Size distribution models of fog and cloud droplets in terms of the modified gamma function

By FRANCESCO TAMPIERI and CLAUDIO TOMASI, *Sezione Microfisica dell'Atmosfera, C.N.R., Bologna, Italy*

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ABSTRACT

Size spectra of water droplets in haze, fogs and clouds of various type measured by several authors have been analyzed by a computational method in terms of the modified gamma distribution function. The preliminary results are shown on the wide ranges that result of the function parameters. These results are synthesized using a graphical method to obtain average and representative size distribution models suitable for reproducing size spectra of water droplets typical for fog and clouds in different evolutionary stages. The examination of the results in terms of skewness and kurtosis suggests typical behaviours of these two shape-parameters which can be taken into account in obtaining multimodal size distributions by the use of the properly weighted models found. The analytical expressions of skewness and kurtosis in the modified gamma distribution function are given in the Appendix.

1. Introduction

Fog layers and clouds play a fundamental role in the thermal balance of the earth attenuating both the incoming solar radiation and the fraction of radiant energy emitted by the earth to space. The application of electromagnetic theory in evaluating the radiation scattering and absorption by the atmospheric water droplets requires the knowledge of their size distribution or the assumption of substitutive analytical models which represent the average characteristics of such distributions.

The size spectra of haze and fog droplets measured in real atmospheres generally show largely variable shapes. Moreover measurements of droplet size spectra within the same cloud in general show large fluctuations of shape and concentration both in time and space. This variability is due to the complexity of the processes which affect the formation and evolution of the fog and cloud droplets. They depend on a great number of parameters (concentration of nuclei and their nature and properties, air temperature, cooling rates in the medium, scale of turbulence and its intensity, etc.).

Therefore in order to be significant representations the analytical models have to be derived from wide sets of experimental data obtained for the various types of fog and cloud. Several analytical expressions have been proposed in the literature as approximations derived from ample collections of experimental data. Some of the most commonly used distributions are Best's (1951) formula, Khrgian & Mazin's (1952) formula, Kolmogorov's log-normal distribution as proposed by Levin (1954), Levin's (1958) gamma distribution and Deirmendjian's (1964) modified gamma distribution function. The latter has one more parameter than the other above-cited expressions. As a result this function is more flexible and covers the whole range of the wing slopes of the distribution curves substituting the empirical size spectra and presents the property of being applicable both to narrow and wide droplet size spectra.

The modified gamma distribution function gives the volume concentration of droplets per unit radius as a function of the radius r in the general form:

$$n(r) = ar^{\alpha} \exp\left[-\frac{\alpha}{\gamma}\left(\frac{r}{r_c}\right)^{\gamma}\right], \quad 0 \leq r < \infty \qquad (1)$$

where a, α, γ, r_c are positive constants. The latter three parameters completely determine the shape of the distribution curve while the constant a is only related to the total number N of droplets per unit volume. It should be noticed moreover that r_c is the radius of maximum frequency and that α can assume only integer values. Some of the above-mentioned expressions are particular cases of Deirmendjian's distribution function and correspond to certain values of the shape-parameters: Levin's gamma distribution can be obtained from eq. (1) for $\gamma = 1$, Khrgian & Mazin's formula for $\alpha = 2$ and $\gamma = 1$. Other properties of the function are reported by Deirmendjian (1969).

The purpose of the present work is to obtain realistic models of the typical size spectra of the water droplets in hazes, fogs and clouds formed through different processes and at various evolutionary stages in terms of the modified gamma distribution function.

First the experimental size spectra given by several authors have been analyzed by a computational method which gives the three parameters α, γ, r_c which correspond to the best-fit solution and by themselves are suitable for reproducing the shape of the experimental size distribution. In order to find the most realistic and representative size distribution models, the preliminary results obtained by the computer have been synthesized on the basis of a graphical method. This procedure allows us to determine the parameter sets which give shapes of the modified gamma distribution function similar to the size spectra most frequently measured in the atmosphere.

2. Computational procedure

Each run of experimental data has been reduced to a set of k pairs (r_i, n_i) where n_i is the droplet concentration per unit radius (expressed in cm^{-3} μm^{-1}) corresponding to the radius r_i (expressed in μm). As the modified gamma distribution function is characterized by four parameters, k is required to be greater or equal to 4. The use of the empirical ratios n_i/n_1 (for $i = 2, \ldots, k$) directly eliminates the constant a. Moreover the use of natural logarithms of the ratios n_i/n_1 results in a wider operational range for the computer.

The three values α, γ, r_c which present the best agreement with the experimental run have been derived by finding a minimum of the function:

$$F(\alpha, \gamma, r_c) = \frac{1}{k-1} \times \sum_{i=2}^{k} \left[\frac{\ln \frac{n_i}{n_1} - \alpha \ln \frac{r_i}{r_1} + \frac{\alpha}{\gamma r_c^\gamma}(r_i^\gamma - r_1^\gamma)}{\ln \frac{n_i}{n_1}} \right]^2 \tag{2}$$

in the range of the integer parameter α from 1 to 10. However, Deirmendjian's function limited on this range of α appears suitable for adequately fitting all the real droplet size spectra of hazes, fogs and clouds. In fact Zuev (1970) suggests the same range of α for representative models based on Levin's gamma distribution which employ only two shape-parameters.

The function F given by eq. (2) represents the average value of the sum of the square of the differences between 1 and the logarithm of the ratio $n(r_i)/n(r_1)$ expressed in terms of the modified gamma distribution function and normalized to the logarithm of the corresponding empirical ratio. In such a way F gives an evaluation of the average deviation of the examined distribution from the empirical data on logarithmic scale.

This computational procedure permits identifying for each integer value of α from 1 to 10 the γ and r_c which minimize the function F and simultaneously calculates the corresponding value of the minimum indicated by f_α. Only for about half of the examined runs did the set of f_α present a minimum we called f_{min}. Correspondingly the three parameters α, γ, r_c represent the best-fit solution.

In the other cases the set of f_α did not show a minimum but the values decreased as α increased in the adopted range. Moreover the differences between two successive values of f_α gradually decreased as α increased step by step, so that some values of f_α were associated with parameters which gave almost equal size distribution shapes. Consequently the good-fit solution has been assumed to be the set (α, γ, r_c) characterized by the lower value of α for which the condition

$$(f_\alpha - f_{\alpha+1}) \leqslant \tan 1^\circ = 0.0175 \tag{3}$$

is satisfied.

This methodology does not require the experimental data relative to the range including the mode radius and does not involve obviously the prespecification of the parameter α.

The computational method used for minimizing the nonlinear function F is the classical Rosenbrock's (1960) algorithm.

3. Analysis of the experimental data and preliminary results

Examining the whole set of the experimental data, one can say that part of the size spectra are undoubtedly consistent with two or three modes centred at quite distinct values of the radius. In such cases it is possible to divide the original data set into separate runs which can be independently analyzed by the computational method. In other cases the size spectra showed indistinct and secondary modes with a number of data insufficient for the calculation; thus the size spectra have been smoothed by averaging the original concentrations on larger radius intervals. However, most of the size spectra gave only one mode and the original data have been taken directly.

In order to present an organized and homogeneous review of all the examined empirical data a classification has been followed taking into account the predominant type of haze and fog formation process as suggested by each author and separately considering the various types of cloud. As regards the fog, however, it would appear realistic to assume that in some cases different mechanisms (for example radiation and advection) could contribute to the formation of the same fog.

Based on this classification the empirical data and the corresponding preliminary results concerning the three significant parameters of the modified gamma distribution are listed as follows:

(a) Water haze

In this class of empirical data we have taken:

1 run relative to haze particles measured with the "spider web" technique by Arnulf et al. (1957) and presented by Eldridge (1966). This author points out that the smaller droplets are not measurable with this technique and that the concentrations have been underestimated. Then the original scattered data have been smoothed on unit radius intervals from 1.5 to 15.5 μm.

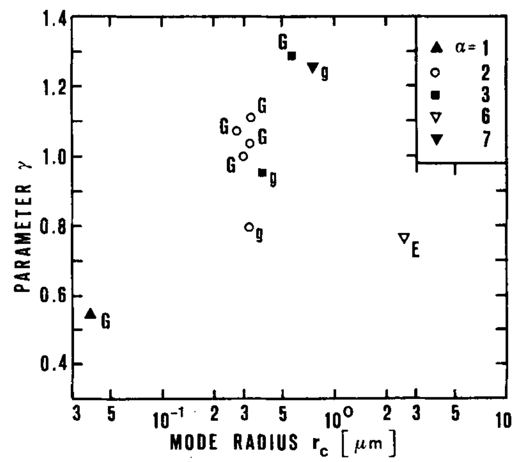

Fig. 1. Sets of the modified gamma distribution parameters obtained from the experimental runs giving size spectra of haze droplets. The mode radius r_c is given in abscissa, the parameter γ in ordinate, while the parameter α is indicated by means of different symbols. The labels refer to authors: E = Eldridge (1966), g = Garland (1971), G = Garland et al. (1973).

3 runs obtained from the original data presented by Garland (1971) and measured with a two-stage impactor. The haze and fog droplets were spread over a celluloid film. These were examined microscopically and classified into logarithmically equal intervals of the diameter. The author suggests that droplets smaller than 2.5 μm were not captured at all. Generally the size spectra were characterized by at least two marked modes, the first located in the range $r < 1$ μm, the others centred at radii more than 5 μm. We have considered as typical haze distribution the part of the size spectra relative to the first mode while the other modes have been attributed to the fog classes.

6 runs obtained from the original data measured by Garland et al. (1973) with the preceding technique. The previous remarks are valid for this group of data.

The parameter sets obtained for this group of 10 runs relative to small water droplets are shown in Fig. 1. A large part of these results (about 80%) appears consistent with Levin's gamma distribution taking into account the sets of parameters in which γ is between 0.75 and 1.25.

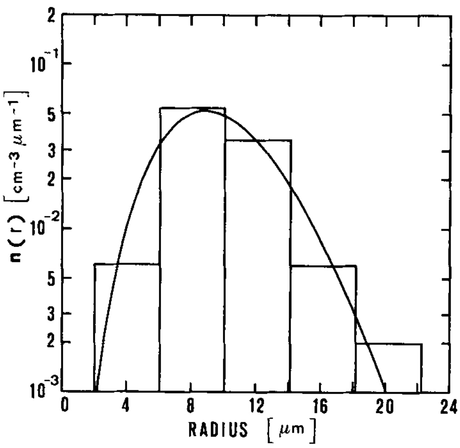

Fig. 2. Comparison between the original size spectrum measured by Vittori & Pesaresi (1962) at Baricella (Po Valley) and the corresponding good-fit modified gamma distribution given by the set $\alpha = 5$, $\gamma = 1.49$ and $r_c = 8.81$ μm and found on the basis of the adopted computational method with $f_{(\alpha=5)} = 3.74 \times 10^{-2}$.

(b) Radiation fog

The following empirical runs have been attributed to this group:

3 runs of data carried out by Webb (1956) with a slide-collection method. The data graphically presented have been numerically reduced in intervals of 2 or 4 μm in the range $r > 4$ μm.

1 run measured by Arnulf et al. (1957) with the "spider web" technique and reported by Eldridge (1966). The remarks and the procedure are identical to those mentioned for the haze data of the same authors.

8 runs measured by Vittori & Pesaresi (1962) with a sampling technique based on the use of glass rods precoated with gelatin films. The collection efficiency has been estimated by these authors to be greater than 40% for droplet radii of more than 2 μm. The original data presented for 4.04 μm radius intervals have been analyzed directly. A comparison between an original size spectrum and the resulting good-fit solution is shown in Fig. 2.

2 runs relative to typical size spectra of radiation fog droplets presented by Tverskoi (1965). The original data have been taken in the 1 to 6 μm radius range.

3 runs reported by Garland (1971) and 6 runs by Garland et al. (1973) which are all relative to different modes than that corresponding to haze. The numerical reduction is identical to that adopted in the previous class for the haze data of the same authors.

All the resulting parameter sets are presented in Fig. 3, from which it appears that about 30%

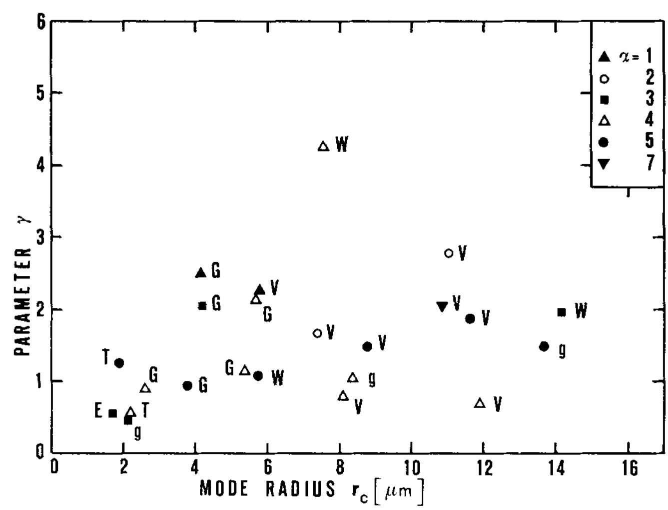

Fig. 3. Sets of the modified gamma distribution parameters obtained from the experimental runs relative to size spectra of radiation fog droplets. The mode radius r_c is given in abscissa, the parameter γ in ordinate, while the parameter α is indicated using different symbols. The labels refer to the authors: W = Webb (1956), V = Vittori & Pesaresi (1962), T = Tverskoi (1965), E = Eldridge (1966), g = Garland (1971), G = Garland et al. (1973).

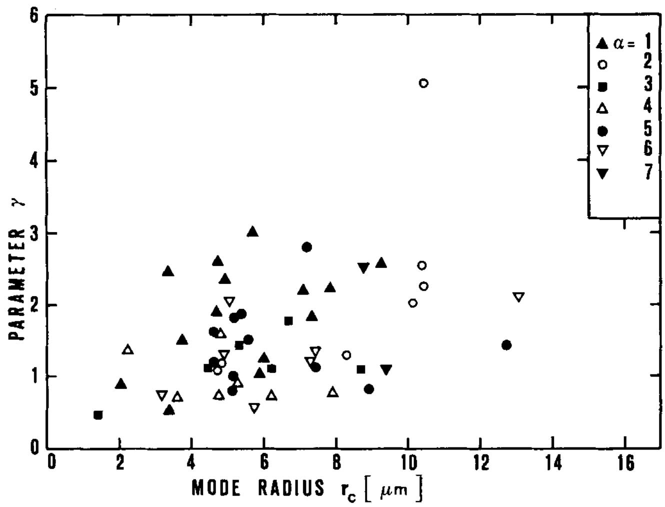

Fig. 4. Sets of the modified gamma distribution parameters found for the experimental size spectra of advection and arctic marine fog droplets. The mode radius r_c and the parameter γ are given respectively in abscissa and in ordinate, while α is indicated by different symbols.

of the data are consistent with Levin's gamma distribution (with the parameter γ in the same tolerance range between 0.75 and 1.25).

(*c*) *Advection fog*

Here a considerable number of empirical data has been examined:

21 runs measured by Kocmond et al. (1971) with the gelatin-coated slide technique. The collection efficiency has to be assumed as being satisfactory for $r > 3$ μm. In this range the original data have been smoothed in intervals of 2 μm.

5 runs relative to measurements carried out by Garland (1971) which concern the second and third mode.

11 runs measured by Kunkel (1971) using a laser hologram camera in the radius range from 2.5 to about 30 μm. The original data have been used.

20 runs carried out by Kumai (1973) at Point Barrow (Alaska) during two periods of arctic marine fog occurrence. The measurements were made with a two-stage impactor containing slides precoated with silicone oil or gelatin. The collection efficiency was about 0.6% for the smallest droplets (with radius of about 3.3 μm) collected at the lowest wind speed. The specimens gave size spectra in the radius range greater than 3.5 μm and for constant intervals of 1.3 μm. The original data have been averaged on intervals of 2.6 μm.

All the parameter sets relative to this class are shown in Fig. 4. About 36% of the results appear to be consistent with Levin's gamma distribution for the γ tolerance range of ± 0.25.

(*d*) *Valley fog*

An ample collection of size spectra relative to valley fogs has been reported by Pilie et al. (1972) on the basis of measurements obtained using a modified Bausch and Lomb slide projector to expose gelatin-coated slides to a stream of foggy air. The impacts of droplets on the slides left "replicas" that were accurately measured under a microscope. The authors point out that errors in these measurements are particularly important for small droplet sizes (radius less than 3 μm) where the number of replicated droplets was limited by small collection efficiencies. The size spectra are given on unit radius intervals and generally cover a wide radius range. In order to obtain more continuous shapes the original data have been averaged in intervals of 2 μm in the radius range greater than 5 μm. In certain cases relative to the initial stage of fog formation (ground fog) the experimental data cover very narrow ranges of radius so that the original data have been taken for $r > 2$ μm. Moreover the size

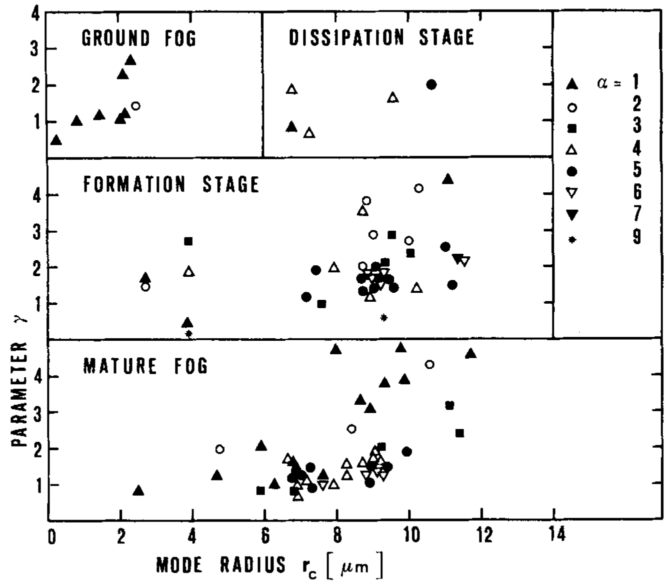

Fig. 5. Sets of the modified gamma distribution parameters obtained from droplet size spectra carried out by Pilie et al. (1972) and concerning four different stages of valley fog. The mode radius r_c is given in abscissa, the parameter γ in ordinate and α is indicated by different symbols.

spectra sometimes show two rather marked modes, which have been separately considered for the computation.

These experimental data have been classified on the basis of their evolutionary stage, as proposed by the authors: 8 runs for ground fog, 38 runs for the formation stage, 45 runs for mature fog and 5 runs for the dissipation stage. On the basis of this classification all the resulting sets of three parameters are reported in Fig. 5. With reference to the previously assumed tolerance range, a large part of the results shown in Fig. 5 are consistent with Levin's gamma distribution: 75% of those relative to the ground fog, 29% for the formation stage, 51% for the mature fog and 40% for the dissipation stage.

(*e*) *Evaporation fog*

Only one run has been examined in regard to a size spectrum presented by Tverskoi (1965) for intervals of unit radius. The resulting parameter set is given by $\alpha = 4$, $\gamma = 0.56$ and $r_c = 1.86$ μm.

(*f*) *Stratocumulus cloud*

With regard to this type of cloud we have analyzed:

3 runs inferred by Eldridge (1957) from infrared transmittance measurements through clouds probably in a growing stage.

2 runs proposed by Tverskoi (1965) as typical size spectra of cloud droplets relative to the base and top layers of the stratocumulus.

2 runs carried out by Jiusto (1967) using an instrumented aircraft for exposing gelatin precoated slides to the airstream in mountain stratocumulus.

4 runs obtained by Fitzgerald & Spyers-Duran (1973) during flights made in several clouds (upwind and downwind).

The resulting sets of the three parameters are given in Table 1 for each examined run together with the corresponding minimum assumed by the function F and indicated by f_{min}. The corresponding value of a (which gives the unit total concentration N of droplets) is also reported.

(*g*) *Stratus cloud*

The following empirical runs have been examined:

2 runs obtained from experimental data carried out by Webb (1956) with a slide collection method.

Table 1. *Results in terms of the modified gamma size distribution parameters for stratocumulus cloud droplets*

Authors	Type		$r_c(\mu m)$	α	γ	a	f_{min}
Tverskoi (1965)	Base		5.33	5	1.19	$2.8230\ 10^{-3}$	0.040
	Top		10.19	2	2.46	$1.9779\ 10^{-3}$	0.150
Eldridge (1957)		(*a*1)	3.59	2	4.98	$4.1872\ 10^{-2}$	0.001
		(*a*2)	4.68	7	2.64	$1.0708\ 10^{-4}$	0.135
		(*a*3)	7.60	8	6.88	$1.1054\ 10^{-7}$	0.152
Jiusto (1967)		(*b*1)	5.49	2	3.46	$1.1854\ 10^{-2}$	0.035
		(*b*2)	6.13	8	2.97	$2.3790\ 10^{-6}$	0.110
Fitzgerald &	Downwind	(*c*1)	2.84	5	1.78	$3.8158\ 10^{-2}$	0.131
Spyers-Duran (1973)		(*c*2)	2.94	8	1.00	$2.0284\ 10^{-1}$	0.017
	Upwind	(*d*1)	4.21	1	3.71	$6.2669\ 10^{-2}$	0.324
		(*d*2)	3.40	1	2.98	$9.2020\ 10^{-2}$	0.053

4 runs by Keily & Millen (1960) using an electric-charge drop-counting and sizing probe.

2 runs suggested by Tverskoi (1965) as representative droplet size spectra of the base and top layer.

The resulting sets of the three parameters are given in Table 2 together with the corresponding values of a and f_{min}.

(h) Nimbostratus cloud

Only two runs have been examined corresponding to the typical droplet size spectra proposed by Tverskoi (1965) for the base and top layers. The results are reported in Table 2.

(i) Orographic cloud

We have analyzed a bimodal size spectrum given by Squires (1958) on the basis of twelve samples taken in Hawaiin orographic clouds. The droplets were collected by exposing glass rods coated with soot to the airstream. The parameter sets describing the two modes independently examined are given in Table 2.

(j) Cumulus cloud

A considerable number of empirical data has been taken:

4 runs obtained by Squires (1958) with the above-mentioned collection technique at four levels of a tradewind cumulus.

18 runs measured by Durbin (1959) with an impaction method at different heights between 1 000 and 2 000 m. The data in the range $r >$ 2.65 μm have been taken.

1 bimodal run carried out by Keily and Mil-

Table 2. *Results in terms of the modified gamma size distribution parameters for stratus, nimbostratus and orographic cloud droplets*

Authors	Type	$r_c(\mu m)$	α	γ	a	f_{min}
	Stratus					
Tverskoi (1965)	Base	4.70	5	1.05	$9.7923\ 10^{-3}$	0.046
	Top	6.75	3	1.30	$3.8180\ 10^{-3}$	0.101
Webb (1956)	(*f*1)	3.41	1	6.59	$1.0820\ 10^{-1}$	0.015
	(*f*2)	7.07	9	1.20	$7.5818\ 10^{-6}$	0.107
Keily &	(*g*1)	0.21	1	0.65	$2.6304\ 10^{1}$	$1.2\ 10^{-4}$
Millen (1960)	(*g*2)	0.90	2	1.18	$4.5182\ 10^{0}$	0.301
	(*g*3)	0.90	1	0.80	$1.2979\ 10^{0}$	0.003
	(*g*4)	7.65	5	2.85	$4.4257\ 10^{-5}$	0.153
	Nimbostratus					
Tverskoi (1965)	Base	6.41	5	1.24	$8.0606\ 10^{-4}$	0.114
	Top	9.67	1	2.41	$1.0969\ 10^{-2}$	0.197
	Orographic cloud					
Squires (1958)	(1st mode)	12.35	4	1.22	$8.1124\ 10^{-5}$	0.026
	(2nd mode)	21.87	3	2.01	$1.9576\ 10^{-5}$	0.344

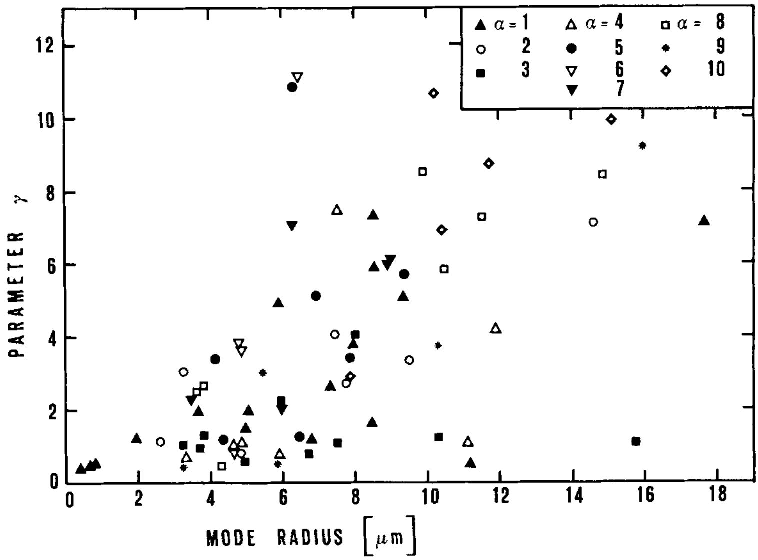

Fig. 6. Sets of the modified gamma size distribution parameters obtained from the experimental runs relative to size spectra of cumulus cloud droplets. The mode radius r_c is given in abscissa, the parameter γ in ordinate, while the parameter α is indicated by means of different symbols.

len (1960) with the previously mentioned instrumental apparatus within a small cumulus cloud.

5 runs proposed by Borovikov et al. (1963) as typical size spectra of cumulus droplets at various heights above the cloud base.

47 runs measured by Warner (1969*a*, 1969*b*, 1973) using an impaction method on sooted glass slides. The author points out that the theoretical collection efficiency of the sampling device is of about 50% for 1.25 μm radius and well over 90% for droplets with 5 μm radius at a true air speed of 65 m s^{-1}.

The resulting sets of the three parameters are shown in Fig. 6 from which it appears that 34% of the results are consistent with Levin's gamma distribution if the tolerance range of the parameter γ is taken to be between 0.75 and 1.25. The parameter set $\alpha = 6$, $\gamma = 1$, $r_c = 4$ μm proposed by Deirmendjian (1964, 1969) as model of droplet size distribution for cumulus clouds results in a satisfactory agreement with some of the found solutions.

An example of best-fit solution found for a multimodal size spectrum relative to cumulus cloud is shown in Fig. 7.

4. Synthesis of the preliminary results

The solutions obtained from the computation procedure in terms of the parameters α, γ, r_c and presented in Figs. 1, 3, 4, 5 and 6 for haze, fogs of various type and cumulus clouds are characterized by a wide dispersion and indicate that the shapes of the water droplet size spectra in the real atmosphere can be subject to large variations so that their analytical representation in terms of Deirmendjian's distribution requires extensive ranges for the fundamental parameters. In the three dimensional space of coordinates α, γ, r_c the points corresponding to the solutions found appear located on a family of parallel planes (of equation α = positive and integer constant) and sometimes gathered in well defined domains. Therefore it appears useful to operate on these groups in order to reduce the parameter sets of each domain to only one set giving a representative and summarizing distribution model.

On the other hand the procedure to be followed for synthesizing the preliminary results for a limited number of significant models should be such as to not change essentially the primary shapes of the distribution curves of each domain. Consequently sets of solution points have been considered to be closely contained in narrow domains of the parameter space, especially with regard to the parameters α and r_c. This approach includes the possibility that points characterized by different values of α result as belonging to the same domain and

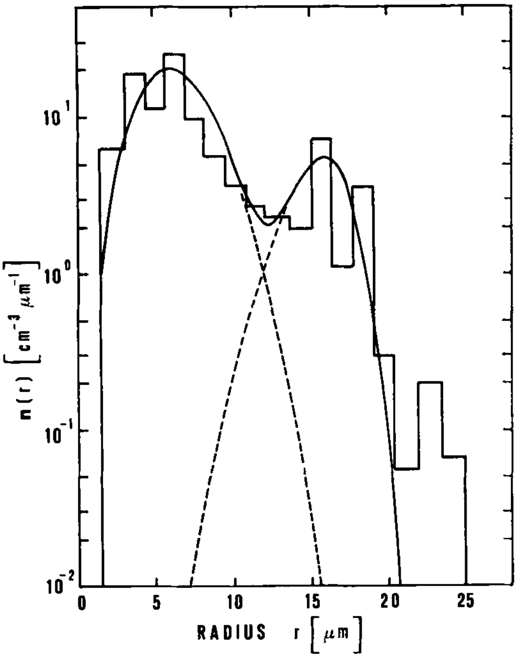

Fig. 7. Example of best-fit solution found on the basis of the modified gamma distribution function for a multimodal size spectrum of cumulus cloud droplets carried out by Warner (1969*a*). The parameter set relative to the first mode is given by $\alpha = 3$, $\gamma = 2.23$ and $r_c = 5.93$ μm with $f_{min} = 0.17$. The parameter set corresponding to the second mode is given by $\alpha = 9$, $\gamma = 9.21$ and $r_c = 15.94$ μm with $f_{min} = 0.13$. The third mode centred at about 23 μm has not been examined because of the insufficient number of data for the computation. The dashed curves represent the overlapping wings of the two distributions independently found.

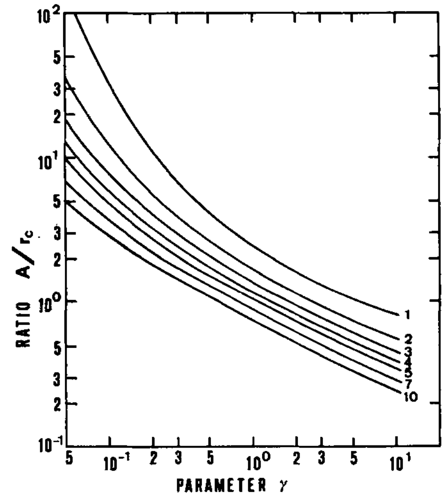

Fig. 8. Ratio A/r_c (between the halfwidth A and the mode radius r_c) plotted as a function of the parameter γ and for different integer values of the parameter α as resulting from computations on the modified gamma distribution function For each isopleth the corresponding value of α is beside reported.

therefore it appears to be necessary sometimes to force all the parameter sets at issue to take the same value of α and preferably that of maximum frequency. But the two parameters α and γ are closely interdependent in shaping the distribution curve centred at a certain value of r_c. Then variations of the parameter α can be correctly imposed only by preserving the original shape of each distribution curve by using balanced variations of the parameter γ.

Assuming the halfwidth A (expressed in μm) as a significant shape parameter we have computed the ratio A/r_c as a function of γ for integer values of α from 1 to 10. The resulting isopleths corresponding to some values of α are drawn in Fig. 8. This diagram immediately allows one to find a new value of γ for which A remains constant (for r_c constant) in spite of the imposed variation of α. However this has to be limited ($|\Delta\alpha| \leq 2$) in order to avoid that the new size distribution could be subjected to too marked variations of the shape with respect to the original size distributions.

For each group of parameter sets relative to a single domain, the following procedure has been adopted:

(i) from the values of α and γ applied to the diagram of Fig. 8 the ratio A/r_c has been evaluated for each solution point of the domain;

(ii) the maximum frequency value of α has been forced on the whole set to be synthesized;

(iii) the new values of γ are defined on the basis of the isopleth relative to the selected value of α and corresponding to the values of A/r_c, as evaluated in (i);

(iv) the average values of γ (as defined in (iii)) and of r_c are calculated for each group of parameter sets belonging to the same domain.

From such a procedure applied to the preliminary results some reliable models have been obtained as analytical interpretations of empirical size spectra in terms of the modified gamma distribution function. The resulting models concerning haze, radiation fog and evapo-

Table 3. *Resulting models in terms of the modified gamma size distribution parameters for haze and radiation fog droplets*

Model	$r_c(\mu m)$	α	γ	a	$\bar{f}_{min}$	Number of parameter sets
Haze						
1	0.04	1	0.54	8.1545 10^{2}	0.001	1
2	0.32	2	1.05	1.1474 10^{2}	0.140	6
3	0.65	5	1.29	6.4822 10^{2}	0.121	2
4	2.48	6	0.76	3.8216 10^{0}	0.285	1
Radiation fog						
1	2.13	4	0.70	4.3237 10^{0}	0.131	5
2	4.98	4	1.23	7.4438 10^{-3}	0.112	7
3	8.06	4	1.77	3.0410 10^{-4}	0.070	5
4	12.22	5	1.62	7.5475 10^{-6}	0.073	6
Evaporation fog	1.86	4	0.56	3.0558 10 ^{1}	0.232	1

ration fog are given in Table 3, those concerning advection fogs in Table 4, those relative to the different stages of valley fog in Table 5, those concerning cumuliform clouds in Table 6.

For each model the following data are reported in the Tables:

(i) the parameters α, γ, r_c which describe the model;

(ii) the value of the parameter a giving a unit total concentration N of droplets;

(iii) the average value $\bar{f}_{min}$ calculated for each domain with respect to the values of f_{min} found for the cases of best-fit solution and the values of f_α selected in the cases of good-fit solution;

(iv) the number of the parameter sets synthesized by the present graphical procedure for each domain.

This procedure has been applied to the domains immediately recognizable in the three dimensional space α, γ, r_c. Moreover the use of narrow ranges of r_c permits to obtain resulting models with well spaced mode radii. The values of $\bar{f}_{min}$ reported in the Tables 3, 4, 5 and 6 in general are less than 15%. This fact indicates that such resulting models appear to be realistic interpretations of the experimental data.

With regard to the stratocumulus, stratus, nimbostratus and orographic cloud droplets the small number of the examined runs does not

Table 4. *Resulting models in terms of the modified gamma size distribution parameters for advection fog droplets*

Model	$r_c(\mu m)$	α	γ	a	$\bar{f}_{min}$	Number of parameter sets
Advection fog						
1	2.75	4	0.85	4.9763 10^{-1}	0.120	5
2	5.04	5	1.17	4.2028 10^{-3}	0.061	17
3	6.20	3	1.05	8.1656 10^{-3}	0.107	2
4	8.10	6	1.47	3.0861 10^{-5}	0.028	7
5	21.16	5	3.09	8.9728 10^{-8}	0.038	4
Arctic marine fog						
1	3.07	1	0.98	1.0645 10^{-1}	0.075	3
2	5.40	2	1.09	2.2833 10^{-2}	0.079	5
3	7.85	3	1.15	2.6209 10^{-3}	0 071	8
4	10.37	2	2.97	1.7966 10^{-3}	0.075	4
5	12.88	5	1.93	3.6783 10^{-6}	0.101	2

Table 5. *Resulting models in terms of the modified gamma size distribution parameters for valley fog droplets*

Model	$r_c(\mu m)$	α	γ	a	$\bar{f}_{min}$	Number of parameter sets
Ground fog						
1	0.86	1	0.89	$1.3822\ 10^{0}$	0.003	3
2	2.22	1	2.04	$2.0338\ 10^{-1}$	0.008	5
Formation stage						
1	3.49	3	1.26	$5.6434\ 10^{-2}$	0.067	6
2	7.50	5	1.40	$2.1192\ 10^{-4}$	0.095	4
3	9.07	5	1.69	$4.0633\ 10^{-5}$	0.108	20
4	10.82	4	2.17	$5.1199\ 10^{-5}$	0.170	11
Mature fog						
1	4.99	1	1.93	$4.0001\ 10^{-2}$	0.178	6
2	7.12	4	1.14	$1.5416\ 10^{-3}$	0.145	14
3	9.00	5	1.47	$6.1487\ 10^{-5}$	0.124	14
4	9.10	1	4.39	$1.3900\ 10^{-2}$	0.191	8
5	11.18	2	3.41	$1.4058\ 10^{-3}$	0.141	5
Dissipation stage						
1	6.92	4	0.94	$3.3437\ 10^{-3}$	0.080	3
2	10.10	4	2.00	$8.0977\ 10^{-5}$	0.148	2

allow us to follow the synthesizing procedure to derive average size distribution models. Nevertheless, as can be seen in Tables 1 and 2, the values of f_{min} relative to each single parameter set are in general less than 15% so that our findings appear to be reliable representations.

5. Discussion of the results

The parameter set α, γ, r_c completely determines the shape of the modified gamma size distribution curve. However its profiles cannot be immediately deduced from the three parameters. Thus it appears appropriate to use other

Table 6. *Resulting models in terms of the modified gamma size distribution parameters for cumulus cloud droplets*

Model	$r_c(\mu m)$	α	γ	a	$\bar{f}_{min}$	Number of parameter sets
1	0.64	1	0.48	$1.2486\ 10^{0}$	0.040	3
2	3.19	3	1.04	$1.1905\ 10^{-1}$	0.043	8
3	3.53	8	2.15	$8.1189\ 10^{-4}$	0.062	4
4	4.80	5	2.16	$1.1046\ 10^{-3}$	0.051	8
5	5.28	3	1.07	$1.4890\ 10^{-2}$	0.165	6
6	5.39	8	1.51	$7.2574\ 10^{-5}$	0.129	4
7	6.92	5	6.60	$4.3996\ 10^{-5}$	0.058	6
8	7.06	8	4.80	$3.0107\ 10^{-7}$	0.248	2
9	7.72	2	2.39	$4.5880\ 10^{-3}$	0.129	11
10	10.26	8	7.41	$7.0591\ 10^{-9}$	0.164	9
11	10.40	4	2.34	$5.6744\ 10^{-5}$	0.229	7
12	15.60	9	11.75	$9.8706\ 10^{-12}$	0.154	4
13	15.98	2	3.97	$4.7580\ 10^{-4}$	0.038	3

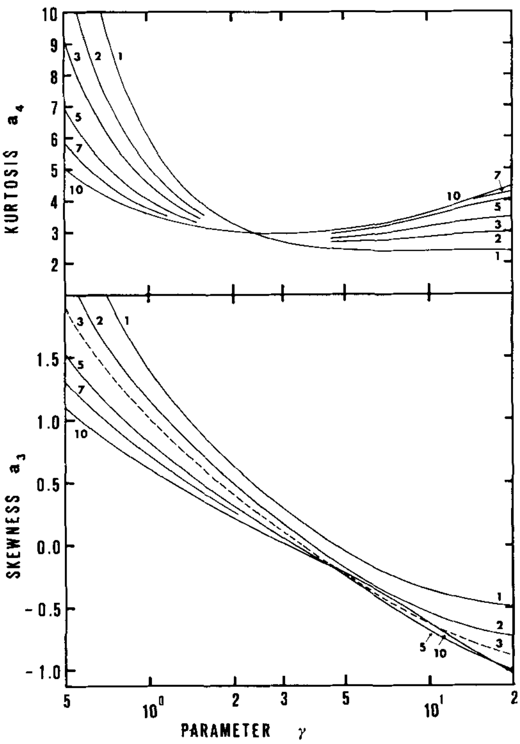

Fig. 9. Moment coefficients of skewness a_3 and kurtosis a_4 plotted as a function of the parameter γ for different integer values of the parameter α on the basis of the analytical expressions proposed in Appendix. For each isopleth the correspodinng value of α is given.

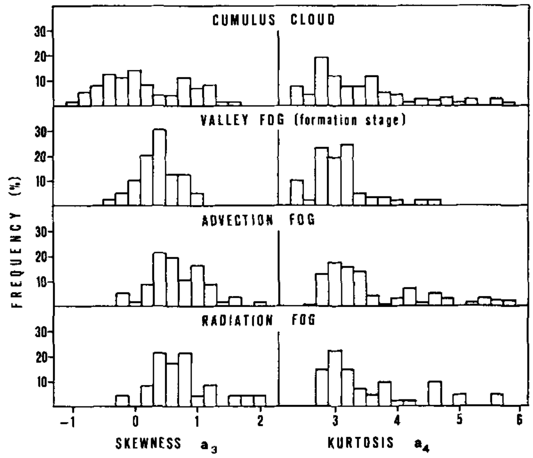

Fig. 10. Histograms of the moment coefficients of skewness and kurtosis relative to the found size distributions of droplets in cumulus clouds and fogs of various type. The class marks of skewness and kurtosis are given in abscissa with class interval size of 0.2. The relative frequency is given in ordinate.

parameters related to the former ones which point out the main graphical features of the curve. A useful shape-parameter is the above mentioned halfwidth A. Other parameters appear to be particularly significant in relating the profiles of the size distribution to the mechanisms which control the formation and growth of the fog and cloud droplets: the moment coefficient of skewness a_3 giving a measure of the degree of asymmetry of the curve and the moment coefficient of kurtosis a_4 which is a measure of the peakedness. Both these parameters can be expressed in terms of the gamma function if they are related to Deirmendjian's distribution function. The definition of the moment coefficients a_3 and a_4 and their analytical expressions in terms of α and γ are given in the Appendix.

Fig. 9 presents the corresponding diagrams which show a_3 and a_4 as a function of γ for different integer values of α. Skewness decreases as α and γ increase while the isopleths giving the kurtosis for some integer values of α all exhibit a minimum (less than 3) localized at different values of γ.

From the preliminary parameter sets entered in these diagrams a_3 and a_4 have been evaluated for the various types of fog and cloud. Fig. 10 presents the histograms of skewness and kurtosis relative to the examined size distributions concerning various types of fog and the cumulus cloud. From the comparison it appears that the fog histograms are characterized by a more limited dispersion, which can be attributed to a more marked stability of the environmental parameters driving the growth mechanisms of the fog droplets. Related to the various types of fog the skewness presents mostly positive values and the tendency to decrease as the mode radius increases. Correspondingly values of the kurtosis greater than 3 characterize the larger part of the fog data indicating a predominance of the leptokurtic shape of the fog droplet size distributions.

Similar trends of skewness and kurtosis are shown also by the stratocumulus and stratus cloud solutions in the mode radius range below 8 μm.

With regard to the cumulus cloud data the values of a_3 and a_4 appear scattered to a larger extent. The values of skewness cover the range from about -1 to 2 with the highest frequencies around 0 and 1. Correspondingly the kurtosis ranges from 2.6 to about 4.4.

In spite of the large scatter of data the behaviour of skewness and kurtosis with respect to the mode radius allows separating roughly the size distributions of cumulus cloud droplets into three classes:

(i) size distributions with high relative concentrations of small droplets and mode radius less than about 6 μm. They are positively skewed and leptokurtic with values of a_3 and a_4 presenting the tendency to decrease rather regularly as the mode radius increases.

(ii) size distributions with a high relative concentration of large droplets and mode radius more than about 10 μm. The corresponding shapes appear negatively skewed and leptokurtic. Skewness a_3 shows the tendency to constantly decrease for increasing mode radius while kurtosis appears increasing.

(iii) intermediate size distributions with mode radius between 6 and 10 μm. In this class the skewness is around zero and the values of kurtosis are close to 3 so that symmetrical and nearly mesokurtic curves appear to be very representative.

With reference to the suggestion of Bartlett & Jonas (1972) the essential differences of trend of the two shape-parameters pointed out by this classification suggest that the size distribution of class (i) can be mainly related to the initial condensation processes which, may be, explain the predominance of the left with respect to the right wing. Conversely the size distributions of class (ii) appear related to a concurrence of condensation with coalescence processes affecting the wing of larger droplets and evaporation responsible for reductions on the left part of the droplet population. The class (iii) can be considered as corresponding to the intermediate developed stage of condensation.

6. Conclusions

The experimental data have been mostly interpreted on the basis of a unique modified gamma distribution for each radius range characterized by a marked mode of the size spectrum. In these cases it appears easy to separate several modes centred at well distinct values of the radius. On the other hand different mechanisms can sometimes contribute to the formation of water droplet size distributions presenting various shapes all closely located within a certain narrow range of the radius. In this case the resulting envelope masks the size distribution features of each single original mode. The interpretation of the composite size spectra on the basis of a linear combination of modified gamma distributions presents remarkable operational difficulties and does not assure reliability in the derived multimodal solution.

Based on the analysis of the experimental data the present paper suggests realistic size distribution models of water droplets in haze, fogs and clouds. Such models can be used both separately and in proper linear combinations for simulating a large part of the size distributions derived from the different formation processes occurring in the real atmosphere.

Although the fog droplet size spectra present mostly a unique marked mode some fog layers can consist of water droplets passing through different evolutionary stages so that their droplet population can be properly represented by means of at least bimodal size distributions derived from our models of Tables 3, 4 and 5. Such models are generally characterized by positive values of the skewness and by a typical dependence of this shape-parameter on the mode radius which appear well consistent with the predominance of growth processes by condensation.

A considerable number of the size spectra measured in cumulus clouds shows the presence of more than one mode. Multimodal structure of the size distribution of cumulus cloud droplets has been associated by Warner (1969*a*) with the potential temperature gradient in the clear air environment and interpreted as the probable result of mixing between the cloud and its environment. In the light of these suggestions realistic representations of size distribution of cumulus cloud droplets can be given by a bimodal model performed through the linear combination of two modified gamma size distributions. In such a model the first size distribution is selected with features proper to the class (i) and associated with the predominance of condensation processes whereas the second one presents the composite features characterizing the class (ii). By varying in a proper way the numerical coefficients of the linear combination of two modes among those of Table 6, bimodal size distributions can be obtained which are suitable for describing the typical evolutionary stages of the cumulus cloud.

The procedure based on different weights of the two modes permits utilizing this model for representing the droplet size distribution at different heights above the cloud base. In conformity with the model proposed by Zaitsev (1950) for the cumulus cloud structure, the first mode has to be provided with greater weight especially if related to the first layers above the cloud base, where the continuous introduction of fresh condensation nuclei appears to be particularly relevant. The second mode has to assume increasing weight through the median and upper layers where Zaitsev's (1950) model suggests the presence of higher relative concentrations of larger droplets.

Appendix

Skewness and kurtosis in the modified gamma distribution function

Skewness is the degree of asymmetry and gives a measure of the departure from symmetry of a distribution curve. Thus the curve has positive skewness when a longer "tail" is presented to the right of the mode than to the left. Otherwise it is said to have negative skewness. Null skewness corresponds to perfectly symmetrical curves. The moment coefficient of skewness a_3 is in general given by the ratio of the third moment about the mean to the 3/2 power of the second moment of the distribution curve.

Kurtosis gives the degree of peakedness of the distribution curve with reference to normal distribution. Curves which have a relatively high peak are called leptokurtic and are characterized by values of kurtosis more than 3. Distributions which are flat-topped are called platykurtic and present kurtosis less than 3. Normal distribution corresponds to kurtosis equal to 3 and is called mesokurtic. The moment coefficient of kurtosis a_4 is defined in general by the ratio of the fourth moment about the mean to the square of the second moment.

The modified gamma distribution function presents the property that the integral of its product with the power r^k over the entire range of radii can be expressed in terms of the gamma function. In general:

$$I_k = a\int_0^\infty r^{\alpha+k} \exp\left[-\frac{\alpha}{\gamma}\left(\frac{r}{r_c}\right)^\gamma\right] dr = \frac{a}{\gamma}\left(\frac{\alpha}{\gamma r_c^\gamma}\right)^{-\frac{\alpha+k+1}{\gamma}} \Gamma\left(\frac{\alpha+k+1}{\gamma}\right) \quad \text{(A 1)}$$

The second, third and fourth moment about the mean are given respectively by the following equations:

$$m_2 = (I_2 - 2\, I_1\bar{r} + I_0\bar{r}^2)/I_0 \quad \text{(A 2)}$$

$$m_3 = (I_3 - 3\, I_2\bar{r} + 3\, I_1\bar{r}^2 - I_0\bar{r}^3)/I_0 \quad \text{(A 3)}$$

$$m_4 = (I_4 - 4\, I_3\bar{r} + 6\, I_2\bar{r}^2 - 4\, I_1\bar{r}^3 + I_0\bar{r}^4)/I_0 \quad \text{(A 4)}$$

where $\bar{r}$ is defined by I_1/I_0.

From eqs. (A 1), (A 2) and (A 3) the moment coefficient of skewness is given by

$$a_3 = \frac{[\Gamma(\varrho_1)]^2\Gamma(\varrho_4) - 3\,\Gamma(\varrho_1)\Gamma(\varrho_2)\Gamma(\varrho_3) + 2[\Gamma(\varrho_2)]^3}{\{\Gamma(\varrho_1)\Gamma(\varrho_3) - [\Gamma(\varrho_2)]^2\}^{3/2}} \quad \text{(A 5)}$$

where

$$\varrho_k = \frac{\alpha+k}{\gamma} \quad \text{(A 6)}$$

From eqs. (A 1), (A 2) and (A 4) the moment coefficient of kurtosis is given by

$$a_4 = \frac{[\Gamma(\varrho_1)]^3\Gamma(\varrho_5) - 4[\Gamma(\varrho_1)]^2\Gamma(\varrho_2)\Gamma(\varrho_4) + 6\,\Gamma(\varrho_1)[\Gamma(\varrho_2)]^2\Gamma(\varrho_3) - 3[\Gamma(\varrho_2)]^4}{\{\Gamma(\varrho_1)\Gamma(\varrho_3) - [\Gamma(\varrho_2)]^2\}^2} \quad \text{(A 7)}$$

with ϱ_k defined by eq. (A 6).

On the basis of eqs. (A 5), (A 6) and (A 7) the moment coefficients of skewness and kurtosis can be evaluated from the parameters α and γ of the modified gamma distribution function. In Fig. 9 these moment coefficients are drawn as a function of γ for different integer values of α.

REFERENCES

Arnulf, A., Bricard, J., Curé, E. & Veret, C. 1957. Transmission by haze and fog in the spectral region 0.35 to 10 microns. *J. Opt. Soc. Amer. 47*, 491–498.

Bartlett, J. T. & Jonas, P. R. 1972. On the dispersion of the sizes of droplets growing by condensation in turbulent clouds. *Quart. J. Roy. Meteor. Soc. 98*, 150–164.

Best, A. C. 1951. Drop-size distribution in cloud and fog. *Quart. J. Roy. Meteor. Soc. 77*, 418–426.

Borovikov, A. M., Gaivoronskii, I. I., Zak, E. G., Kostarev, V. V., Mazin, I. P., Minervin, V. E., Khrgian, A. Kh. & Shmeter, S. M. 1963. *Cloud physics*. Jerusalem, Israel Program for Scientific Translations, pp. 65–123.

Deirmendjian, D. 1964. Scattering and polarization properties of water clouds and hazes in the visible and infrared. *Appl. Opt. 3*, 187–196.

Deirmendjian, D. 1969. *Electromagnetic scattering on spherical polydispersions*. Elsevier, New York, pp. 75–119.

Durbin, W. G. 1959. Droplet sampling in cumulus clouds. *Tellus 11*, 202–215.

Eldridge, R. G. 1957. Measurements of cloud dropsize distributions. *J. Meteor. 14*, 55–59.

Eldridge, R. G. 1966. Haze and fog aerosol distributions. *J. Atmos. Sci. 23*, 605–613.

Fitzgerald, J. W. & Spyers-Duran, P. A. 1973. Changes in cloud nucleus concentration and cloud droplet size distribution associated with pollution from St. Louis. *J. Appl. Meteor. 12*, 511–516.

Garland, J. A. 1971. Some fog droplet size distributions obtained by an impaction method. *Quart. J. Roy. Meteor. Soc. 97*, 483–494.

Garland, J. A., Branson, J. R. & Cox, L. C. 1973. A study of the contribution of pollution to visibility in a radiation fog. *Atmos. Environ. 7*, 1079–1092.

Jiusto, J. E. 1967. Aerosol and cloud microphysics measurements in Hawaii. *Tellus 19*, 359–368.

Keily, D. P. & Millen, S. G. 1960. An airborne cloud-drop-size distribution meter. *J. Meteor. 17*, 349–356.

Khrgian, A. Kh. & Mazin, I. P. 1952. The size distribution of droplets in clouds. *Trudy Ts. A. O.*, No. 7, 56.

Kocmond, W. C., Pilie, R. J., Eadie, W. J., Mack, E. J. & Leonard, R. P. 1971. Investigation of warm fog properties and fog modification concepts. *NASA Contractor Report CR-1731*, Washington, D.C.

Kumai, M. 1973. Arctic fog droplet size distribution and its effect on light attenuation. *J. Atmos. Sci. 30*, 635–643.

Kunkel, B. A. 1971. Fog drop-size distributions measured with a laser hologram camera. *J. Appl. Meteor. 10*, 482–486.

Levin, L. M. 1954. Size distribution functions of cloud and rain drops. *Doklady AN SSSR 94*, No. 6.

Levin, L. M. 1958. Distribution functions of cloud droplets. *Izvestiya AN SSSR, geophys. ser.*, No. 10, 1211.

Pilie, R., Eadie, W., Mack, E., Rogers, C. & Kocmond, W. 1972. Project fog drops. I. Investigations of warm fog properties. *NASA Contractor Report CR-2078*, Washington, D.C.

Rosenbrock, H. H. 1960. An automatic method for finding the greatest or least value of a function. *Computer J. 3*, 175–184.

Squires, P. 1958. The microstructure and colloidal stability of warm clouds. I. The relation between structure and stability. *Tellus 10*, 256–261.

Tverskoi, P. N. 1965. *Physics of the atmosphere*. Israel Program for Scientific Translations, Jerusalem, pp. 314–343.

Vittori, O. & Pesaresi, R. 1962. Spettro dei diametri di particelle e proprietà ottiche delle nebbie. *CENFAM Report S.R.*, No. 3.

Warner, J. 1969*a*. The microstructure of cumulus cloud. I. General features of the droplet spectrum. *J. Atmos. Sci. 26*, 1049–1059.

Warner, J. 1969*b*. The microstructure of cumulus cloud. II. The effect on droplet size distribution of the cloud nucleus spectrum and updraft velocity. *J. Atmos. Sci. 26*, 1272–1282.

Warner, J. 1973. The microstructure of cumulus cloud. V. Changes in droplet size distribution with cloud age. *J. Atmos. Sci. 30*, 1724–1726.

Webb, W. L. 1956. Particulate counts in natural clouds and fogs. *J. Meteor. 13*, 203–206.

Zaitsev, V. A. 1950. Liquid water content and distribution of drops in cumulus clouds. *Trudy glav. Geofiz. Obs. 19*, 81–122.

Zuev, V. E. 1970. *Atmospheric transparency in the visible and in the infrared*. Israel Program for Scientific Translations, Jerusalem, pp. 92–100.

МОДЕЛИ РАСПРЕДЕЛЕНИЯ ПО РАЗМЕРАМ ТУМАННЫХ И ОБЛАЧНЫХ КАПЕЛЬ В ТЕРМИНАХ МОДИФИЦИРОВАННОЙ ФУНКЦИИ ГАММ

Спектры размеров водяных капель, тумана и облаков различных типов, измеренных многими авторами, были анализированы вычислительным методом в терминах модифицированной функции тамм распределения. Предварительные результаты показаны в широкой области параметров функции.

Синтез этих результатов дан на основе графического метода для получения средних и показательных моделей распределения по размерам. Эти модели пригодны для воспроизведения размерных спектров водяных капель типичных для тумана и облаков разных стадий развития.

Исследования результатов в терминах асимметрии и относительной высоты распределения показывают типичные поведения этих двух параметров формы, которые могут быть взяты во внимание для достижения размерного многомодового распределения, используя, для этого, найденные модели специально взвешенные.

Аналитические выражения для асимметрии и относительной высоты в терминах модифицированной функции гамма-распределения, даются в Приложении.