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On the coalescence of water droplets in turbulent clouds

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ABSTRACT

It is shown that of the nine components of the flow velocity gradient tensor, only the two which represent the vertical gradient of the horizontal velocity components contribute significantly to increase the collision rate of small droplets above the still-air value. Hence, to predict the rate at which droplets collide, the turbulent motion within a developing cloud is modelled by a uniform vertical shear flow which varies randomly in time.

1. Introduction

Laboratory experiments of Jonas & Goldsmith (1972) show that a uniform vertical shear flow produces a significant increase in the rate of coalescence of small water droplets of comparable size above the coalescence rate in still air. These results can be explained if it is assumed that there is an interaction region associated with the wake of a droplet such that other droplets inside this region eventually collide with that droplet (Manton, 1974). The simplest model of such an interaction region is a cylindrical volume of uniform circular cross-section of radius r_i and of length L extending vertically behind a falling droplet. Thus, any other droplet swept into the cylindrical interaction region of a given droplet by a relative shear flow eventually collides with it.

Now the air motion within a developing cloud is invariably turbulent; that is, it consists of an unsteady and non-uniform vorticity distribution. Hence, it would appear that the motion cannot be modelled by a simple uniform shear flow. This is because the turbulent time scale is much smaller than the expected time between droplet collisions (Manton, 1974). On the other hand, the flow may be modelled by a spatially uniform but temporally unsteady shear if the linear dimension of the swept-out volume for a collision is small compared with the length scale of the turbulence. Then colliding droplets are embedded in essentially the same turbulent eddy at any instant. It is shown below that this is indeed the case for droplets of comparable size.

When the droplet radii differ greatly, the expected initial separation is much larger than the turbulent length scale. However, in this case, coalescence is primarily due to the greater terminal velocity of the larger droplet, which therefore catches up to the smaller. Because the turbulent motion conserves volume, the coalescence process is unaffected by the turbulence (Manton, 1974). Thus, it remains to consider the effect of turbulence on the coalescence of droplets of comparable size.

Although the relative motion of colliding droplets may be described adequately by assuming the air motion to be an unsteady, uniform shear flow, the velocity gradient tensor $\partial u_i/\partial x_j$ contains nine components in general and so the analysis could become quite complex. But it is found that of the six off-diagonal terms of $\partial u_i/\partial x_j$, only the two vertical gradients produce significant coalescence (Manton, 1974). The first task of the present work is to show that convergences and divergences within the flow give rise to negligible contributions to the rate of coalescence. Thus, in addition to the horizontal gradient off-diagonal terms, all the diagonal terms of $\partial u_i/\partial x_j$ may be neglected in the analysis. That is, the effect of turbulence on the rate of coalescence of droplets may be modelled by an unsteady, uniform vertical shear flow. Consequently, the expected rate of coalescence is calculated below.

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2. Droplet coalescence in a converging flow

The velocity vector $\mathbf{u}$ describing a uniform convergence in an incompressible fluid is given by

$$u_1 = C_1 x_1, \quad u_2 = C_2 x_2, \quad u_3 = -(C_1 + C_2)x_3, \quad (2.1)$$

where (x_1, x_2, x_3) represents a rectangular coordinate system such that x_3 increases vertically upwards; C_1 and C_2 are constant flow strain rates. The velocity vector $\mathbf{v}$ for the motion of a small droplet (radius of order $10\,\mu$m) within a developing cloud is described well by the equation (Manton, 1974)

$$v_i = u_i - W\delta_{i3}, \quad (2.2)$$

where W is the terminal velocity of the droplet and δ_{ij} is the Kronecker delta. Equation (2.2), which results from a balance of the Stokes drag and the gravitational force on a droplet, is valid except when droplets are close enough together for mutual interaction to become important (i.e. when one is inside another's interaction region). When this occurs, the droplet motion is independent of the external fluid flow, and the interaction is as for droplets in still air. That is, a shear flow cannot increase the probability of a collision between two neighbouring droplets. It can at most increase the number of times that droplets pass within a given distance of each other.

From (2.1) and (2.2), it is seen that the position $\mathbf{x}$ at time t of a droplet in a convergence is

$$\begin{aligned} x_1 &= x_{10}\, e^{C_1 t}, \\ x_2 &= x_{20}\, e^{C_1 t} \\ x_3 &= \{x_{30} + W/(C_1 + C_2)\}\, e^{-(C_1+C_2)t} - W/(C_1 + C_2), \end{aligned} \quad (2.3)$$

where $\mathbf{x} = (x_{10}, x_{20}, x_{30})$ at $t = 0$. To determine the motion of a droplet of radius R_2 relative to a larger droplet of radius R_1, we introduce the relative position vector

$$y_i = x_i^{(2)} - x_i^{(1)}, \quad (2.4)$$

where $\mathbf{x}^{(j)}$ is the position vector for the droplet of radius R_j. Equations (2.3) and (2.4) give

$$\begin{aligned} y_1 &= y_{10}\, e^{C_1 t}, \\ y_2 &= y_{20}\, e^{C_1 t}, \quad (2.5) \\ y_3 &= U/(C_1 + C_2) + \{y_{30} - U/(C_1 + C_2)\}\, e^{-(C_1+C_2)t}, \end{aligned}$$

where $y_{i0} = x_{i0}^{(2)} - x_{i0}^{(1)}$ and $U = W_1 - W_2$ is the relative terminal velocity of the droplets.

Now collisions may occur by two different means. The first method is simply due to the larger droplet overtaking the smaller. Thus, a collision occurs for all small droplets passing through the circular surface

$$y_3 = 0 \quad \text{with} \quad y_1^2 + y_2^2 < r_0^2, \quad (2.6)$$

where r_0 is given by the usual collision efficiency E_0 in still air; that is,

$$E_0 = r_0^2/(R_1 + R_2)^2.$$

Droplets passing through the surface (2.6) at time t originate from the elliptic surface

$$y_{30} = \{U/(C_1 + C_2)\}\{1 - e^{(C_1+C_2)t}\},\ y_{10}^2\, e^{2C_1 t} + y_{20}^2\, e^{2C_2 t} < r_0^2.$$

This surface has an area of $\pi r_0^2\, e^{-(C_1+C_2)t}$, and so the swept-out volume in time t is

$$\begin{aligned} V_0(t) &= -\int_0^t \pi r_0^2\, e^{-(C_2+C_2)t}\, dy_{30} \\ &= \pi r_0^2\, Ut, \end{aligned} \quad (2.7)$$

independently of the shear flow.

The second way in which a collision can occur is for a small droplet to be swept by the shear flow into the interaction region of the large droplet. For simplicity, the interaction region is taken to be the volume

$$0 < y_3 < L \quad \text{with} \quad y_1^2 + y_2^2 < r_i^2. \quad (2.8)$$

We consider the swept-out volume V_s in time t due to three types of convergence; namely, (i) $C_1 > 0$ and $C_2 > 0$, (ii) $C_1 < 0$ and $C_2 < 0$, and (iii) C_1 and C_2 of opposite signs.

It is seen from Fig. 1 that if $L \leq U/(C_1 + C_2)$ in case (i), then V_s equals zero and collisions occur only by the large droplet overtaking smaller ones. On the other hand, if $L > U/(C_1 + C_2)$ then there is a flux of droplets through the surface

$$y_3 = L \quad \text{with} \quad y_1^2 + y_2^2 < r_i^2.$$

From (2.5), we see that droplets passing through this surface at time t, initially occupy the elliptic surface

$$y_{30} = \{L - U/(C_1 + C_2)\}e^{(C_1+C_2)t} + U/(C_1 + C_2)$$
$$\text{with } y_{10}^2\, e^{2C_1 t} + y_{20}^2\, e^{2C_2 t} < r_i^2.$$

Therefore,

$$\begin{aligned} V_s &= \int_0^t \pi r_i^2\, e^{-(C_1+C_2)t}\, dy_{30}, \\ &= \pi r_i^2\{L(C_1 + C_2) - U\}t. \end{aligned} \quad (2.9)$$

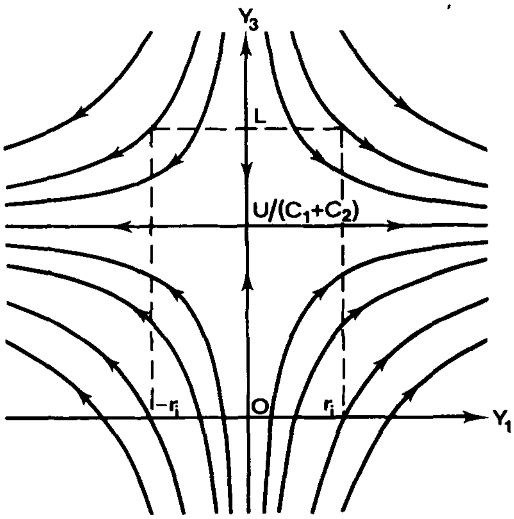

Fig. 1. Path lines of small droplets relative to larger droplet in uniform convergence for $C_1 > 0$ and $C_2 > 0$.

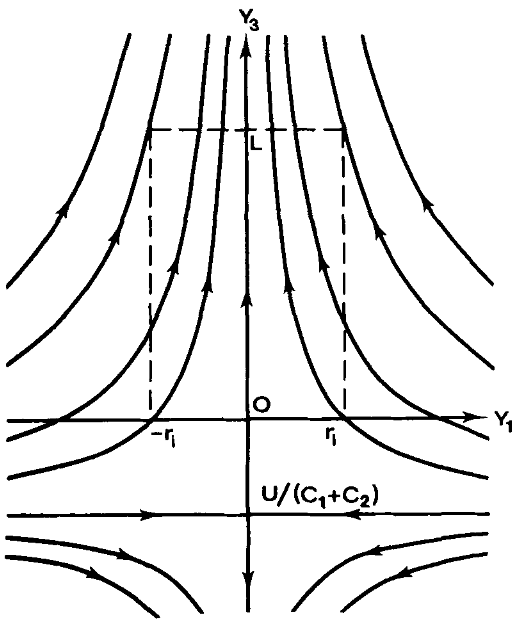

Fig. 2. Path lines of small droplets relative to larger droplet in uniform convergence for $C_1 < 0$ and $C_2 < 0$.

Hence, for $C_1 > 0$ and $C_2 > 0$, the total swept-out volume by a large droplet is

$$V_T = V_0 + V_s = \begin{cases} \pi r_0^2\, Ut & \text{for} \quad L \le U/(C_1 + C_2). \\ \pi r_i^2 L(C_1 + C_2)t + \pi Ut\,(r_0^2 - r_i^2) & \text{for} \quad L > U/(C_1 + C_2). \end{cases} \qquad (2.10)$$

Droplet path lines corresponding to case (ii) are

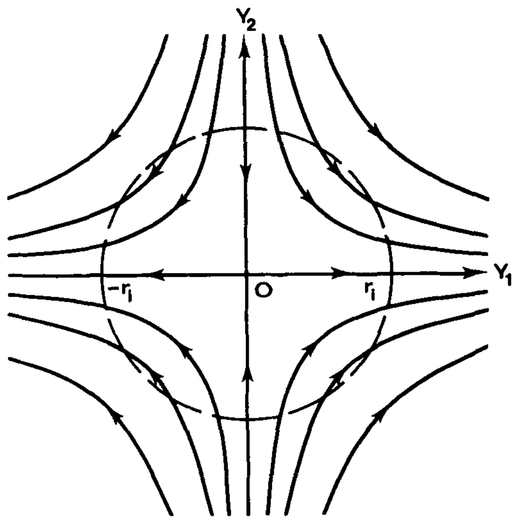

Fig. 3. Path lines of small droplets relative to larger droplet in uniform horizontal convergence for $C_1 = -C_2 = C$.

shown in Fig. 2. From this, it is clear that the swept-out volume V_s induced by the convergent flow is given by the flux through the surface $y_3 = L$, $y_1^2 + y_2^2 < r_i^2$ minus the flux through the surface $y_3 = 0$, $y_1^2 + y_2^2 < r_i^2$. Using (2.5) and (2.7), we find that

$$V_s = -\pi r_i^2 L(C_1 + C_2)t,$$

and hence the total swept-out volume when $C_1 < 0$ and $C_2 < 0$ is

$$V_T = \pi r_0^2\, Ut - \pi r_i^2 L(C_1 + C_2)t. \qquad (2.11)$$

Case (iii) when C_1 and C_2 are of opposite signs is generally more complex than the previous cases. Thus, we consider the particular situation in which $C_1 = -C_2 = C$, say. This corresponds to a purely horizontal convergence. The neglect of vertical convergence is allowable because its effect is found from cases (i) and (ii). Fig. 3 shows that the shear flow sweeps small droplets into the interaction region through the surfaces

$$0 < y_3 < L, \quad y_2 > 0, \quad |y_1| < r_i/2^{1/2} \quad \text{with} \quad y_1^2 + y_2^2 = r_i^2$$

and

$$0 < y_3 < L, \quad y_2 < 0, \quad |y_1| < r_i/2^{1/2} \quad \text{with} \quad y_1^2 + y_2^2 = r_i^2$$

Hence, the cross-section of the swept-out volume V_s is as shown in Fig. 4. Therefore, from (2.5),

$$V_s = 2r_i^2 LCt,$$

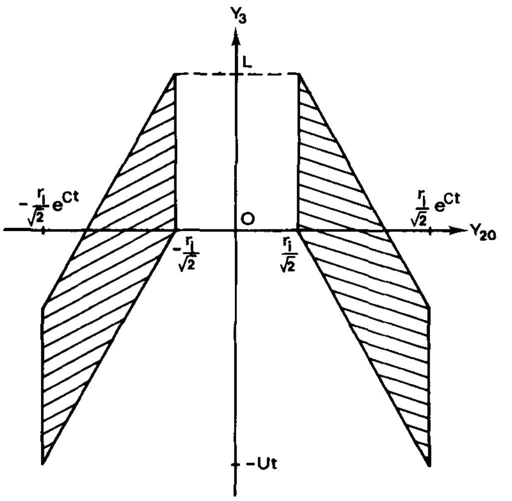

Fig. 4. Cross-section of swept-out volume V_s due to horizontal convergence.

and so,

$$V_T = \pi r_0^2\, Ut + 2r_i^2 LCt. \tag{2.12}$$

From eqs. (2.10), (2.11) and (2.12), we see that a convergence of order C produces an increase in the swept-out volume V_T of order $r_i^2 LCt$ above the still-air value. On the other hand, a uniform vertical shear ($u_i = Cx_3\,\delta_{i1}$, say) increases V_T by $r_i L^2 Ct$ (Manton, 1974). It is found from the observations of Jonas and Goldsmith (1972) that L/r_i could be of order 8 (Manton, 1974), and so it would seem that convergences within a flow have a small effect on the coalescence rate of droplets compared with the effect of vertical shears. Also, Manton (1974) shows that horizontal shears of the forms $u_i = Cx_2\,\delta_{i1}$ and $u_i = Cx_1\,\delta_{i3}$ give rise to increases in the swept-out volume of order $r_i^2\, LCT$ and $r_i^3\, Ct$, respectively. It therefore appears that when estimating the effect of a complex shear flow upon the coalescence rate of small droplets, only the vertical gradients of the horizontal velocities must be considered. The neglected terms then yield an error of order r_i/L.

3. Droplet coalescence in a turbulent flow

The turbulence within a developing cloud typically has a velocity scale u_* of order 50 cm sec^{-1} and a length scale l_T of order 20 m. Such motions produce a rate of dissipation of turbulent energy ϵ of order u_*^3/l_T; that is, $\epsilon \sim$ 60 cm^2 sec^{-3}. For isotropic turbulence, the mean square value of one component of the turbulent strain rate is given by

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = \epsilon/15\nu,$$

where ν is the kinematic viscosity of air. The length scale of motions associated with the turbulent shear is the Taylor microscale l_0 where

$$l_0 = u_* \left\{\overline{\left(\frac{\partial u}{\partial x}\right)^2}\right\}^{-1/2}$$

Thus, for a developing cloud, the turbulent shear rate $\{\overline{(\partial u/\partial x)^2}\}^{1/2}$ is of order 5 sec^{-1} and the length scale l_0 is of order 10 cm.

It has been shown in Section 2 that of the nine components of the velocity gradient tensor $\partial u_i/\partial x_j$ only the two off-diagonal components $\partial u_1/\partial x_3$ and $\partial u_2/\partial x_3$ should contribute significantly to the rate of coalescence of droplets within a flow. Also, the shear flow dominates the coalescence process only for droplets of comparable size. When colliding droplets are of unequal size, the larger droplet simply overtakes the smaller; and the rate of coalescence is essentially unchanged from the still-air value. Thus, we consider the relative motion of droplets of equal size in a randomly unsteady, vertical shear flow. The random shear produces relative horizontal motions of the droplets, so that, on the average droplets lying within the conical volume

$$0 < y_3 < L \quad \text{with} \quad y_1^2 + y_2^2 < L_1^2 (y_3/L)^2$$

are swept into the interaction region of a given droplet. The radius L_1 increases with time in a manner dependent upon the statistics of the shear flow. Thus, the average swept-out volume at some instant is

$$\langle V_s \rangle = \int_0^L \pi L_1^2 (y_3/L)^2\, dy_3 = \tfrac{1}{3}\pi L_1^2 L.$$

Only a single droplet lies within the swept-out volume when $n\,\langle V_s\rangle = 1$, where n is the number density of the droplets. Therefore, the expected radius of the swept-out volume associated with a single collision is given by

$$L_1 = (3/\pi nL)^{1/2}.$$

For a developing cloud, n is typically of order 10^2 cm^{-3} and the droplet radius R is of order 10 μm. Thus, taking $L \sim 8R$, we find that $L_1 \sim$

1 cm; that is, colliding droplets have an initial separation of about 1 cm. But the length scale l_0 of the turbulent motions associated with the shear is of order 10 cm. Therefore, because $L_1/l_0 \sim 0.1$, it is clear that the collision-inducing turbulence may be approximated by a simple shear.

We now consider the collision of a droplet of radius R_2 with another of radius R_1, where $R_1 \geqslant R_2$. The motion of a droplet within a developing cloud is described by eq. (2.2), viz.

$$v_i = u_i - W\delta_{i3} \quad (3.1)$$

Introducing the relative position vector y_i, given by (2.4), we see that (3.1) becomes

$$\frac{dy_i}{dt} = u_i(\mathbf{x}^{(2)}, t) - u_i(\mathbf{x}^{(1)}, t) + U\delta_{i3}, \quad (3.2)$$

where $\mathbf{x}^{(j)}$ is the position vector for the droplet of radius R_j, and $U = W_1 - W_2$ is the relative terminal velocity of the droplets.

By expanding $u_i(\mathbf{x}^{(2)}, t)$ in a Taylor series, (3.2) may be written

$$\frac{dy_i}{dt} = y_j\frac{\partial u_i}{\partial x_j}(\mathbf{x}^{(1)}, t) + y_j y_k \frac{\partial^2 u_i}{\partial x_j \partial x_k}(\mathbf{x}^{(1)}, t) + \ldots + U\delta_{i3} \quad (3.3)$$

But we have shown that the expected separation of the droplets is much smaller than the turbulence length scale ($L_1/l_0 \sim 0.1$). Hence, y_i is adequately described by retaining only the first term in the Taylor expansion of (3.3). Also, we found that only two of the nine components of $\partial u_i/\partial x_j$ contribute significantly to the collision process. Thus, eq. (3.3) may be reduced to

$$\frac{dy_i}{dt} = C_{13}y_3\delta_{i1} + C_{23}y_3\delta_{i2} + U\delta_{i3}, \quad (3.4)$$

where $C_{ij} = \partial u_i/\partial x_j\,(\mathbf{x}^{(1)}, t)$ is a random function of time representing the turbulent shear rate at the larger droplet. The volume $V_s(t)$ of fluid which is swept through the interaction region,

$$0 < y_3 < L \quad \text{with} \quad y_1^2 + y_2^2 < r_i^2,$$

by the relative shear over a time interval $(0, t)$ can be calculated from (3.4).

In the interval $(t, t + dt)$, the shear-induced swept-out volume is increased by the amount

$$dV_s = \int_0^L dA\, dy_3, \quad (3.5)$$

where dA is the horizontal area element at a given vertical position y_3. Fig. 5 shows the projection

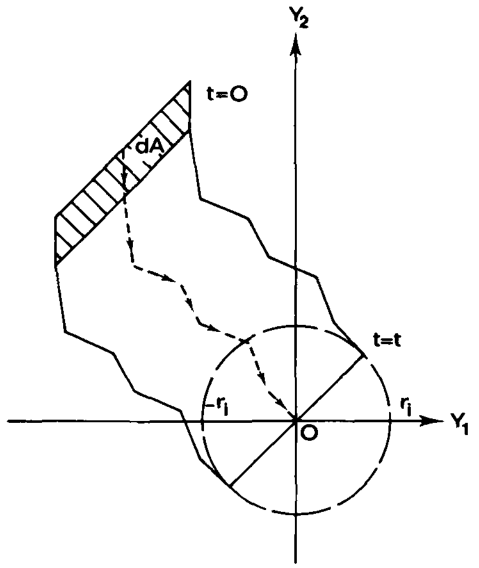

Fig. 5. Horizontal area element dA which contributes to the shear-induced swept-out volume at time t in a turbulent flow.

onto a horizontal plane of the path lines of small droplets which enter the interaction region at time t. Because the model velocity field is uniform in the horizontal (y_1, y_2) plane at any instant in time, we see that dA is the area of a parallelogram whose orientation is determined by the direction of the horizontal velocity vector at time t. That is,

$$dA = 2r_i\, dt\, |\mathbf{w}(t)\cdot\mathbf{w}(0)|\,/\,|\mathbf{w}(t)|, \quad (3.6)$$

where $\mathbf{w} = (dy_1/dt,\ dy_2/dt)$. Equations (3.5) and (3.6) implicitly assume that the area elements at different times do not overlap. This would seem to be a valid assumption provided that the expected time between collisions is large compared with the time scale of the turbulence.

Equation (3.4) implies that

$$w_1(t) = (y_{30} + Ut)C_{13}(t)$$

and (3.7)

$$w_2(t) = (y_{30} + Ut)C_{23}(t),$$

where $y_3 = y_{30}$ at time $t = 0$. Putting (3.7) into (3.6), we find that

$$dA = 2r_i y_{30} dt\, |C_{13}(t)C_{13}(0) + C_{23}(t)C_{23}(0)|\,/\{C_{13}^2(t) + C_{23}^2(t)\}^{1/2}, \quad (3.8)$$

independently of the relative terminal velocity U.

The average swept-out volume is calculated from (3.5) and (3.8) to be

$$\langle V_s(t)\rangle = r_l L^2 \int_0^t dt \langle |C_{13}(t) C_{13}(0) + C_{23}(t) C_{23}(0)| / \{C_{13}^2(t) + C_{23}^2(t)\}^{1/2} \rangle, \tag{3.9}$$

where $< >$ denotes an average value. To evaluate (3.9), the integrand must first be determined.

An approximation to the integrand of (3.9) is obtained by replacing C_{l3}^2 by its average value $\langle C_{l3}^2\rangle$ in the denominator; that is, we let

$$\langle |C_{13}(t) C_{13}(0) + C_{23}(t) C_{23}(0)| / \{C_{13}^2(t) + C_{23}^2(t)\}^{1/2}\rangle \simeq \langle |C_{13}(t) C_{13}(0) + C_{23}(t) C_{23}(0)|\rangle / \{\langle C_{13}^2(t)\rangle + \langle C_{13}^2(t)\rangle\}^{1/2}. \tag{3.10}$$

If the turbulence is assumed to be stationary and isotropic, then C_{13} and C_{23} are independent but identically distributed stationary random variables with zero mean. Thus,

$$\langle C_{13}^2(t)\rangle = \langle C_{23}^2(t)\rangle = C^2, \tag{3.11}$$

the constant mean square velocity gradient. Also, $C_{13}(t) C_{13}(0)$ and $C_{23}(t) C_{23}(0)$ are independent, identically distributed random variables, where

$$\langle C_{13}(t) C_{13}(0)\rangle = \langle C_{23}(t) C_{23}(0)\rangle = C^2 F(t).$$

$F(t)$ is the "Lagrangian" correlation function of the velocity gradients following a droplet, such that $F(0) = 1$ and $F \to 0$ as $t \to \infty$. The coherence time scale for the turbulent fluctuations is of order C^{-1}. Therefore, for $Ct \gg 1$, F is essentially zero; that is, $C_{13}(t)$ and $C_{13}(0)$ are uncorrelated. Thus, we take the distribution of $C_{13}(t) C_{13}(0)$ to be Gaussian with zero mean and variance C^4. Now it can be shown from (3.10) and (3.11) that

$$\langle |C_{13}(t) C_{13}(0) + C_{23}(t) C_{23}(0)| / \{C_{13}^2(t) + C_{23}^2(t)\}^{1/2}\rangle \simeq (2/\pi)^{1/2} C. \tag{3.12}$$

Equation (3.12) may be used in (3.9) provided that $Ct \gg 1$. Then the assumption that $C_{13}(t)$ and $C_{13}(0)$ are uncorrelated is valid over most of the domain of integration.

Putting (3.12) into (3.9), we find that the expected shear-induced swept-out volume is given by

$$\langle V_s(t)\rangle = (2/\pi)^{1/2} r_l L^2 Ct.$$

Therefore, the expected rate at which collisions occur is

$$f_s = (2/\pi)^{1/2} r_l L^2 nC, \tag{3.13}$$

where n is the number density of the smaller droplets.

The validity of (3.12) and hence of (3.13), requires that

$$C/f_s \gg 1;$$

that is, from (3.13),

$$(2/\pi)^{1/2} r_l L^2 n \ll 1.$$

Taking the typical values $r_l \sim R_1$, $L \sim 8R_1$, $n \sim 10^2$ cm^{-3} and $R_1 \sim 10^{-3}$ cm, we find that $C/f_s \sim 2 \times 10^5$. Hence, the turbulence time scale is much less than the expected time between collisions.

In addition to the shear-induced collisions, coalescence can also occur by a larger droplet catching up to a smaller one by virtue of the greater terminal velocity of the former. That is, collisions occur for all small droplets passing through the surface

$$y_3 = 0 \quad \text{with} \quad y_1^2 + y_2^2 < r_0^2,$$

where πr_0^2 is the still-air cross-section for a collision between a droplet of radius R_1 and another of radius R_2. Because the fluid velocity u_i is a random function of time and position, it is seen from (3.2) that the expected swept-out volume corresponding to this collision process is

$$\langle V_0(t)\rangle = \pi r_0^2 Ut,$$

independently of the turbulence. Hence, the expected rate at which collisions occur is

$$f_0 = \pi r_0^2 nU. \tag{3.14}$$

From (3.13) and (3.14), we see that the total rate of coalescence between droplets in a turbulent flow is given by

$$f = f_0 + f_s = \pi r_0^2 nU + (2/\pi)^{1/2} r_l L^2 nC.$$

The expression f_s in (3.15) for the shear-induced collision rate is not necessarily valid for droplets of unequal size. However, as U increases from zero for $R_1 = R_2$, the collision rate f_0 increases while f_s is essentially constant. Hence, the former tends to dominate the overall collision rate f when R_1 and R_2 are unequal.

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О СЛИЯНИИ ВОДЯНЫХ КАПЕЛЬ В ТУРБУЛЕНТНЫХ ОБЛАКАХ

Показано, что из девяти компонент тензора градиента скорости потока только два, представляющих вертикальный градиент горизонтальных компонент скорости, дают существенный вклад в увеличение скорости соударений маленьких капелек по сравнению со случаем покоящегося воздуха. Поэуому для предсказания скорости соударений турбулентное движение внутри развивающегося облака моделируется потоком с однородным вертикальным сдвигом скорости, случайно меняющимся во времени.