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The Gulf Stream: Inertia and friction

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ABSTRACT

The inertial theory of the Gulf Stream (Charney, 1955) is extended to include vertical friction in the cyclonic shear zone (the western side) of the stream. The vertical friction is assumed to be controlled by local Froude conditions.

1. Introduction

The Florida current is characterized by two typical zones; the western (inshore) zone where the shear is cyclonic and the eastern anticyclonic zone. The eastern zone is reproduced by a model of Charney (1955) which however, in conflict with observations, predicts super critical flow in respect to Froude conditions in the western zone, and maximum tangential velocities at the western wall.

Charney suggested that "an internal frictional boundary layer should be introduced to reduce the velocities to zero on the inshore edge of the current". Webster (1965), and Hallock (1973) indicated that in the cyclonic shear zone, horizontal eddies support the main stream. This rules out horizontal eddy viscosity and leaves vertical friction as a possible dominant suppression mechanism.

In an earlier work (Assaf & Hecht, 1974) we noticed that vertical friction can account for the sloping of the interface in sea straits, where Froude and Reynolds numbers have similar magnitude to that of the cyclonic Gulf Stream. It is suspected that when the flow approaches critical conditions, vertical friction will be induced by the growth and breaking of internal waves, a phenomenon investigated in non-rotating fluid (Lofquist, 1960; Turner, 1973).

The application of this kind of friction to the rotating Gulf Stream is not obvious. Stern (1972), has shown that in a uniform flow in a rotating channel, the criterion for the onset of instability is the same as in non-rotating fluid. The Gulf Stream is not a uniform current. (It flows above an inclined front with a slope of five permill and this slope may be important with respect to meandering of the current.) Nonetheless, the friction assumed in this paper is related to local instability of internal waves at the Brunt Vaisala frequency range. The growth and decay of these waves is assumed to be controlled by local Froude–Richardson stability criteria, and thus, it is not influenced by rotation. Our approach here resembles the modelling of Ekman boundary layer (Thompson, 1973). In both cases, rotation is not included in the stability criteria; nonetheless, it is an essential part of the flow.

From Lofquist (1960) experiments, it is possible to conclude that for a given Reynolds number, R_e the drag coefficient C_D increases exponentially with Froude number F_r (Fig. 1). Although the exact

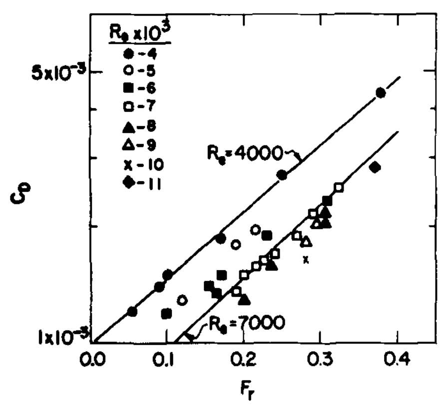

Fig. 1. The drag coefficient C_D vs. Froude number F_r for a different Reynolds number R_e. The figure is based on Lofquist experiments (1960, Table 1), Reynolds numbers are rounded to thousands.

nature of friction for larger Reynolds numbers is debatable, it is commonly believed that friction increases considerably with Froude number (see Turner, 1973, Fig. 9.5, and remember that U_e/U_1 is a measure of C_D, Abraham & Eysink, 1971). In the Gulf Stream the Froude number of the incoming water is less than 0.01. It increases to more than 0.20 in the cyclonic shear zone; it is expected that friction also increases considerably.

The friction assumed in this paper is $C_D v^2/h$ where the velocity is v, and the depth is h (eq. 2). Friction of this nature is expected to regulate the flow when it is comparable with the inertial terms, i.e. with v^2/Y. Here, Y is the horizontal scale of the current along the stream lines (approximately north) in the acceleration zone. In that case, any further increase in velocity due to inertial acceleration will increase the drag coefficient (Fig. 1) and decrease the depth (Charney, 1955). This friction is expected to control the flow when $C_D/h > 1/Y$, or in the case of the Gulf Stream, when $C_D > 5 \times 10^{-4}$ (here, $h = 500$ m and $Y = 10^6$ m).

Our primary concern is to present a mechanism for suppression of the inertial acceleration and keeping the flow in a subcritical condition. This mechanism reduces the velocity at the western zone, but it does not bring it to zero at the wall.

The overall view of the present model can be summarized as follows: Water with constant potential vorticity enters the stream, and behaves as in Charney's scheme as long as critical Froude conditions are not approached. When the flow on the western side approaches critical velocity, friction is induced and the velocity stabilized. The frictional boundary layer is defined as that portion of the stream where the induced friction maintains the flow at some constant internal Froude number. It is identified with the bulk of the cyclonic shear zone of the current. The model is limited to Florida Current between 23° N and 35° N.

2. Formulation

2.1. *General view*

The Gulf Stream is approximated by a homogeneous layer with density ρ flowing above an inactive deep ocean of density $\rho + \Delta\rho$. The equations of motion and continuity are:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g'\frac{\partial h}{\partial x} \tag{1}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g'\frac{\partial h}{\partial y} - \frac{C_D v^2}{h} H(v^2 - g'ah) \tag{2}$$

$$\frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0 \tag{3}$$

Here C_D is the drag coefficient, and H is the Helmholtz step function

$$H = \begin{cases} 0, & v^2 < g'ah \\ 1, & v^2 \gtreqless g'ah \end{cases}$$

h the depth, of the upper layer, $g' = g\Delta\rho/\rho$ the reduced gravity and f Coriolis parameter.

The inertial boundary layer, or the inertial zone, is defined as the domain where $H = 0$, while the frictional zone is defined as the region where $H = 1$. The parameter $a < 1$, is related to the proximity of the current within the frictional boundary layer to critical Froude conditions (at $a = 1$, the flow is critical).

2.2. *The frictional zone*

In the frictional zone it is assumed that:

$$v^2 = g'ah \gg u^2 \tag{4}$$

The energy function $G_f = (u^2 + v^2)/2 + g'h$ is then approximated by:

$$G_f = g'(1 + a/2)h \tag{5}$$

Adding (1) and (2) after multiplying by u and v respectively, we obtain the analog of Bernoulli equation for the frictional zone:

$$u\frac{\partial}{\partial x}(1 + a/2)h + v\frac{\partial}{\partial y}[(1 + a/2)h + aC_D\bar{y}] = 0 \tag{6}$$

Here, $\bar{y} = y - y_m$ measures the distance that the stream line has been under frictional influence, and (x_m, y_m) is the matching boundary between the frictional and the inertial zones. At the coastal edge, $u = 0$ and (6) reduces to:

$$\frac{dh_0}{dy} = \frac{-aC_D}{1 + a/2} \tag{7}$$

where $h_0 = h(0, y)$.

The east–west forces are assumed to be in geostrophic balance. Equation (4) can thus be written as:

$$\frac{g'}{f}\frac{\partial h}{\partial x} = \sqrt{ag'h} \tag{8}$$

Solving for h, we find that:

$$h(x,y) = h_0(y) + \left(\frac{h_0}{L}\right)^{1/2} x + \frac{x^2}{4L} \tag{9}$$

where $L = g'/af^2$.

2.3. *The inertial zone*

As a first approximation, it is assumed that in the inertial boundary layer, the potential vorticity F is conserved along a stream line (this assumption will be modified later on). Furthermore, following Stommel (1966, p. 112) the potential vorticity of the inflowing water is approximated to be constant for all the stream lines. The equation of motion for the inertial zone (Stommel, 1966, p. 111), can be written as:

$$\frac{\partial^2 h}{\partial \bar{x}^2} - \lambda^2 h = -\frac{1}{L} \tag{10}$$

Given our boundary conditions,

$$\frac{g'}{f}\frac{\partial h_m}{\partial x} = \sqrt{g'ah_m}$$

the solution is:

$$h(\bar{x}) = \left(h_m - \frac{f}{F}\right)\cosh \lambda\bar{x} + \left(\frac{a h_m f}{F}\right)^{1/2} \sinh \lambda\bar{x} + \frac{f}{F} \tag{11}$$

where, $\lambda = (Ff/g')^{1/2}$, $\bar{x} = x - x_m$, $h_m = h(x_m, y_m)$.

In the inertial zone, one may write $\partial G/\partial\Psi = F$ (Fofonoff, 1954) or, as F is constant

$$G = G_0 + F\Psi \tag{12}$$

where $G = (u^2 + v^2)/2 + g'h$ is the energy function, and Ψ, the stream function.

2.4. *Streamfunction and potential vorticity*

The northward flow is assumed to be geostrophic in the frictional and the inertial zones. The stream function can then be written as:

$$\Psi = \frac{g'}{2f}[h^2(x,y) - h_0^2(y)] \tag{13}$$

which is valid for the two zones.

Equations (1) and (2) are now coupled with the continuity equation (3) to give the rate of change of the potential vorticity:

$$\frac{dF}{dt} = \frac{1}{h}\frac{\partial}{\partial x}[a g' C_D H(v^2 - ag'h)] \tag{14}$$

is the characteristic width of the stream, and $\hat{h}$ the characteristic depth. For a steady state, eq. (14) is therefore approximated by:

$$\frac{\partial F}{\partial y} = a g' C_D/(\hat{h}\hat{x}\hat{v}) \tag{15}$$

3. Calibration

On the average, the Gulf Stream is stable with respect to internal Froude or Richardson conditions. Current profiles for the straits of Florida as given by Pillsbury (1891) or Düing (1975), indicate that in the cyclonic shear zone, the average speed is about 1 m/s and the current depth is about 200 m. This gives $a = \frac{1}{2}$.

For the computations, the following set of parameters was used:

The reduced gravity $g' = 0.02$ m s^{-2}.
The Coriolis parameter $f = 1.45 \sin\phi \times 10^{-4}$ s^{-1}.
The latitude ϕ varied between 23° and 35°.

The potential vorticity was estimated from the depth of the 10 °C isotherm south-east of Florida current, say 800 m at 25° N (see Stommel, 1966, p. 112), from which: $F = f(25°)/800 = 0.75 \times 10^{-7}$ s^{-1} m^{-1}. The initial energy G_0 was estimated from the depth of 10 °C isotherm in the Gulf of Mexico, $h^* = 500$ m. Neglecting the kinetic energy, compared with the potential energy of the incoming water: $G_0 = g'h^* = 10$ m^2 s^{-2}. The drag coefficient $C_D = 10^{-3}$, as in straits flow (Assaf & Hecht, 1974).

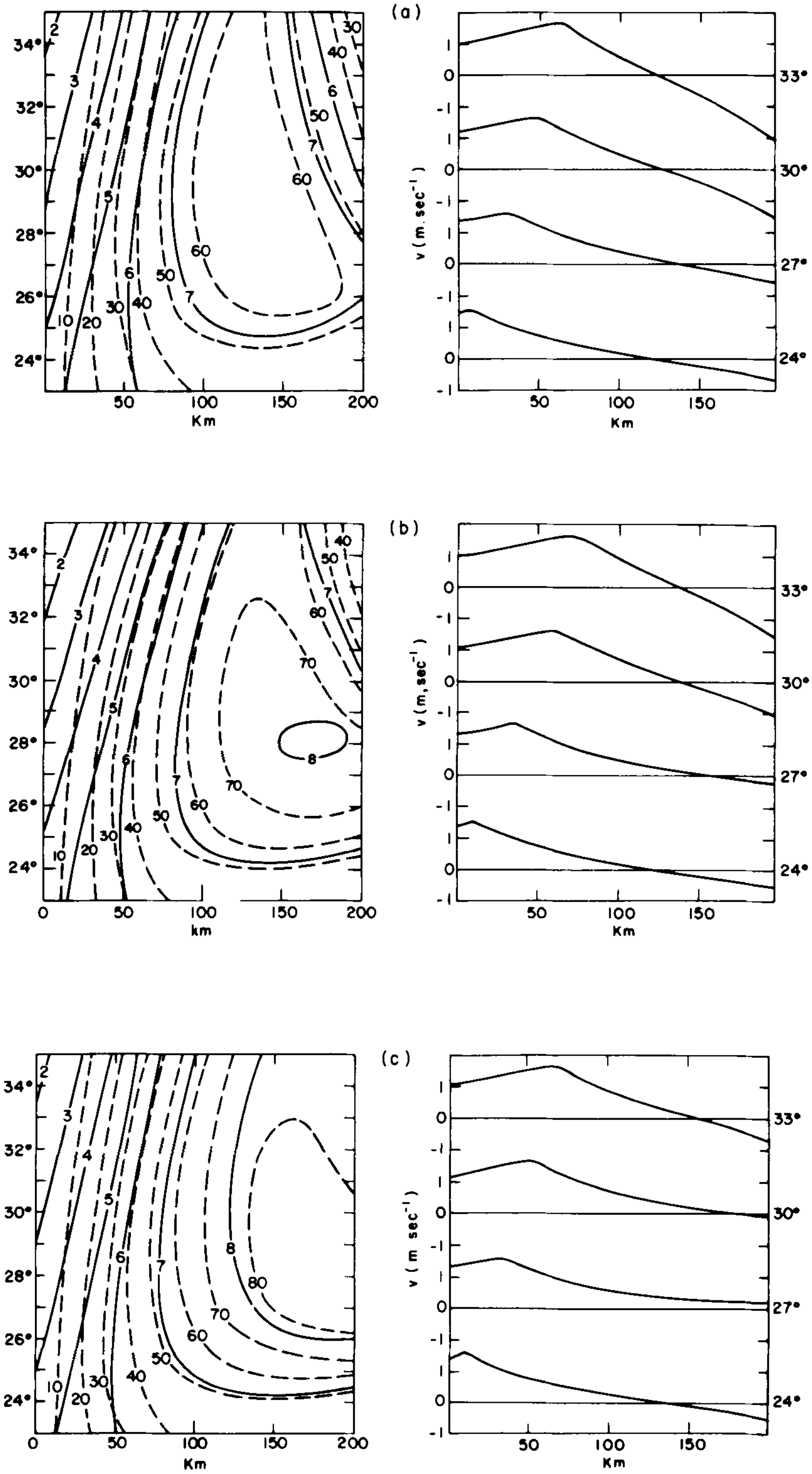
(a)
(b)
(c)
34°
32°
30°
28°
26°
24°
0
50
100
150
200
Km
v (m.sec⁻¹)
33°
30°
27°
24°

Table 1. *Sensitivity of the model for a variation of the parameters*

ϕ	a	F $m^{-1} s^{-1} \times 10^{-7}$	C_D $\times 10^{-3}$	X_M km	Ψ_M $m^3 s^{-1} \times 10^6$	h_M m	V_e $m s^{-1}$
24°	0.25	0.75	0.8	121	43	662	−1.0
34°	0.25	0.75	0.8	111	57	718	−3.2
24°	0.25	0.75	1.4	180	63	739	−0.05
34°	0.25	0.75	1.4	170	86	824	−0.37
24°	0.25	0.70	1.2	100	39	639	−2.1
34°	0.25	0.70	1.2	140	62	719	−2.3
24°	0.25	0.75	1.2	150	56	713	−0.26
34°	0.25	0.75	1.2	160	77	796	−0.82
24°	0.3	0.70	1.2	110	50	675	−1.3
34°	0.3	0.70	1.2	130	63	719	−2.4
24°	0.3	0.725	1.2	150	59	713	−0.44
34°	0.3	0.725	1.2	150	66	740	−1.6

X_M the width of the northward current
Ψ_M the transport of the northward current (Ψ at $x = X_M$)
h_M the depth of the current at $x = X$
V_e the northward velocity at the eastern end (at $x = 200$ km).

The computation procedure can be summarized as follows (see Fig. 2a, b): It is assumed that at the southern edge the flow is inertial except for the point in the south-west corner where the flow is on the verge of frictional balance. For a given G_0,g' and a, $h(0,0)$ is calculated from (5) assuming that $G_f(0, 0) = G_0$. At the southern boundary, $h(x, 0)$ is then computed from (11). Farther north, we apply (7) to compute $h_0(y)$. Equation (9) is then used to calculate $h(x, y)$. At each step after $h(x, y)$ is computed Ψ and G_f are calculated (13 and 5). The matching between the frictional and the inertial zones are found by equating the calculated $G_f(h)$ with $G[\Psi(h)]$. When $G_f(h) = G(\Psi)$, $x = x_m$ and $y = y_m$. Equation (11) is then applied to compute $h(\bar{x}, y)$.

The model was tested for a larger drag coefficient (Fig. 2b) and for nonconstant potential vorticity (Fig. 2c). With the same initial energy G_0 and the same domain of integration. The figures also present the velocity distribution at different latitudes. Further tests of the model for variations of parameters are given in Table 1.

4. Discussion

The results are given in Figs. 2a, b and c. One can see that some of the essential features of the Gulf Stream are reproduced by the model, e.g. the volume transport, the acceleration in the southwest and even a well-defined tendency for separation in the north.

The countercurrent in the north is too large. Nevertheless, one may notice that in Fig. 2c, where the potential vorticity slightly increases with latitude, the countercurrent is closer to the observed one.

The relaxation of the constraint of constant potential vorticity may look at first somewhat arbitrary; it is therefore worth explaining the following two points, (a) and (b):

(a) The volume transport of the Gulf Stream is much larger than expected from the wind field (Stommel, 1966, p. 99). This larger volume transport is due to a return flow of water originating in the Gulf Stream itself (Defant, 1941; Munk & Carrier, 1950). It seems reasonable that a

Fig. 2. The Gulf Stream as predicted by the model for three different conditions:
(a) Constant potential vorticity $F = 7.5 \times 10^{-8}$ $m^{-1} s^{-1}$ and drag coefficient $C_D = 10^{-3}$.
(b) Potential vorticity as in (a) and $C_D = 1.1 \times 10^{-3}$.
(c) Potential vorticity changes linearly between 7.5×10^{-8} at the south ($\phi = 23°$ N) and 8.1×10^{-8} in the north ($\phi = 35°$ N), $C_D = 10^{-3}$.
Solid lines are depth contours of the current in hundreds of meters. Dashed lines are the stream function contours in 10^6 $m^3 s^{-1}$. The velocity profiles across the stream at different latitudes are given at the right.

portion of the returning flow has been exposed to the frictional effects of the Gulf Stream and therefore, increased its potential vorticity. For this reason, the returning flow is expected to be characterized by a larger potential vorticity than the flow from the south.

(b) Equation (15) relates the expected change in the potential vorticity with the frictional coupling. For the above estimate of a, g' and C_D, and for the $\hat{x} = 10^5$ m, $\hat{h} = 10^3$ m and $v = 1$ m/s, one obtains:

$$\frac{\partial F}{\partial y} = 5 \times 10^{-14}\ \mathrm{m^{-2}\,s^{-1}}$$

Or, for a 1000-km downstream distance, the change in the potential vorticity is expected to be 5 $\times$ 10^{-8} $\mathrm{m^{-1}\,s^{-1}}$. This change of potential vorticity takes place at the matching boundary, where the water enters the frictional boundary layer and changes the sign of its relative vorticity.

It is plausible that some of the change in potential vorticity takes place throughout the inertial boundary layer, due to horizontal coupling with the frictional boundary layer. In Fig. 2c, it was assumed that this change of the potential vorticity at the inertial boundary layer is one-tenth of the expected change in the matching zone.

The model with constant potential vorticity is sensitive to variations in F. A 30% change in the volume transport follows a 7% change in F, as seen from lines 5–8 in Table 1. The model is less sensitive to variation in C_D and a.

Excluding the returning flow for the model with constant potential vorticity for reasons discussed above, and limiting ourselves to the northward flow, we can summarize the tolerance of the model in the following ranges of plausible values (Fig. 2 and Table 1):

$0.8 < C_D < 1.4 \times 10^{-3}$
$0.25 < a < 0.3$
$0.7 < F < 0.75 \times 10^{-7}\ \mathrm{m^{-1}\,s^{-1}}$

Comparing Fig. 2a and 2b, one can notice that an increased drag coefficient increases the volume transport. This is somewhat against the intuitive tendency to relate friction with suppression of motion. Closer examination of the model reveals that increased friction extends the width and the depth of the frictional boundary layer and with it, the energy of the inertial zone. Thus, the larger volume transport is associated with accumulation of the potential energy in the Sargasso Sea. This increase in the potential energy, is in fact, induced by friction.

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ГОЛЬФСТРИМ: ИНЕРЦИЯ И ТРЕНИЕ

Инерционная теория Гольфстрима (Чарни, 1955) обобщается путём включения вертикального трения в зону циклонического сдвига скорости (на западной стороне) струи. Предполагается, что вертикальное трение контролируется локальным числом Фруда.