An example of the modification of ocean currents by bottom topography

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ABSTRACT

A mathematical model is presented for steady currents in ocean basins which become deeper towards the poles. Relatively large currents occur in an interior layer at the line where $\partial/(\partial y)\,(f/h)$ changes sign, (f is the Coriolis parameter, h the depth and y a northward co-ordinate). The flow in the interior layer is described quantitatively and the work thus complements recent studies by Kamenkovich & Mitrofanov (1971) and Brink et al. (1973).

Introduction

The purpose of this note is to discuss an issue recently examined by Welander (1968), Kamenkovich & Mitrofanov (1971), Johnson et al. (1971), Fandry & Leslie (1972) and Brink et al. (1973). The flow in an ocean basin depends on the parameter f/h, where f is the Coriolis parameter and h is the depth, and wind-driven circulations have well-known properties if f/h increases either to the north or the south—for example the familiar westward intensification of currents if f/h increases northward. The case where $\partial/(\partial y)(f/h)$ changes sign (y is the northward co-ordinate) is more contentious, however, and is the subject of the cited references.

Welander (1968) and Johnson et al. considered this problem first and presented solutions which were subsequently shown to be deficient by Fandry & Leslie (1972). In the meantime, Kamenkovich & Mitrofanov (1971) independently had derived the solution in detail, although they did not present complete calculations for the currents. Finally, Brink et al. (1973) described an elegant laboratory simulation of the problem confirming the arguments of Fandry & Leslie and Kamenkovich & Mitrofanov. The present note investigates an alternative simulation of the phenomenon examining, in particular, motions in the free boundary layer adjacent to the critical line $\partial/(\partial y)(f/h) = 0$. The calculations confirm the streamlines sketched by Fandry & Leslie and Kamenkovich & Mitrofanov and are complementary to the work of Brink et al.

Ocean circulations are sometimes simulated by steady driven flows in rotating cylinders with variable bottom topography. For example, Pedlosky & Greenspan (1967) have discussed a "sliced cylinder" configuration, in which the plane bottom of a cylindrical container is inclined to the horizontal and the fluid is driven by a slightly faster rotation of the top surface. In the present case, driven motions are considered in a rotating cylinder whose bottom is shaped as a parabolic trough. This geometry has the property that the depth parallel to the axis of rotation has a maximum on an interior vertical plane in the cylinder. It is therefore an "inverted" analogue of the geophysical case where $\partial/(\partial y)\,(f/h) = 0$ along an interior line in an ocean basin. The motions will again be driven by a slightly faster rotation of the top surface of the container.

Linear theory

If the top and bottom surfaces of the container are given by $d(x, y) = 1$ and $b(x, y) = \frac{1}{2}\alpha x^2$ respectively, the modified pressure $P_0(x, y)$ is given by the approximating linear equation (see, for example, Carrier, 1965, or Barton, 1975*b*)

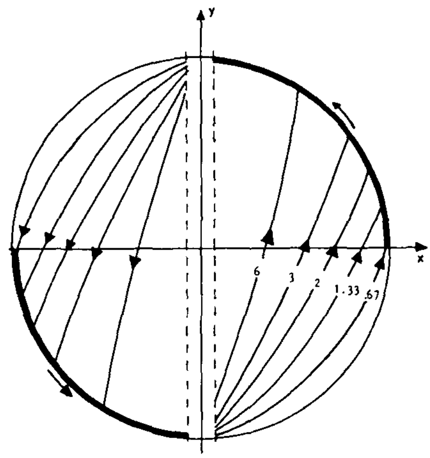

Fig. 1. Streamlines of the outer solution; numerical values represent $-(E\alpha^{-2})^{-1/2}P_0/\Omega$. The interior layer and the mass-return boundary layers are also shown.

$$E^{1/2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)P_0-\alpha x\frac{\partial P_0}{\partial y}=E^{1/2}\hat{\mathbf{k}}\cdot\nabla\times\mathbf{U} \qquad (1)$$

Here, P_0 is non-dimensional and is related to the fluid pressure p by

$$P_0=\{p+\varrho gz+\tfrac{1}{2}\varrho\omega^2(x^2+y^2)\}/(\varrho\omega LU) \qquad (2)$$

x and y are non-dimensional co-ordinates fixed in the container which is rotating at $\omega\hat{\mathbf{k}}$ about a vertical axis, and E is the Ekman number of the flow

$$E=\nu/(\omega L^2) \qquad (3)$$

The co-ordinates x and y have been non-dimensionalized by the length L (taken as the diameter of the cylinder) and U denotes a typical value of the velocity $\mathbf{U}$ applied at the top surface of the cylinder.

Suppose the fluid is driven by the additional rotation of the top surface at an angular velocity $\Omega\hat{\mathbf{k}}$ with respect to the rotating frame. Eq. (1) thus becomes

$$E^{1/2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)P_0-\alpha x\frac{\partial P_0}{\partial y}=2\Omega E^{1/2}+O(E^{1/2}\alpha^2) \qquad (4)$$

if the bottom slopes are small ($0<\alpha<<1$), and this equation is to be solved under the boundary condition

$$P_0(x,y)=0 \quad \text{at} \quad \sqrt{x^2+y^2}=\tfrac{1}{2}. \qquad (5)$$

The horizontal velocity components u and v are given by the geostrophic formulae

$$u=-\tfrac{1}{2}P_{0,y}+O(E),\quad v=\tfrac{1}{2}P_{0,x}+O(E)$$

There is an obvious similarity between eq. (4) and the analogous equation considered in an oceanographical context by Kamenkovich & Mitrofanov (1971). The solution of (4) is constructed separately in various regions throughout the cylinder. In the bulk of the fluid away from the cylinder's walls, eq. (4) has the approximating reduced form

$$-\alpha x\frac{\partial P_0}{\partial y}=2\Omega E^{1/2} \qquad (6)$$

with solution

$$P_0(x,y)=-2\Omega x^{-1}(E\alpha^{-2})^{1/2}(y\pm\sqrt{\tfrac{1}{4}-x^2}),\quad x\gtrless 0. \qquad (7)$$

The direction of integration of (6) to obtain (7) is given by standard arguments (see, for example, Greenspan, 1968). The solution (7) is shown in Fig. 1, and it is apparent that the flow does not satisfy condition (5) on the entire boundary of the cylinder. A boundary layer analysis in the manner of Pedlosky & Greenspan (1967) can be used for this problem to give an improved approximation for the pressure:

$$P_0^*=P_0+E^{1/2}\tilde{p} \qquad (8)$$

where $\tilde{p}$ is found to be the boundary layer function

$$\tilde{p}(\varrho,\theta,z)=-\frac{8\Omega}{\alpha\sqrt{3}}\tan\theta \times\operatorname{Im}\left(\exp\left\{-\frac{i\pi}{3}-\left|\frac{1}{2}\alpha\sin 2\theta\right|^{1/3}\varrho\, e^{i\pi/3}\right\}\right)\times\left(\frac{\sin 2\theta+|\sin 2\theta|}{2\sin 2\theta}\right),$$

$$(\varrho=(\tfrac{1}{2}-\sqrt{x^2+y^2})E^{-1/3},\quad \theta=\tan^{-1}(y/x)). \qquad (9)$$

This $E^{1/3}$ boundary layer describes a return of fluid enabling global mass conservation, but the solution (8) still does not satisfy a non-slip condition at the cylinder walls. A weaker $E^{1/4}$ boundary layer adjacent to the entire boundary is required for this purpose.

The interior layer

The solution (8) is inappropriate near $x = 0$ where both P_0 and $\tilde{p}$ become singular. A new balance of terms in eq. (4) is required in this "interior layer" region, and it is easily found that the proper governing equation near $x = 0$ is (in stretched variables)

$$\psi_{\xi\xi} - \xi\psi_\eta = 2\Omega \tag{10}$$

where ξ, η and ψ are defined by

$$\xi = (E\alpha^{-2})^{-1/6}x, \quad \eta = \tfrac{1}{2} + y, \quad \psi = (E\alpha^{-2})^{-1/3}P_0 \tag{11}$$

The velocity components u and v are seen to be relatively large in the interior layer, being of $O((E\alpha^{-2})^{1/3})$ and $O((E\alpha^{-2})^{1/6})$ respectively, compared with the magnitude $O((E\alpha^{-2})^{1/2})$ away from $x = 0$. The scaled pressure ψ is required to satisfy the conditions

$$\left.\begin{aligned} &\psi(\xi, \eta) = \psi(-\xi, 1-\eta) \\ &\psi(\xi, \eta) \sim -2\Omega\eta/\xi + O(\xi^{-4}), \quad \xi \to \infty \\ &\psi(\xi, 0) = 0 \\ &\psi(0, \eta) = f(\eta) \end{aligned}\right\} \tag{12}$$

where $f(\eta)$ is to be determined so that the velocity components are continuous at $\xi = 0$.

The solution of eqs. (10) and (12) was found by Kamenkovich & Mitrofanov (1971) and independently by the author (Barton, 1973), although Kamenkovich & Mitrofanov did not calculate the stream functions. A brief description of the interior layer solution is now given for reference.

The solution ψ is found to be the sum of three terms

$$\psi(\xi, \eta) = \psi_1(\xi) + \psi_2(\xi, \eta) + \psi_3(\xi, \eta) \tag{13}$$

where

$$\psi_1(\xi) = \Omega\xi^2,$$

$$\psi_2(\xi, \eta) = c_1\xi\eta^{1/3}M(-\tfrac{1}{3}, \tfrac{4}{3}, -\xi^3/(9\eta)),$$

$$c_1 = -\Omega\Gamma(\tfrac{5}{3})\, 3^{2/3}/\Gamma(\tfrac{4}{3}),$$

$$\psi_3(\xi, \eta) = c_2\int_0^\eta f(t)\,\xi(\eta - t)^{-4/3}\exp\left\{-\frac{\xi^3}{9(\eta - t)}\right\}dt,$$

$$c_2 = (3^{2/3}\Gamma(\tfrac{1}{3}))^{-1}.$$

Here M is Kummer's confluent hypergeometric function. The requirement that $\partial\psi/\partial\xi$ is continuous at $\xi = 0$ is found to give an integral equation for $f'(\eta)$

$$\int_0^1 f'(u)\,\mathrm{sgn}(\eta - u)\,|\eta - u|^{-1/3}du = \frac{c_1}{3c_2}(\eta^{1/3} + (1-\eta)^{1/3}), \quad 0 < \eta < 1 \tag{14}$$

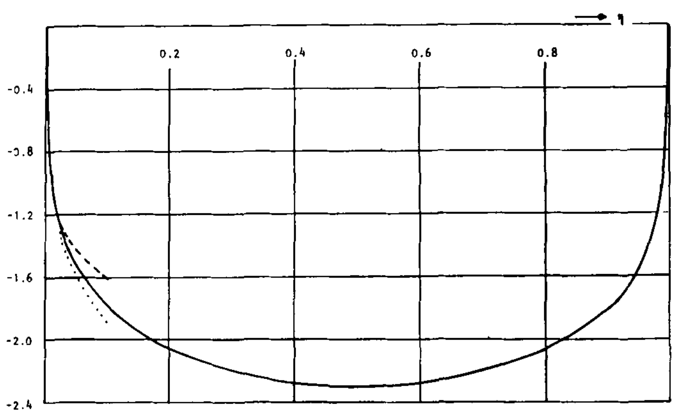

Fig. 2. Pressure at the centre-line: ——, $f(\eta)/\Omega$; - - - - -, $-2.35\eta^{1/6}$; ......, $-2.35\eta^{1/6} - 1.43\eta^{2/3}$.

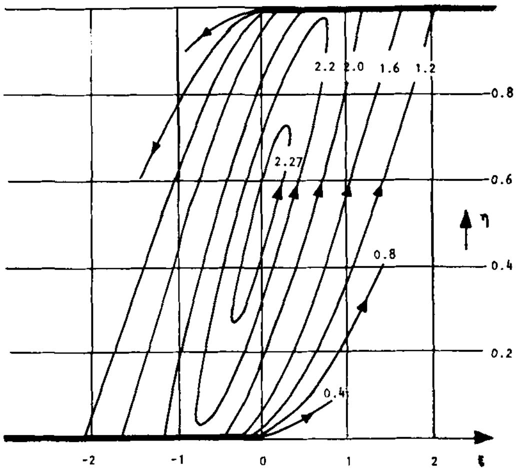

Fig. 3. Streamlines in the interior layer; numerical values represent $-(E\alpha^{-2})^{-1/3}P_0/\Omega$.

whose solution may be written as

$$f(\eta) = \frac{1}{2\pi}\sin\frac{\pi}{3}\int_0^\eta R(t)\,(\eta-t)^{-2/3}dt$$
$$+\pi^{-2}\sin^2\frac{\pi}{6}\int_0^\eta t^{-1/6}(1-t)^{-1/6}(\eta-t)^{-2/3}$$
$$\left\{(P)\int_0^1 \frac{\tau^{1/6}(1-\tau)^{1/6}R(\tau)\,d\tau}{\tau-t}\right\}dt$$

using a modification of a technique suggested by Carrier et al. (1966, p. 426). Alternatively, a numerical solution of (14) may be employed (Barton, 1973, 1975*a*) to give the form for $f(\eta)$ shown in Fig. 2.

Finally, the calculation of the lines $\psi =$ constant gives the stream function diagram shown in Fig. 3. These stream functions confirm the sketches of the flow pattern given by Kamenkovich & Mitrofanov (1971).

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ПРИМЕР МОДИФИКАЦИИ ОКЕАНСКИХ ТЕЧЕНИЙ ТОПОГРАФИЕЙ ДНА

Представлена математическая модель стационарных течений в океанических бассейнах, углубляющихся к полюсам. Относительно большие течения наблюдаются во внутреннем слое на линии, где величина $\partial(f/h)/\partial y$ меняет знак (f — параметр Кориолиса, h — глубина и y — координата, направленная на север). Течение во внутреннем слое описывается количественно и тем самым, данная работа дополняет недавние исследования Каменковича и Митрофанова (1971) и Бринка и др. (1973).