A problem in the parameterization of large-scale eddies

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ABSTRACT

The large-scale eddies in the extratropics are assumed to be in exact geostrophic and hydrostatic balance. An attempt is made to parameterize these eddies through arithmetic mean and root mean square values of the horizontal and vertical slopes of the zonal harmonic components. The implications of the two types of parameterization for the meridional transfers of momentum and sensible heat and the partitioning of the zonal variances of momentum and temperature in terms of their availability for poleward transport (which involves the proper depiction of the eddies) are discussed.

List of symbols

a	mean radius of the Earth
c_p	specific heat at constant pressure of dry air
C_n	amplitude of the nth zonal harmonic of height perturbations on an isobaric surface
f	Coriolis parameter
g	acceleration due to Earth's gravity
n	denotes the nth zonal harmonic and is a positive integer
p	pressure
R	gas constant for dry air
S_n	phase of the nth zonal harmonic of height perturbations on an isobaric surface
T	(virtual) temperature
u	zonal component of the (geostrophic) wind
v	meridional component of the (geostrophic) wind
z	height of an isobaric surface
λ	longitude
ϕ	latitude
$\partial S_n/a\partial\phi$	horizontal slope of the nth harmonic of height perturbations
$\partial S_n/\partial \ln p$	vertical slope of the nth harmonic of height perturbations
$-$ (bar)	denotes weighted arithmetic averaging with respect to wave-numbers
$\sim$	denotes square root of the sum of squares; summation is with respect to wave-numbers
$\langle - \rangle$	denotes weighted root mean square averaging with respect to wave-numbers
[ ]	denote zonal averaging
*	denotes the difference of a local value from the zonal mean
$\lvert x \rvert$	modulus of x

In a previous paper (Srivatsangam, 1975) the author considered the partitioning of the zonal variances of virtual temperature and zonal momentum in terms of their availability for horizontal transport processes. That analysis was done by expressing z, T, v and u in terms of zonal harmonics, the numerical calculations having been peeformed for the first 10 harmonics of the standing eddies of January 1970 and the transient eddies of the first three days of that month. Here an attempt will be made to fractionise zonal variances through parametric methods so that computations in terms of individual harmonics need not be performed.

In the following we shall denote by 'mean' arithmetic mean and by 'average' both arithmetic mean and root mean square (rms) values.

As in Srivatsangam (1975) we shall express z^* by

$$z^* = \sum_{n=1}^{\infty} C_n \cos (n\lambda - S_n) \tag{1}$$

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whence

$$[z^{*2}] = \sum_{n=1}^{\infty} \tfrac{1}{2} C_n^2 \qquad (2)$$

through Parseval's Theorem (PT).

From eq. (1) by geostrophic approximation and PT

$$[v^{*2}] = (g/fa \cos \phi)^2 \sum_{n=1}^{\infty} \tfrac{1}{2} n^2 C_n^2 \qquad (3)$$

If an average wave-number is represented by $\overset{\langle - \rangle}{n}$, eq. (3) becomes

$$[v^{*2}] = (g\overset{\langle - \rangle}{n}/fa \cos \phi)^2 [z^{*2}] \qquad (4)$$

by virtue of eq. (2).

From eq. (1) the zonal-mean geostrophic momentum transport by eddies is given by

$$[v^* u^*] = \sum_{n=1}^{\infty} [v_n^* u_n^*]$$

$$= (g^2/f^2 a \cos \phi) \overset{\langle - \rangle}{n} \sum_{n=1}^{\infty} \tfrac{1}{2} C_n^2 (\partial S_n / a \partial \phi) \qquad (5)$$

where $\overset{\langle - \rangle}{n}$ has been substituted from eq. (4).

Similarly the zonal-mean transport of sensible heat due to eddies is

$$c_p [v^* T^*] = c_p (g^2/Rfa \cos \phi) \overset{\langle - \rangle}{n} \sum_{n=1}^{\infty} \tfrac{1}{2} C_n^2 (\partial S_n / \partial \ln p) \qquad (6)$$

where T^* is obtainable from eq. (1) through hydrostatic approximation and the equation of state.

If now average slopes can be defined by $\partial \bar{S}_n / a \partial \phi$ and $\partial \bar{S}_n / \partial \ln p$, eqs. (5) and (6) become

$$[v^* u^*] = (g^2/f^2 a \cos \phi) \overset{\langle - \rangle}{n} (\partial \bar{S}_n / a \partial \phi) [z^{*2}] \qquad (7)$$

and

$$[v^* T^*] = (g^2/Rfa \cos \phi) \overset{\langle - \rangle}{n} (\partial \bar{S}_n / \partial \ln p) [z^{*2}] \qquad (8)$$

respectively. From eqs. (7) and (8) $\partial \bar{S}_n / a \partial \phi$ and $\partial \bar{S}_n / \partial \ln p$ can be evaluated. Since all the quantities in eqs. (7) and (8) except the slopes are positive everywhere in the Northern Hemisphere, the sign of the value of $[v^* u^*]$ is determined solely by that of $\partial \bar{S}_n / a \partial \phi$ and the sign of the value of $[v^* T^*]$ by that of $\partial \bar{S}_n / \partial \ln p$.

From eq. (1) we may also obtain the following equation for the zonal variance of geostrophic zonal wind speed

$$[u^{*2}] = (g/f)^2 \sum_{n=1}^{\infty} \tfrac{1}{2} C_n^2 (\partial S_n / a \partial \phi)^2 + \tfrac{1}{2} (g/f)^2 \sum_{n=1}^{\infty} (\partial C_n / a \partial \phi)^2 \qquad (9\text{a})$$

$$\equiv [u^{*2}]^{(1)} + [u^{*2}]^{(2)} \qquad (9\text{b})$$

through geostrophic approximation and PT, and for the zonal variance of virtual temperature the equation

$$[T^{*2}] = (g/R)^2 \sum_{n=1}^{\infty} \tfrac{1}{2} C_n^2 (\partial S_n / \partial \ln p)^2 + \tfrac{1}{2} (g/R)^2 \sum_{n=1}^{\infty} (\partial C_n / \partial \ln p)^2 \qquad (10\text{a})$$

$$\equiv [T^{*2}]^{(1)} + [T^{*2}]^{(2)} \qquad (10\text{b})$$

through hydrostatic approximation and PT.

In eqs. (9a, b) and (10a, b) the first right hand side (rhs) terms alone contribute to meridional transports since only the slope-dependent terms of u^* and T^* enter into zonal-mean meridional transfer equations.

In terms of the average slopes given by eqs. (7) and (8)

$$[u^{*2}]^{(1)} = (g/f)^2 (\partial \bar{S}_n / a \partial \phi)^2 [z^{*2}] \qquad (11)$$

and

$$[T^{*2}]^{(1)} = (g/R)^2 (\partial \bar{S}_n / \partial \ln p)^2 [z^{*2}] \qquad (12)$$

and knowing the values of $[u^{*2}]$ and $[T^{*2}]$ then $[u^{*2}]^{(2)}$ and $[T^{*2}]^{(2)}$ can be obtained by differencing. These values have been obtained for the standing eddies of January from the 5-year mean statistics of Oort & Rasmusson (1971) and presented in Table 1. The data analysed pertains to the zone 20° N to 75° N (where the geostrophic approximation is pertinent) and the 100; 200; 300; 500; 700; 850 and 1 000 mb levels. It may be mentioned that the Oort & Rasmusson data are for actual winds and temperature but these are not very different from geostrophic wind data and virtual temperature statistics, respectively, in the zone and month we are dealing with.

In Srivatsangam (1975) the value of $[u^{*2}]^{(1)}$: $[u^{*2}]^{(2)}$ was obtained as 1:0.97 for the zone

Table 1. *Partitioning of zonal variances as derived from the Oort & Rasmusson (1971) statistics on January standing eddies and eqs. (11) and (12)*

$[u^{*2}]^{(1)}$ m^2s^{-2}	$[u^{*2}]^{(2)}$ m^2s^{-2}	$[u^{*2}]^{(1)}:[u^{*2}]^{(2)}$
3	22.8	1:7.6
$[T^{*2}]^{(1)}$ °K²	$[T^{*2}]^{(2)}$ °K²	$[T^{*2}]^{(1)}:[T^{*2}]^{(2)}$
4.1	10.8	1:2.63

25° N to 75° N and the levels 100; 200; 300; 400; 500; and 700 mb. The weighted mean of $[T^{*2}]^{(1)}:[T^{*2}]^{(2)}$ obtained in that paper was 1:0.9 for the zone 20° N to 80° N and the levels 200; 300; 400 and 500 mb.

The data in Table 1 are also weighted averages, the weights for the latitudes being the cosine of those angles and the 100; 200; 300; 850 and 1 000 mb levels having a weight of 1 each and the 500 and 700 mb levels having 2.5 and 2 as weights respectively. A comparison of the data in Table 1 with the values given in the above paragraph shows that eqs. (11) and (12) underestimate the components of variances available for meridional transport and overestimate those which are not available for this purpose. We shall discuss this further below.

Another method for fractionising zonal variances is as follows:

Let us define an average amplitude of height perturbations $\tilde{C}_n$ so that

$$[z^{*2}] = \tfrac{1}{2}\,\tilde{C}_n^2 \tag{13}$$

Differentiating eq. (13) with respect to $a\phi$ and squaring both sides of the resulting equation

$$\{\partial[z^{*2}]/a\partial\phi\}^2 = 2[z^{*2}](\partial\tilde{C}_n/a\partial\phi)^2 \tag{14}$$

Similarly

$$\{\partial[z^{*2}]/\partial \ln p\}^2 = 2[z^{*2}](\partial\tilde{C}_n/\partial \ln p)^2 \tag{15}$$

Substituting for $(\partial\tilde{C}_n/a\partial\phi)^2$ and $(\partial\tilde{C}_n/\partial \ln p)^2$ in the expressions for $[u^{*2}]^{(2)}$ and $[T^{*2}]^{(2)}$ (see eqs. (9a, b) and (10a, b))

$$[u^{*2}]^{(2)} = \tfrac{1}{4}(g/f)^2[z^{*2}]^{-1}\{\partial[z^{*2}]/a\partial\phi\}^2 \tag{16}$$

and

$$[T^{*2}]^{(2)} = \tfrac{1}{4}(g/R)^2[z^{*2}]^{-1}\{\partial[z^{*2}]/\partial \ln p\}^2 \tag{17}$$

The partitioning of zonal variances as obtained from eqs. (16) and (17) is presented in Table 2, and it is readily seen that the values of $[u^{*2}]^{(1)}:[u^{*2}]^{(2)}$ and $[T^{*2}]^{(1)}:[T^{*2}]^{(2)}$ in Table 2 are in accord with the values for these ratios quoted earlier from Srivatsangam (1975).

Why do the data in Tables 1 and 2 differ so drastically?

In obtaining the average slopes $\partial\bar{S}_n/a\partial\phi$ and $\partial\bar{S}_n/\partial \ln p$ we have used a weighted arithmetic averaging procedure, so that, as mentioned above, the signs of the values of $[v^*u^*]$ and $[v^*T^*]$ solely depend upon $\partial\bar{S}_n/a\partial\phi$ and $\partial\bar{S}_n/\partial \ln p$ respectively. Now if $[v^*u^*]$ is zero ($f \neq 0$), then $\partial\bar{S}_n/a\partial\phi = 0$. But the individual harmonics might transport positive and negative quantities of momentum which cancel each other while the value of $[u^{*2}]^{(1)}$, being a squared quantity, is non-zero. Under such conditions eq. (11) will yield $[u^{*2}]^{(1)} = 0$. In fact it can be readily seen that

$$|\partial\bar{S}_n/a\partial\phi| \leqslant |\partial\overset{\langle - \rangle}{S_n}/a\partial\phi| \tag{18}$$

where $\partial\overset{\langle - \rangle}{S_n}/a\partial\phi$ is the weighted root mean square value of the horizontal slopes of the harmonics. *The equality sign in eq. (18) holds if all the harmonics have the same amplitude and horizontal slope, and the inequality sign when there is any difference between the amplitudes or the slopes of different harmonics. The inequality sign in eq. (18) is generally associated with atmospheric eddies, indicating the structural and causal differences between the many harmonics. Hence the values of $[u^{*2}]^{(1)}$ given by eq. (11) are almost always underestimates.* A similar argument ap-

Table 2. *Partitioning of zonal variances as derived from the Oort & Rasmusson (1971) statistics on January standing eddies and eqs. (16) and (17)*

Weights are as indicated in the text

$[u^{*2}]^{(1)}$ m^2s^{-2}	$[u^{*2}]^{(2)}$ m^2s^{-2}	$[u^{*2}]^{(1)}:[u^{*2}]^{(2)a}$
17.0	10.8	1:0.64
$[T^{*2}]^{(1)}$ °K²	$[T^{*2}]^{(2)}$ °K²	$[T^{*2}]^{(1)}:[T^{*2}]^{(2)b}$
7.5	4.5	1:0.6

[a] Data are for the zone 25° N to 70° N and the 100; 200; 300; 500; 700; 850 and 1 000 mb levels.
[b] Data are for the zone 20° N to 75° N and the levels 200; 300; 500; 700 and 850 mb.

plies to the values of $[T^{*2}]^{(1)}$ obtained from eq. (12). One advantage of eqs. (11) and (12) is that they are consistent with the meridional transports of momentum and sensible heat respectively.

On the other hand $\tilde{C}_n$, as defined by eq. (13), takes into account the differences between the values of C_n. The values of $[u^{*2}]^{(2)}$ and $[T^{*2}]^{(2)}$ derived from eqs. (16) and (17) also thus include the differences in the amplitudes of u_n^* and also T_n^*. The magnitudes of the average slopes derivable from the data in Table 2 are also readily shown to be rms values. These do not correspond to momentum and sensible heat transport (vide eq. (18) supra).

Thus the parametric representation of eddies in terms of mean slopes (eqs. (11) and (12)) will underestimate $[u^{*2}]^{(1)}$ *and* $[T^{*2}]^{(1)}$ *and overestimate* $[u^{*2}]^{(2)}$ *and* $[T^{*2}]^{(2)}$. *The former implies the elimination of the degree of freedom enjoyed by the atmosphere in employing divers harmonics to accomplish the observed transports, and the latter abnormal variations in the average amplitude of atmospheric eddies with height and latitude. The parameterization of eddies in terms of rms slopes (eqs. (16) and (17)), contrariwise, partitions the zonal variances properly; but the horizontal transports derived from these will not correspond to observed transfers.*

Finally a few words about the possibility of parameterizing transient eddies through monthly-mean statistics such as are presented by Oort and Rasmusson (op. cit.): eqs. (11) and (12) will yield seriously underestimated values of $[u^{*2}]^{(1)}$ and $[T^{*2}]^{(1)}$ and drastic overestimates of $[u^{*2}]^{(2)}$ and $[T^{*2}]^{(2)}$ since in this case $\partial\bar{S}_n/a\partial\phi$ and $\partial\bar{S}_n/\partial \ln p$ are arithmetic averages with respect to wave-numbers as well as time. Eqs. (16) and (17) will indeed apportion zonal variances correctly if the daily values of the components of variances are averaged in time. But if monthly-mean statistics of transient-eddy variances are used, the resulting fractionation will be in error unless the daily values of the zonal variances of height are invariant in time, which is in contradiction to the very definition of transient eddies. Hence the parameterization of transient eddies through average slopes and the use of monthly-mean statistics is more problematic than the parameterization of standing eddies.

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ОДНА ЗАДАЧА В ПАРАМЕТРИЗАЦИИ КРУПНОМАСШТАБНЫХ ВИХРЕЙ

Крупномасштабные вихри вне тропической области предполагаются находящимися в точном геострофическом и гидростатическом балансе. Сделана попытка параметризовать эти вихри через среднеарифметическую величину и через дисперсию горизонтальных и вертикальных наклонов зональных гармонических компонент. Обсуждаются следствия двух типов параметризации для меридиональных переносов количества движения и тепла и распределение зональных вариаций количества и температуры в терминах их доступности для переноса по направлению к полюсам (что включает необходимое описание вихрей).