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The spectral balance equation

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ABSTRACT

The formula for generating stream function coefficients in terms of height coefficients developed from the linear balance equation wherein the dependent variables are expanded in solid harmonics has been analyzed and results show that for non-zonal waves the kinetic energy derived increases with decreasing scale. A new set of basis functions has consequently been developed in which the geopotential field may be expanded to yield energetically consistent stream coefficients and simultaneously satisfy the balance condition. A relationship is determined whereby those stream coefficients may be determined from the height coefficients of a solid harmonic expansion and these coefficients are shown to be consistent with those previously derived for the zonal case.

1. Introduction

The balance equation has received scant attention in the last few years when compared to the intensity with which it was studied in the 1950's. This tendency may well reflect the current interest in the primitive equations for atmospheric flow prediction as well as four-dimensional analysis for initialization. Nevertheless, both the theoretical and observational validation of balance in the larger scales of atmospheric flow remain undisputed and that knowledge should be utilized when applicable.

There appears to be a need for this information when interpreting atmospheric observations and assessing kinetic energy distributions. Although wind data is available regularly on a hemispheric basis its vertical structure is not detailed, whereas geopotential height fields are sampled with considerably more vertical resolution. Thus if the rotational part of the wind field incorporates the bulk of atmospheric kinetic energy in the large scale flow (including synoptic scales), the linear balance equation will provide us with energy data based on observations of geopotential height alone. Because there are indications from theoretical analyses (see Charney, 1971) that energy (both kinetic and available potential) should show a statistical distribution dependent on scale, and that this scale should reflect the three dimensionality of the atmosphere, it is imperative that we investigate the scale dependence of energy from observations. It is for such analyses, given the greater quantity of geopotential observations, that we may benefit from our understanding of the balance conditions.

The balance equation, and in particular its linear form, has of course been used frequently for the purpose indicated. Because of our focus on scale dependence, we prefer to consider the balance equation in spectral form wherein the stream function and geopotential are expanded in series of associated Legendre functions. The first application in this domain seems to have been presented by Eliasen & Machenhauer (1965), and they have provided a recurrence relationship for generating the stream coefficients (representing rotational kinetic energy) from the height coefficients which are determined from observations. Since their formula is not self-starting for the zonal coefficients, i.e. a condition is missing, Eliasen & Machenhauer (1965) have ingeniously provided the condition by determining the lowest coefficient from the mean geostrophic angular momentum.

Merilees (1968) subsequently showed that this condition imposed by Eliasen and Machenhauer is the only one which will provide a convergent energy distribution, and implies a termination for the zonal series of stream function coefficients. The non-zonal formulae require no additional condition and consequently lead to non-terminating series. Merilees has shown that these series are not convergent and hence the energy in each wave is unbounded.

We propose to show that the formula of Eliasen and Machenhauer when applied to height coefficients for the non-zonal planetary waves leads to energy distributions with scale which depend predominantly on the equation itself rather than on the data. This result obviously suggests that the recurrence formula is not complete and is in concurrence with Merilees' hesitation over its use. To circumvent this difficulty we propose that the expansion of the height field is performed in a different set of functions, to be determined from the solution of the balance equation on the assumption that the stream function is expanded in associated Legendre polynomials. We shall show that these functions are generated from Legendre polynomials and that the stream coefficients may be determined from height coefficients, even when the height coefficients are given in terms of an expansion in associated Legendre polynomials. However, the equations relating the coefficients will be seen to be complete and the energy will be bounded. Finally, the relations to be derived will also be fully consistent with the zonal constraint imposed by Eliasen and Machenhauer, and justified by Merilees.

2. Expected energy distribution

The linear balance equation considered herein may be written

$$\nabla \cdot \mu \nabla \psi = \nabla^2 \Phi \tag{1}$$

where ψ represents the stream function and Φ the geopotential height. We use surface spherical coordinates λ (longitude) and μ ($\sin \phi$ — latitude) and the dependent variables are nondimensionalized by the earth's radius, rate of rotation and gravity. Following Eliasen & Machenhauer (loc. cit.) both Φ and ψ may be expanded in a series of associated Legendre polynomials such that

$$\psi = \sum_\alpha \psi_\alpha Y_\alpha(\lambda, \mu); \quad \Phi = \sum_\alpha \Phi_\alpha Y_\alpha(\lambda, \mu) \tag{2}$$

where $\alpha = n_\alpha + im_\alpha$, a complex wave number with m denoting the planetary wave number and n representing a two-dimensional scale index (see Baer, 1972) which also refers to the order of the Legendre polynomial when $Y_\alpha = \exp(im_\alpha \lambda) P_\alpha(\mu)$. We may substitute (2) into (1) and generate the recurrence formula established by Eliasen and Machenhauer assuming that the solid harmonics are normalized as well as orthogonal. The relation between the coefficients Φ_α and ψ_α then becomes in the notation of Merilees (loc. cit.)

$$\psi_{\alpha+1} = \frac{n_{\alpha+1}}{n_{\alpha+2}} \frac{\Phi_\alpha}{\epsilon_{\alpha+1}} - \frac{n_\alpha^2 - 1}{n_\alpha(n_{\alpha+2})} \left(\frac{\epsilon_\alpha}{\epsilon_{\alpha+1}} \right) \psi_{\alpha-1}$$

$$\epsilon_\alpha \equiv \left[\frac{n_\alpha^2 - m_\alpha^2}{4n_\alpha^2 - 1} \right]^{1/2} \tag{3}$$

and evidently $\alpha \pm 1 = n_\alpha \pm 1 + im_\alpha$. Since α is clearly defined, we shall find it convenient on subsequent occasions to suppress the α notation when referring to m or n. The limitations posed by eq. (3) when $n_\alpha = m_\alpha = 0$, as indicated in the introduction, should be apparent. We shall, however, only consider (3) for $m \neq 0$. Furthermore, since little data is available in the Southern Hemisphere, we shall assume that the height field is represented by even functions, and the corresponding stream field by odd functions, thereby completing the representation over the sphere. The arguments to be presented may easily be extended to more general parity considerations.

Since we are interested in the dependence of the stream coefficients on the height coefficients (which are known from observations), we apply (3) repeatedly and finally arrive at the following series where $\psi_m^n \equiv \psi_\alpha$;

$$\psi_n^m = a(\alpha) \sum_{j=0}^{\frac{n-m-1}{2}} b_j(m) \Phi_{m+2j}^m \tag{4}$$

$$a(\alpha) \equiv \frac{n}{(n+1)(n-m-2)!!(n+m-2)!!} * \left[\frac{(n-2)!!}{(n-1)!!} \right]^2 *$$

$$* \left[\frac{(2n+1)(n-m-1)!(n+m-1!}{n^2-m^2} \right]^{1/2}$$

$$b_j(m) \equiv (-)^{(n-m-1-2j)/2} (2j-1)!!(2m+2j-1)!!$$

$$* \left[\frac{(m+2j)!!}{(m+2j-1)!!} \right]^2 * \left[\frac{2m+4j+1}{(2j)!(2m+2j)!} \right]^{1/2}$$

Inspection of (4) clearly indicates what Merilees has stated, that there is no termination of the stream series despite a termination of the height series. If we consider the height coefficients to vanish for $n_\alpha > N$, then we see that the ratio of stream coefficients for $n_\alpha > N$ is given by the ratio of the $a(\alpha)$ alone, and does not depend on the height coefficients. We shall consider the implications of this on the energy structure subsequently. We shall first attempt to extend this argument into the domain of nonvanishing height coefficients.

There is some indication that the height coefficients Φ^m_{m+2j} decrease with increasing j, that the significantly largest values are given for the smallest values of j. This indication comes both from the variance calculations of Eliasen & Machenhauer (1965) and the writer's own experience with height expansions. If this is indeed so, the independence of the ratio of stream coefficients from height coefficients may well be extended into the domain $n < N$ dependent of course on the characteristics of the coefficients $b_j\,(m)$ (see eq. (4)). Calculations of b_{j+1}/b_j shows it to be in the domain $0.79 \leq b_{j+1}/b_j \leq 1.13$ for the range of values m and j, $15 \leq n = m + 2j \leq 25$. We have chosen this range because it reflects the scales in which energy statistics comparable to theory might be found (see Baer, 1972). These values for the ratio of terms suggest that the magnitude of the coefficients Φ^m_{m+2j} themselves and not their coefficients b_j determine the significance of each successive term to the series (4). Thus so long as the ratio $\Phi^m_{m+2j+2}/\Phi^m_{m+2j} < 1$, as indicated by observations, each additional term in the series (4) after the first few (and assuming that the first few are the largest) does not contribute significantly to the series.

We may conclude from the previous discussion that successive coefficients of the stream function ψ_α as given by (4) will depend only on one another and the factors $a(\alpha)$ except for those with the smallest allowed values of n which will indeed depend on the height coefficients. We shall now attempt to establish the energy distribution implied by these stream coefficients. The energy in each wave component α is given by E_α and the energy in each two dimensional scale element n_α is E_n defined as

$$E_\alpha = n(n+1)\psi_\alpha\psi^*_\alpha; \quad E_n = \Sigma\, E_\alpha \tag{5}$$

Presuming now that we are dealing with those elements of ψ_α for which $\Sigma b_j \Phi^m_{m+2j}$ does not change significantly, we take the ratio of energies in two successive scales given from (5) and utilizing (4) for the stream coefficients we find,

$$E_{\alpha+2} = \frac{n}{n+3}\left(\frac{n+2}{n+1}\right)^3 \frac{2n+5}{2n+1} \frac{(n+1)^2-m^2}{(n+2)^2-m^2} E_\alpha \tag{6}$$

We must now perform a summation over all allowed m-values, $m < n$ to determine the energy in terms of n alone, as seen from (5). If $m \ll n$ this clearly would create no difficulty but for several values of m such as $n-1$ and $n-3$ the weighting of the energy coefficients E_α should be considered. Note that we have chosen $n-m$ odd. For simplicity of argument let us assume that the m-distribution of energy for a given n-scale is the same in any two sucessive n-scales (say n and $n+2$) and that the energy in any component can be represented by some mean value. Thus we would have that

$$\sum_m m^2 E_\alpha \approx \bar{E}_\alpha \sum m^2 = \frac{n(n-1)}{3} E_n \; (n \text{ odd})$$
$$= \frac{n^2-1}{3} E_n \; (n \text{ even}) \tag{7}$$

If the energy decreases with m-scale as anticipated from observations our sum will be overestimated. However, if our assumption of equivalence of distribution holds, that overestimate will be the same in both n-scales and will tend to compensate. If we now perform a sum over all allowed m-values in (6) after first multiplying by $(n+2)^2 - m^2$, and utilizing the result of (7), we get,

$$E_{n+2} = E_n * \begin{cases} \dfrac{n}{n+1}\left(\dfrac{n+2}{n+1}\right)^2 & n \text{ odd} \\ \dfrac{2n}{n+3}\,\dfrac{2n+5}{2n+1}\,\dfrac{(n+2)^4}{(n+1)^2} & \\ \quad \times \dfrac{1}{2n^2+8n+9} & n \text{ even} \end{cases} \tag{8}$$

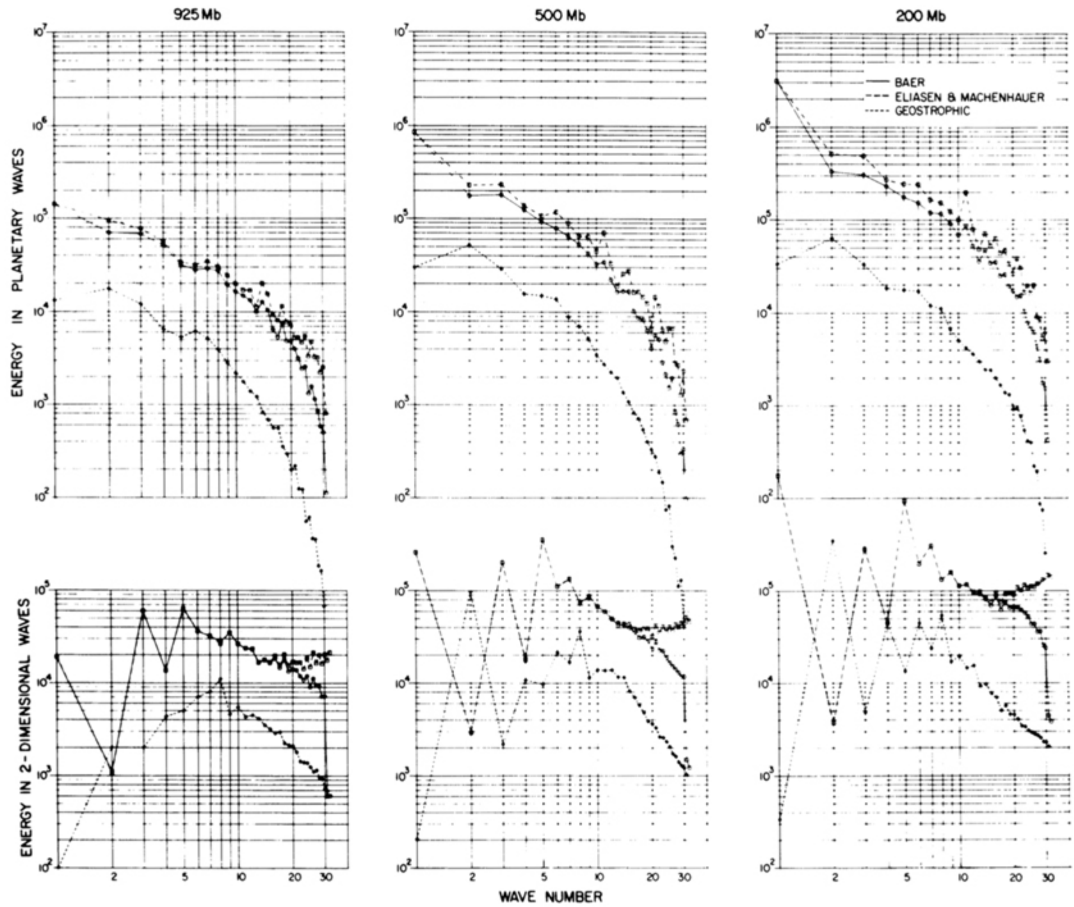

Fig. 1. Time mean energy as a function of wave number (both planetary and two-dimensional) at three pressure levels for the winter, 1970, utilizing three variants of the balance equation.

Now if (8) were consistent with the expected theory for isotropic scale characteristics, the energy should fall off with index n according to $E_n \propto n^{-3}$ which is clearly not the case. Indeed if we assume a distribution for E_n of the form $E_n = An^b$ and compare the derivative to the finite difference given from (8), we have for the slope,

$$b = \frac{n}{2}\left[\frac{n(n+2)^2}{(n+1)^3} - 1\right] \tag{9}$$

where we have used the odd relationship from (8) for simplicity. The exponent b gives a limiting value for $b \to 0.5$ for large n. Indeed, for n as small as 10 we find $b > 0.40$.

Let us consider another alternative to calculating the energy distribution. Assume that E_{α} decays at a rate steeper than m^{-2} with m for given n, an assumption again borne out by observations. We may with this condition legitimately ignore all reference to m^2 in (6) under summation and come immediately to an alternate form of (8) as,

$$E_{n+2} = \frac{n}{n+3}\,\frac{n+2}{n+1}\,\frac{2n+5}{2n+1}\,E_n \tag{8a}$$

and a corresponding slope value of

$$b = -\tfrac{3}{2}n[(n+3)(n+1)(2n+1)]^{-1} \tag{9a}$$

We note that in this case $b < 0$, but its value is extremely small going to zero as n^{-2} and having a value less than 10^{-2} for $n = 15$.

These results indicate that formula (3) implies essentially no decay of energy with increasing scale, and that distribution is independent of the observed data. We have analyzed two months of height data (January–February 1970) and calculated the energy therefrom utilizing (3). The results describing the time mean energy (eq. (5)) in terms of the two-dimensional wave number n may be seen on the dashed curves of the lower part of Fig. 1 for three different pressure levels. In all cases we see that the energy amplitude tends to increase with increasing wave number for the shortest scales, as anticipated by the above analysis. For comparison we show in the upper part of Fig. 1 the energy distribution in terms of the planetary wave number (the more conventional representation). Here we note that the energy continues to decay with increasing wave number when using (3). The implications of this section thus suggest that we must find an alternative to (3) if we hope to establish the scale dependence of atmosphere energy from a balance condition.

3. Alternative expansion functions

We have already indicated the convenience for calculating energy associated with an expansion of the stream function in solid harmonics. We need not, however, confine ourselves to such an expansion for the height field. Indeed, an alternate approach would be to establish those functions in which the height field should be expanded such that the balance equation is satisfied for an expansion of the stream function in solid harmonics. Let us consider then the following representations:

$$\psi = \Sigma\psi_\alpha e^{im_\alpha\lambda}P_\alpha(\mu); \quad \Phi = \Sigma\Phi_\alpha e^{im_\alpha\lambda}H_\alpha(\mu) \tag{10}$$

The difference between these equations and those of (2) is simply that we have introduced the functions H_α, as yet unknown. For simplicity in the subsequent discussion the Legendre polynomials (P_α) will be considered unnormalized. We shall now attempt to establish the functions H_α. Substitution of (10) into the balance equation (1) and integration over λ after multiplying by $\exp(-im_\alpha\lambda)$ yields an equation for each value of m dependent on latitude alone;

$$\sum_{n_\alpha} \Phi_\alpha\left\{\frac{d}{d\mu}(1-\mu^2)\frac{dH_\alpha}{d\mu} - \frac{m_\alpha^2}{1-\mu^2}H_\alpha\right\}$$

$$= \sum_{n_\alpha}\psi_\alpha\left\{\frac{d}{d\mu}\mu(1-\mu^2)\frac{dP_\alpha}{d\mu} - \frac{m_\alpha^2\mu}{1-\mu^2}P_\alpha\right\} \tag{11}$$

If we now choose the H_α to satisfy (11) for each n_α, the balance equation will be satisfied identically and the stream function coefficients will be given directly from the height coefficients. The differential equation for H_α which we must solve is,

$$L(H_\alpha) \equiv [(1-\mu^2)H_\alpha']' - \frac{m^2H_\alpha}{1-\mu^2}$$

$$= [\mu(1-\mu^2)P_\alpha']' - \frac{m^2\mu P_\alpha}{1-\mu^2} \tag{12}$$

where the prime denotes differentiation with respect to μ. The P_α are known, and for boundary conditions on the H_α we will require regularity at the poles. Eq. (12) is an inhomogeneous Sturm–Liouville equation and may be solved by utilization of the appropriate Green's function. Although our primary interest is for solution H_α in the wave domain ($m_\alpha \neq 0$) we shall include $m = 0$ for completeness.

The solution to (12) can be shown to be,

$$H_\alpha = -\int_{-1}^{1} G(\mu, z)\,[(z(1-z^2)P'(z))'$$

$$-\frac{m^2z}{1-z^2}P_\alpha(z)]dz \tag{13}$$

where the Green's function used in (13) is given as

$$G(\mu, z) = \frac{1}{2m}\left\{\frac{(1\pm\mu)(1\mp z)}{(1\mp\mu)(1\pm z)}\right\}^{m/2} \qquad m \neq 0$$

$$G(\mu, z) = -\tfrac{1}{2}\ln[(1\mp\mu)(1\pm z)] + \ln 2 - \tfrac{1}{2} \qquad m = 0 \tag{14}$$

where the upper sign holds for $\mu \le z$ and the lower sign for $\mu \ge z$. Carrying out the integrations of (13) using (14) and the definition of the Legendre polynomials leads to the solution for H_α in terms of Legendre polynomials as follows;

$$(2n+1)H_\alpha = \frac{n-m+1}{n+1}nP_{\alpha+1}$$

$$\times\frac{(n+m)(n+1)}{n}P_{\alpha-1}(1-\delta_{n,0}) \tag{15}$$

The H_α are thus linear combinations of the P_α; they are complete, including the zonal functions. Details leading to (15) may be found in a report by the writer (Baer, 1975).

We note that this alternate representation provides us with an expansion series which will satisfy the balance equation. It should also be apparent from (15) that these functions have opposite parity to the particular wave vector under consideration.

4. Calculation of stream coefficients

We are now in a position to expand the observed height field in a series given by (10) where the expansion functions H_α are defined in terms of known functions (15). There are a number of ways in which this expansion may be performed computationally but we must deal with the fact that the H_α are not orthogonal over the domain. Once the observation points in latitude are established, an orthogonal set of functions may be generated by a Gramm–Schmidt procedure (see Fougere, 1963, for example). Since we are accustomed and programmatically adapted to data field expansion in terms of solid harmonics, it is probably most direct—as well as enlightening—to utilize the coefficients of such an expansion for determination of the coefficients in the new basis functions, H_α. We shall proceed in this vein.

Let us now assume that the height field (on any pressure surface and at any time) has been expanded in solid harmonics and that the coefficients are known; i.e.,

$$\Phi = \sum_\alpha \hat{\Phi}_\alpha Y_\alpha(\lambda, \mu) \tag{16}$$

where the notation remains the same. Although it is customary to perform such expansions with normalized functions, our discussion will presume the functions to be unnormalized.

Consistent with our previous discussion, we shall choose the expansion (16) in terms of even functions ($n + m$ even) and furthermore utilize triangular truncation, terminating the series at a maximum scale N such that $n_\alpha \le N$ and $\alpha_{\max} = N(1 + i)$. We now equate (16) to a series for the stream function. Because of the nature of the H_α (they satisfy the balance equation) we may expand the stream coefficients in these functions directly and we find,

$$\Phi = \sum_\beta \psi_\beta H_\beta(\mu) e^{im_\beta \lambda} \tag{17}$$

where to avoid confusion we have chosen β as the wave vector. Note from (15) that if we plan to equate (17) with (16), the β vectors must be odd, i.e., $n_\beta + m_\beta$ odd. Since the ψ_β are as yet unknown, the truncation of series (17) remains to be established. Note also that the longitudinal (λ) dependence of both (16) and (17) are identical. We thus find that the stream coefficients for a given planetary wave will depend only on the height coefficients of that wave in the form,

$$\sum_{k+0}^{[N-m/2]} \hat{\Phi}^m_{m+2k} P^m_{m+2k} = \sum_{j=0} \psi^m_{m+2j+1} H^m_{m+2i+1} \tag{18}$$

where we have utilized the parity conditions just stated and the fact that $n \ge m$. The series limit on k must be chosen as the integer part of the ratio stated.

The best values of ψ^m_n will be determined from (18) by a least-squares minimizing procedure. We shall simultaneously presume the error to be distributed over the same number of stream coefficients as there are height coefficients. Thus for each planetary wave m we may generate a set of stream coefficients equal in number to the given height coefficients. The least-squares procedure leads to a set of equations for the stream coefficients which may be established by multiplication of (18) by each of the H^m_n and integration over the domain (from pole to pole). The number of such equations, as implied above, will be determined exclusively by the values of N and m. By virtue of the definition of H_β (eq. (15)) the process outlined will lead to equations which will relate at most three successive stream coefficients to at most two successive height coefficients. These results may be seen readily as follows. Rewriting (15) in terms of m and i,

$$H^m_{m+2i+1} = a_i P^m_{m+2i} + b_i P^m_{m+2i+2}$$

$$a_i \equiv \frac{(2m+2i+1)(m+2i+2)}{(m+2i+1)(2m+4i+3)} (1 - \delta_{m+i,0})$$

$$b_i \equiv \frac{(2i+2)(m+2i+1)}{(2m+4i+3)(m+2i+2)} \tag{19}$$

multiplication of (18) by H^m_{m+2i+1} and integrating leads to the equation,

$$\gamma^m_{i,i+1}\psi^m_{m+2i+3} + \gamma^m_{i,i}\psi^m_{m+2i+1} + \gamma^m_{i,i-1}\psi^m_{m+2i-1} = \sigma^m_{i,i}\hat{\Phi}^m_{m+2i} + \sigma^m_{i,i+1}\hat{\Phi}^m_{m+2i+2}$$

$$\gamma^m_{i,i+1} \equiv a_{i+1} b_i K_{i+1}; \quad \gamma^m_{i,i} \equiv a_i^2 K_i + b_i^2 K_{i+1}$$

$$\gamma^m_{i,i-1} \equiv a_i b_{i-1} K_i; \quad \sigma^m_{i,i} \equiv a_i K_i;$$

$$\sigma^m_{i,i+1} \equiv b_i K_{i+1}$$

$$K_i \equiv \int_{-1}^{1} (P^m_{m+2i})^2 d\mu = \frac{1}{2m+4i+1} \frac{(2m+2i)!}{(2i)!} \tag{20}$$

Combining the equations for all allowed values of i given by (20) leads to a matrix equation for the coefficients ψ_n^m in the form

$$A^m \Psi^m = B^m \Phi^m$$

$$\Psi^m \equiv \begin{Bmatrix} \psi_{m+1}^m \\ \vdots \\ \psi_{m+2i+1}^m \\ \vdots \end{Bmatrix}; \quad \Phi^m \equiv \begin{Bmatrix} \hat{\Phi}_m^m \\ \vdots \\ \hat{\Phi}_{m+2i}^m \\ \vdots \end{Bmatrix} \tag{21}$$

$$A^m \equiv \{\gamma_{i,j}^m\} \quad ; \quad B^m \equiv \{\sigma_{i,j}^m\}$$

Note in (21) that $\gamma_{i,j}^n = 0$ unless $j = i$, $i \pm 1$, and that $\sigma_{i,j}^m = 0$ unless $j = i$, $i \pm 1$. The solution of (21) will be completed by multiplication with the inverse matrix of A^m. We need not dwell on this computational point, but we do recognize that A^m is a tridiagonal matrix which makes the solution to (21) particularly simple (see, for example, Forsythe & Wasow, 1960).

The stream function coefficients derived in (21) are the alternative coefficients to those given by formulae (3) as originally derived by Eliasen and Machenhauer. They differ from (3) for waves $m \neq 0$ and have convergent energetic properties, as well as satisfying the balance equation. Although the solution to (3) will satisfy (21), they will do so only for an infinite set. Since we are constrained both observationally and computationally to a finite set of energy coefficients, (21) will yield a different set from (3). This may be seen from the bottom curves of Fig. 1, where the solid curves describe the energy distribution with two-dimensional scale based on the stream field generated from (21). Note that the energy decays with decreasing scale to the limit N. For comparison we include the energy structure calculated from the geostrophic assumption (f constant) as the dotted curves. The geostrophic energy also decreases with decreasing scale.

For the zonal case ($m = 0$), (21) yields results identical to those derived from (3), for which energetic consistency has been proved (Merilees, loc. cit.). We may see this correspondence in the zonal coefficients as follows. The stream coefficients which we have derived stem from the variational equation,

$$\delta \int (\nabla^2\Phi - \nabla\cdot\mu\nabla\psi)^2 d\lambda d\mu = 0 \tag{22}$$

where the variation takes place on the stream function. With the stream function given by the series (2), and assuming that the variation of the coefficients $(\delta\psi_\alpha)$ must be arbitrary, (22) becomes

$$\int (\nabla\cdot\mu\nabla Y_\alpha)(\nabla^2\Phi - \nabla\cdot\mu\nabla\psi) d\lambda d\mu = 0$$

$$= c_\alpha \int Y_{\alpha+1}(\nabla^2\Phi - \nabla\cdot\mu\nabla\psi) d\lambda d\mu + d_\alpha \times \int Y_{\alpha-1}(\nabla^2\Phi - \nabla\cdot\mu\nabla\psi) d\lambda d\mu \tag{23}$$

utilizing the recursion formulae for Legendre polynomials. We now recall that formulae (3) was derived from the following integral,

$$\int Y_\alpha (\nabla^2\Phi - \nabla\cdot\mu\nabla\psi) d\lambda d\mu = 0 \tag{24}$$

As noted by Merilees, (24) gives rise to a non-terminating divergent series for ψ for non-zonal waves ($m \neq 0$). Thus our conclusion that the solutions to (21) will differ from those of (3). However, the choice of the first zonal coefficient to represent the mean angular momentum of the flow as utilized by Eliasen and Machenhauer was precisely the condition which also terminated the *zonal* series for ψ and yields a finite number of equations from (24). These equations clearly also satisfy (23) and thus, if we terminate the stream series developed by (23) at the same point as (24), the two results will be entirely equivalent, as suggested earlier.

5. Conclusions

The spectral form of the linear balance equation in common use has been investigated for non-zonal scales with regard to the scale dependence of kinetic energy derived therefrom. Results of the analysis indicate that the energy tends to increase with scale as one proceeds to shorter scales rather than to decrease as theory would expect.

Based on this inconsistency, we have reinvestigated the characteristics of the balance equation, and have generated a new set of basis functions which satisfy the equation and in which the observed geopotential field may be expanded to yield a set of stream coefficients for solid harmonics. These new functions are relatively simple, consisting of the sum of two associated Legendre polynomials, and should yield readily to data expansion.

If, however, one chooses to expand the geo-

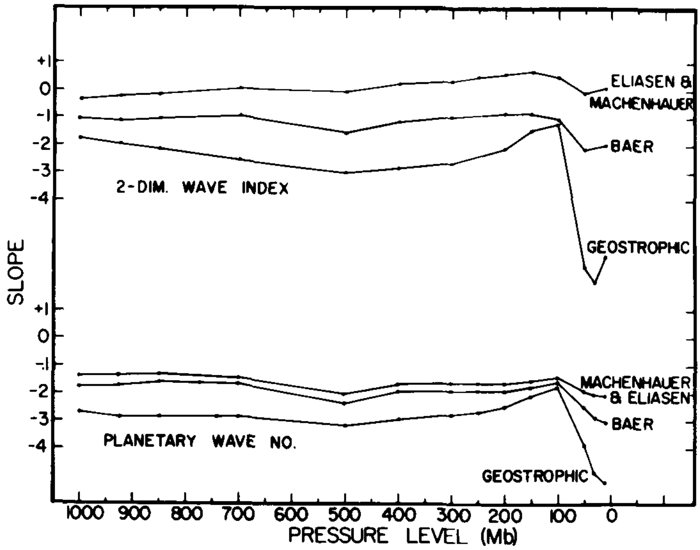

Fig. 2. Statistical slope of energy decay with wave number based on a power law for time mean energy data (winter, 1970) at 14 pressure levels utilizing three variants of the balance equation. Figure describes the slope in both the planetary wave number and the two-dimensional spherical wave number.

potential field in terms of solid harmonics as is conventionally done, a consistent set of stream function coefficients may still be developed. We have provided a matrix equation for generating these stream coefficients from the given height coefficients in an energetically consistent fashion and for each planetary wave independently. The results are also shown to be consistent with the zonal equation in accepted use, one that has been shown to contain the proper energy constraints.

Two months of data (January–February 1970) at a number of pressure levels have been utilized to obtain stream coefficients by the conventional procedure (Eliasen & Machenhauer, 1965), by the method recommended herein (Baer) and by geostrophic balance (f constant). These results, averaged in time, have been converted to energy as a function of n (two-dimensional wave number) and have been fit to a logarithmic decay law. The range wherein the law is applicable was chosen as $15 \leq n \leq 25$. The best fit of the power at which the energy decays with wave number is plotted for a number of pressure levels on Fig. 2. We see that at most levels the slope based on the formula of Eliasen and Machenhauer is greater than zero whereas the slope based on the technique recommended herein is in the vicinity of -1. The slope based on geostrophic balance is still more negative. For contrast we describe on Fig. 2 the slopes for energy decay of the same data set based on the planetary wave index (m) in the range $8 \leq m \leq 18$. Although the relative distribution of these methods are similar to those for the two-dimensional structure, the slopes are systematically steeper.

If theoretical expectations are indicative of what the data should show, our new formula is a step in the right direction since a slope of -3 might be anticipated. However, it is clearly far from -3 as seen from Fig. 2. Current analyses are under way to determine what the spectral structure of the rotational part of the energy is when taken directly from wind observations. These results should determine whether further refinements in calculating balanced energetics are necessary.

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СПЕКТРАЛЬНОЕ УРАВНЕНИЕ БАЛАНСА

Анализируется формула, выражающая коэффициенты функции тока через коэффициенты высоты изобарической поверхности, выведенная из линейного уравнения баланса, в котором зависимые переменные разложены в ряд по сферическим гармоникам. Результаты показывают, что для незональных волн кинетическая энергия возрастает с убыванием масштаба. Выведен новый набор базисных функций для разложения поля геопотенциала, приводящий к энергетически согласованным коэффициентам функции тока и удовлетворяющий условию баланса. Определяется соотношение, с помощью которого находятся эти коэффициенты функции тока по коэффициентам разложения высоты в ряд по сферическим гармоникам. Показано, что найденные коэффициенты согласуются с полученными ранее для зонального случая.