us to determine ΔS in (4.1) and (4.2) as a function of the ice surface temperature. The results are summarized in Table 1. For a sea-water salinity of 35‰, the properties of sea-water near the freezing point are $\gamma = 0.8$, $= 0.02$ cm²/s and $\kappa_S =$ 7.10^{-6} cm²/s.

We observe that, even for such small temperature perturbations as considered here, haline convection commences relatively quickly; the characteristic time being of order 1 min. or so. The

Table 1. *Onset of haline convection for various values of the surface temperature $T_S = T_0 - Q$. ΔS denotes the induced salinity increase at the ice-water interface, t_c and λ_c are the critical time and wavelength, respectively, and δ_c, ζ_c are the salinity boundary layer and ice thickness at $t = t_c$*

Q(°C)	ΔS(‰)	t_c(s)	λ_c(mm)	δ_c(mm)	ζ_c(mm)
0.2	3.7	112.0	1.86	1.31	0.034
0.4	7.3	70.7	1.48	0.99	0.052
0.6	11.0	54.1	1.29	0.83	0.065
0.8	14.6	44.7	1.17	0.73	0.076
1.0	18.2	38.6	1.09	0.65	0.085

critical wavelength is roughly between 1 and 2 mm, and is a little less than twice the boundary-layer thickness. It is worth noting that these wavelengths are about the same order of magnitude as those observed by Foster (1969), although his experiments were performed at much lower surface temperature than that for which the present analysis is valid. Since, however, the initial wavelength basically varies as $\Delta S^{-1/3}$, which is a rather slow variation, this result may not be so surprising after all. Experiments with haline convection in a Hele-Shaw cell of "infinite" depth with $\Delta S = 25‰$ (Green & Foster, 1975), also show that the horizontal cell size of the primary convection is just a few millimetres.

It should be noted from Table 1 that the ice-sheet is extremely thin at the onset of convection. This again makes it clear that there is no trapping of brine at this stage.

As mentioned before, formulae (4.1) and (4.2) have been derived under the assumption of a fixed upper boundary. Taking κ_S/δ_c to represent the characteristic velocity at slightly supercritical convection, we find that this is of the same order of magnitude as the downward velocity of the ice–water interface at time t_c. Accordingly, the results in Table 1 should be considered as order of magnitude estimates only.

The early stage of haline convection under ice, with the body of sea-water at its freezing point and initially at rest, has not yet been subject to careful laboratory observations. From a theoretical point of view, one would expect three-dimensional disturbances to be dominant, since the basic salinity profile changes. In fact, by analogy to the findings of Krishnamurti (1968), convection cells with hexagonal planform should be observed.

Recently Farhadieh & Tankin (1975) have reported some interferometric measurements of the freezing of sea-water. In their case strong thermal convection was maintained in the water during the freezing process. The fact that thermal convection dominated the fluid motion, probably explains why they did not observe any effects of haline convection like those reported by Foster (1969).

It may also be worth pointing out that haline convection in the salinity boundary layer below the growing ice sheet may affect the morphological stability of the ice–water interface. For morphological instability to occur, constitutional supercooling must be produced ahead of the advancing interface; see Serkerka (1973) for a comprehensive review. If, in a simplified model, the phase boundary is assumed to move with constant velocity W, a stationary solution for the salt distribution in the fluid is

$$S = S_0 + S_0 \frac{(1-\kappa^*)}{\kappa^*} \exp\left\{-\frac{W}{\kappa_S}(z-\zeta)\right\}, z \geq \zeta \tag{4.3}$$

while the corresponding temperatures in the ice and water are approximately linear for the case of interest. In (4.3) κ^* is the distribution coefficient (the ratio of salt in the solid to salt in solution at the interface). Linear stability analyses of this solution then yield a criterion for the onset of morphological instability, i.e. dendritic growth (Mullins & Serkerka, 1964). The salinity distribution (4.3), however, varies typically over a distance

$\delta \sim \kappa s/W$ and will be convectively unstable (Howard, 1966) when

$$R_S = \frac{g\gamma(S(\zeta) - S_0)\delta^3}{\nu \kappa_s} \gtrsim 10^3 \tag{4.4}$$

When sea-water freezes, nearly all the salt is rejected back into the water, and k_* is accordingly very small. Estimates suggest $k_* < 10^{-4}$ (Harrison & Tiller 1963). For $S_0 = 35‰$, (4.4) gives that haline convection will occur when

$$W \lesssim 10^{-2}\,\text{cm/s} \tag{4.5}$$

This is contrary to morphololological instability which is favoured by increasing W (Serkerka, 1973). Despite the roughness of the above considerations, the results indicate that for small interface velocities, i.e. not too strong freezing, haline convection may occur prior to dendritic growth. This will probably result in a delay of the latter since convection mixes up the fluid adjacent to the interface and hence reduces the degree of supercooling.

5. Later stages of ice formation

For larger times and/or freezing rates, convection below the growing ice-sheet may develop in the form of long streamers (Foster, 1969), or the motion may become intermittent. In the latter case the qualitative behaviour of the flow is described by Howard's (1966) heuristic theory of blobs breaking away from an unstable boundary layer; the process repeats itself in a cyclic manner (see also Sparrow et al., 1970; Foster, 1971). Arguing along the same lines as Palm & Weber (1973) for thermal convection, it may be shown that the salt transported by descending heavy blobs is sufficient to account for the total downward salt transfer. Assuming then, that the intermittent blob mechanism effectively removes the salinity excess at the ice–water interface, the latter can be considered as a constant temperature boundary. Mathematically this means we take $\alpha = 0$ in (2.15). The removal of salt may also be aided by the presence of turbulence from other sources, currents, thermal convection, etc., in the sea. It should be mentioned that at this stage of sea-ice formation, the ice–water interface, under natural growing conditions, is quite corrugated with growing ice platelets and drainage channels (see Lake & Lewis 1970). The brine drainage from these channels also contributes to the turbulent picture in the water below. Accordingly, there is a complex transition zone rather than a sharp, smooth boundary.

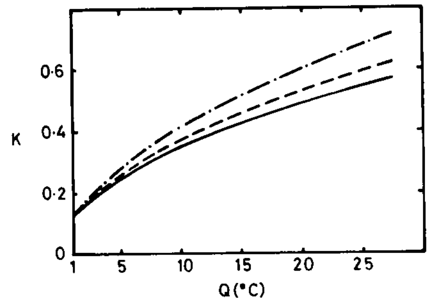

Fig. 2. The non-dimensional growth rate coefficient for sea-ice as a function of surface temperature for various values of the ice salinity. ——, $S_i = 0$, ----, $S_i = 4‰$; —·—·—, $S_i = 4\,(Q/°C)^{1/2}‰$.

In Fig. 2 we have displayed the non-dimensional growth rate coefficient K for ice as a function of the drop in surface temperature Q. The solid curve corresponds to pure ice ($S_i = 0$), while the dotted curves have been obtained by assuming $S_i = 4‰$ and $S_i = 4(Q/°C)^{1/2}‰$, respectively. The first value of S_i is about the average for sea-ice, while the latter expression takes into some account that the trapping of brine increases with increasing growth rate; which is essentially proportional to $Q^{1/2}$. In these computations we have chosen $\kappa_T = 0.008$ cm²/s to account for a smaller k_T due to the presence of air bubbles, etc. The results, however, differ very little from those obtained with $\kappa_T = 0.01$ cm²/s. It is observed that sea-ice at this stage has a larger growth rate than pure ice, and that the growth rate increases with the salt content. Computations show that the growth rate actually decreases due to the variation (the increase) of c_p (eq. 2.5). This decrease, however, is more than compensated for by the increase due to the variation of the latent heat of fusion, see (2.6). In total, then, the presence of salt in the ice leads to increased growth.

The fact that the growth rate increases with decreasing L is also obvious from Stefan's (1890) approximate formulae for ζ. When the ice is thin, the temperature distribution in the ice is approximately linear. Hence heat balance at the ice–water interface (2.7) directly implies $k_T Q/\zeta = L\rho\dot{\zeta}$, so that $\zeta \propto L^{-1/2}$. Why the relatively large increase in

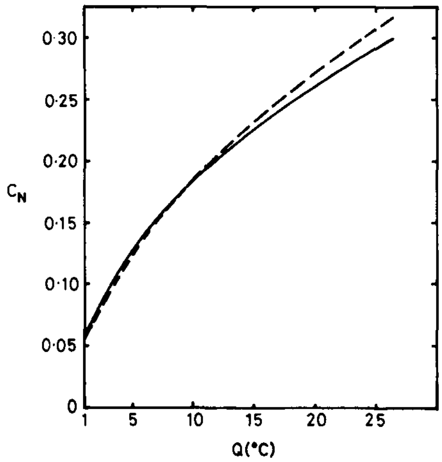

Fig. 3. The non-dimensional heat flux coefficient c_N in eq. (5.1) as a function of the ice surface temperature for $S_i = 0$ (solid line) and $S_i = 4‰$(broken line).

c_p (see (2.5)) has so little effect, is probably due to the fact that c_p basically varies locally (near the interface). Hence the integrated effect turns out to be only moderate. From Fig. 2 it follows that for $Q = 18$ °C, i.e. a surface temperature of −20 °C, and with $S_i = 17‰$, 8.6 cm of ice forms in 8 h. This is about the average growth rate reported by Foster (1972) for rapidly growing ice in the Wedell Sea.

The heat flux to the atmosphere during the freezing process may conveniently be expressed in non-dimensional terms as

$$N_u \equiv \left(\frac{\partial \theta}{\partial z}\right)_{z=0} = c_N t^{-1/2} \tag{5.1}$$

In Fig. 3 we have plotted c_N as a function of Q. The coefficient did not vary much with the salt content in the ice, but a slight increase at larger Q was found, as illustrated by the dotted curve for $S_i = 4‰$.

In this case, where the ice growth is assumed to be independent of conditions in the sea, the computations easily may be extended to account for a time-dependent drop of the surface temperature. Let $q = q(t)$ for $t > 0$. The "smoothing" condition (3.5) is no longer valid, which reduces the number of coefficients in the temperature profile by one. Accordingly, the temperature is taken to be parabolic in z. This leads to

$$\theta = \frac{q}{\zeta}(z - \zeta) + \frac{Cz}{\zeta^2}(z - \zeta) \tag{5.2}$$

We then finally obtain (see the appendix for details)

$$\zeta = \frac{\left(\int_0^t 12AB\,dt\right)^{1/2}}{B} \tag{5.3}$$

where $A(t)$ and $B(t)$ are defined by (A.7). For $S_i = 0$, formulae (5.3) reduces to the one reported by Goodman (1958). In this case, when q is small, (5.3) gives

$$\zeta = \left(2\int_0^t q\,dt\right)^{1/2} \tag{5.4}$$

which is Stefan's formulae in non-dimensional form.

6. Summary and concluding remarks

We have investigated the early stage of sea-ice formation taking into account the coupling between the heat transfer in the ice and the salt transfer in the underlying water. Assuming that ice, water and salt are in thermo-dynamic equilibrium at the ice–water interface, it is shown that the ice growth is substantially reduced due to the lowering of the freezing point caused by salt rejection during the freezing process. It should be emphasized that the analysis is valid only for small perturbations of the surface temperature from the initial freezing point.

Using Foster's (1968) results, we find that the salinity boundary layer just below the ice relatively quickly (of order 1 min) becomes gravitationally unstable, and haline convection commences. The critical wavelength is typically between 1 and 2 mm.

At larger times and freezing rates, intermittent convection may develop. Assuming that the salt excess is effectively removed by downward blob motion or turbulence in general, the ice–water interface can be regarded as a constant temperature boundary. In this case we have determined the ice thickness and the heat flux to the atmosphere as functions of the surface temperature. The ice was assumed to have a constant concentration of salt. Particularly the growth rate coefficient is found to increase with increasing sea-ice salinity. Basically this is due to the associated decrease of the latent heat of fusion. A formula for the ice growth when the surface temperature is a function of time, has also been derived.

Finally it may be argued that the assumption of a given temperature at the upper ice surface may be unrealistic; especially for longer periods of time. The surface temperature in general is determined not only by the air temperature, but also by radiation, snow cover, etc. The present paper, however, does not intend to cope with the annual heat budget for the polar regions. The analysis of section 5 is merely aimed at rapidly growing ice at high latitudes. Under such circumstances it seems reasonable to assume a given surface temperature. Since the production of salt surface water is especially large in the early stage of rapidly freezing polynyas and leads, in which ice may form in very large flat sheets, even in the presence of strong wind (Foster, 1972), the presented results are hoped to give some relevant information about the salt flux into the polar seas.

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8. Appendix

Substitution of (3.9) and (3.10) into (2.16), (2.19) and (3.1), finally yields

$$(\delta^2 + 4\zeta\delta)\dot{\zeta} - 12\tau\zeta = 0 \qquad \text{(A1)}$$

$$l_m^{-1}(q + 2C - 4\alpha\zeta/\delta)^2 + 6C\varphi_m^{-1} +$$
$$+ \alpha[l_m^{-1}(q + 2C - 4\alpha\zeta/\delta)(-12\zeta^2/\delta^2)$$
$$+ 24\tau\zeta^3/\delta^3] = 0 \qquad \text{(A2)}$$

$$\zeta\frac{d}{dt}\zeta(q/2 + C/4 + 2\alpha\zeta/\delta - \beta I)$$
$$+ \left(l_m - 4\alpha\zeta/\delta - \frac{\beta}{\Theta_0 - 4\alpha\zeta/\delta}\right)\zeta\dot{\zeta} - q + C$$
$$+ 4\alpha\zeta/\delta = 0 \qquad \text{(A3)}$$

Here

$$\beta = G_1 S_i/(L^0/c_p^0)^2, \quad \Theta_0 = T_0/(L^0/c_p^0)$$

$$l_m = 1 + \frac{G_2 S_i}{(L^0/c_p^0)(\Theta_0 - 4\alpha\zeta/\delta)} \qquad \text{(A4)}$$

$$\varphi_m = 1 + \frac{\beta}{(\Theta_0 - 4\alpha\zeta/\delta)^2}$$

$$I = \int_0^1 \frac{d\xi}{-C\xi^3 - (q - C - 4\alpha\zeta/\delta)\xi + q - \Theta_0}$$

When $\alpha = 0$, as assumed in section 5, insertion of (5.2) into (2.19) readily gives

$$C = (2qb_0 + b_0^2)^{1/2} - q - b_0 \qquad \text{(A5)}$$

where

$$b_0 = l_m/\varphi_m$$

(A3) may then be integrated to give

$$\zeta = \frac{\left(\int_0^t 12AB\,dt\right)^{1/2}}{B} \qquad \text{(A6)}$$

Here

$$A = 2q + b_0 - (2qb_0 + b_0^2)^{1/2}$$
$$B = 2q + 6l_m - b_0 + (2qb_0 + b_0^2)^{1/2} - \beta(I_0 + 1/\Theta_0) \qquad \text{(A7)}$$

and

$$I_0 = \int_0^1 \frac{d\xi}{-C\xi^2 - (q - C)\xi + q - \Theta_0}$$

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ПЕРЕНОС ТЕПЛА И СОЛИ, СВЯЗАННЫЙ С ФОРМИРОВАИЕМ МОРСКОГО ЛЬДА

С помощью интегрального метода теплового баланса теоретически исследуется процесс формирования льда на море, вызываемый заданным постоянным падением температуры на поверхности моря и связанным с ним измененнем солености морской воды из-за опреснения льда. Температура воды считается постоянной. Лед, вода и соль предполагаются находящимися в термодинамическом равновесии на поверхности раздела лед-вода. Анализ справедлив дкя малых темпертаурных возмущений. Найдено, что скорость роста льда значительно уменьшается благодаря понижению точки замерзания, вызываемого опреснением. С помощью результатов Фостера (1968) найдено, что после характерного времени порядка 1 минуты начинается халинная конвекция с типичной длиной волны 1-2 мм. Представлены также результаты для проблемы роста льда, когда граница раздела лед-вода может рассматриваться как граница с постоянной температурой. Учитываются также некоторые эффекты переменности термнческих свойств льда благодаря образованию в нем полостей с рассолм. Рассматривается также случай температуры поверхности, зависящей от ремени.