The entropic energy of geophysical fluid systems

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ABSTRACT

The main objective of this work is to develop an integral theory for a wide class of fluid systems that not only includes the concepts of balance of mass, momentum, and energy, but also of entropy and the second law of thermodynamics. This theory provides an explicit relationship between the integrals of kinetic energy and of entropy in terms of a new energy form, the entropic energy.

Linearly viscous fluids whose entropy functions are strictly concave are defined as equilibrium regular fluids. A fluid system consists of an equilibrium regular fluid of fixed mass M, whose body force is a consequence of a steady, positive, monotonic potential field, and whose bounds satisfy certain specified conditions. Geophysical fluid systems are a non-degenerate set of fluid systems.

In geophysical fluid systems there exists a unique state for given values of system mass and system total energy E, called the associated equilibrium state, that is motionless, hydrostatic, and isothermal (Theorem 2.1). The product of the temperature T_0 of this state and the difference $S_0 - S$ in system entropies between it and a natural state is equal to the entropic energy N, the sum of the system kinetic energy K and a static energy form $T_0\Sigma$. The entropic energy is a positive measure, resembling an $\mathcal{L}_2$-norm, of the deviation of the natural state from its associated equilibrium state, and thus the associated equilibrium state is thermodynamically stable in Gibbs sense (Theorem 2.2).

If a geophysical fluid system is in isolation ($\dot{E} = 0$, $\dot{S} \geqslant 0$) entropic energy is bounded by its value at the time isolation occurs and its rate of change can never become positive for as long as the system remains in isolation (Theorem 2.3). When a system is energetically isolated ($\dot{E} = 0$), the rate of change of circulation intensity (as measured by $\dot{N}$) is controlled only by the rate of entropy production, and necessary conditions for an increase or maintenance of intensity in certain physically-realistic systems can be formulated solely in terms of the average temperatures in regions of heating and cooling (Theorem 3.3).

Finally, suppose the sum of the convergences of radiation and heat fluxes vanishes somewhere in the domain of a geophysical fluid system. Then, whether in energetic or complete isolation or neither, the system must contain accelerations or motions at a given instant if its rate of change of entropic energy is non-zero at that instant (Theorem 3.1). Moreover, the system must be in motion on part of an interval of time if its entropic energy is constantly changing on that interval (Theorem 3.2).

1.0. Introduction

Convection in geophysical fluid systems evolves through complex interactions between the mass, thermal, and motion fields that are governed by both the laws of mechanics and thermodynamics. The separation of the thermodynamic and mechanistic aspects of fluid behavior is often valuable, but only through a simultaneous consideration of all the laws consistent with experience can understanding of the origins and trends of fluid convection be obtained.

The classical approach to a unified comprehension of these laws is the synthesis of the equations of conservation of momentum, mass, and thermodynamic energy into the law of conservation of energy. While this view of the global, or integral, nature of fluid motion has provided satisfactory answers to some of the questions concerning fluid circulation, it has

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failed in two related ways. First, it contains a decided bias towards mechanical controls of convection, because the concepts of entropy and its governing law, the second law of thermodynamics, have not been explicitly included in its construction. Second, energy theories only delineate what energy balances are necessary in fluid systems, and provide no information about probable or necessary modes or patterns of response. The first deficiency of these theories may account for the second, because it is generally believed that entropy, not energy, plays the dominant role in the evolutionary control of thermally-driven fluid systems.

Thus it seems not only natural, but necessary, to incorporate the concept of entropy and the second law of thermodynamics into any rational theory of global[1] convection by finding an exact law relating entropy S and kinetic energy K, the two most important integral quantities in thermodynamics and mechanics respectively.

An important step in the incorporation of entropic principles into dynamical theory was taken by Coleman & Greenberg (1967) when they established the dynamical consequences of Gibbs' concept of stability under isolation. Gibbs (1873, 1875) asserted that a body in equilibrium would be stable in a thermostatic sense if and only if its entropy exceeded that of any other static state of the body compatible with isolation. Coleman & Greenberg applied this concept to thermodynamic systems in which "regular" fluid bodies are isolated from all external influences so that their volume and the sum of internal and kinetic energies are constant. They showed that there exists a unique, uniform static state of the isolated fluid body that is thermodynamically stable in the Gibbs sense and that the deviations from this state are bounded (in an $\mathcal{L}_1$-norm) for all time $t \geqslant \tau$, by the deviations at time τ. The degree of isolation imposed on the fluid bodies renders the theory too restrictive for direct application to geophysical problems; specifically, external body forces were not present and volume, rather than mass, was conserved. Furthermore, the difference between the total entropy of the Gibbs stable state and other natural states of the isolated fluid body was not explicitly determined.

The application of the concept of Gibbs stability to a more complicated physical system was discussed recently by Dutton (1973). Stability theorems were developed for a planetary atmosphere consisting of an ideal gas, isolated in the sense that mass and the sum of kinetic, internal, and potential energies are conserved and in the sense that the second law implies that the total entropy is non-decreasing. Moreover, an exact expression for the difference in entropies between a natural state and the Gibbs stable state was derived, thereby producing a new energy form called "entropic energy". The rate of change of this integral provided a new view of the significance of entropy to the evolution and maintenance of the global atmospheric circulation.

Before undertaking the task of explicitly relating S and K, it must be noted that the integral of entropy by itself has little significance, whereas its deviation $S_0 - S$ from the entropy S_0 of some physically-meaningful equilibrium state is constrained by the second law and may lead to the desired integral relationship. Thus this article will begin with the definition of such a state, realizable in geophysical fluid systems in general, and continue with the determination of its properties. These physical properties will then be exploited in the formulation of the explicit relationship between $S_0 - S$ and K.

It seems clear from the works described above that the appropriate equilibrium state is one whose entropy exceeds that of any other natural state with the same total mass and the sum of kinetic, potential, and internal energies —a Gibbs stable state. Coleman & Greenberg showed that Gibbs' concepts are applicable to thermodynamics and Dutton's development suggests exploring whether exact formulations of these concepts are possible for complex geophysical fluid systems other than ideal gas planetary atmospheres.

What is proposed here is the extension of Coleman & Greenberg's theory to fluid bodies that are not completely isolated from an external environment while retaining their generality, and to use Dutton's techniques to obtain more explicit results. This task entails more than the application and combination of these two works, however. First of all, in order

[1] "Global" is used throughout as an antonym to "local".

to construct a mathematical abstraction of geophysical fluid systems that is relevant to the realistic description of such diversified situations as the atmosphere and laboratory experiments, more complications must be built into the abstract model than were necessary for Coleman & Greenberg's studies. Provisions for an external body force, a variety of spatial configurations, and some freedom of thermodynamic interaction with the external universe must be made. Care must be taken to identify and exclude those systems for which the theory is inapplicable; for example, it has not yet been possible to include systems with truly mobile free surfaces.

A more difficult problem than the specification and treatment of fluid boundaries and external interactions is the inclusion of fluids with a variety of equations of state. Atmospheric scientists have long realized the simplifications inherent in the assumption of an ideal gas equation of state. The study of ideal gas systems does, however, provide clues to the behavior of other fluids and suggests possible approaches to the analysis of this behavior. This in effect is the contribution of the earlier analysis of an ideal gas to this work, even though the model of an ideal gas atmosphere obscures by its very simplicity the fundamental properties that made the theory possible. Thus the generalization here must both isolate the crucial fluid properties while establishing their compatibility with geophysical systems in general, and then rigorously bridge the analytical gaps.

With the successful resolution of these difficulties and the completion of the development in the next chapter, a new global theory with a decided thermodynamic flavor becomes available for the study of a broad class of geophysical fluid systems. Some implications of the theory to fluid convection will be examined in Chapter 3. The development throughout will be as general as possible so that it may embrace many geophysical fluids of interest in a variety of spatial configurations and interacting in a number of ways with their external environments. It is hoped that a view of global convection can be developed that will uniformly apply not only to the atmosphere and the oceans, but also to laboratory experiments, and that the latter will take on an added significance and relationship to the former within the framework of the theory.

2.0. The theory of entropic energy and the global stability of fluid systems

The development of the entropic theory of convection begins with the determination and exploitation of the Gibbs stable state. Such a configuration will be found, and will be shown to be uniquely determined by the mass and total energy of the fluid system under certain conditions. A relationship between global entropy and global kinetic energy then follows immediately from the formation of the difference $S_0 - S$, and from this equation a stability theorem for fluid systems under specified conditions is obtained.

Attention is given first to fluids free from variations in chemical composition. The class of fluids and the physical environments for which the succeeding theorems apply will now be stated explicitly.

2.1. *Fluid systems*

A *fluid body*[1] is defined to be a smooth manifold of material points $\mathbf{X}$ imbedded in a fixed three-dimensional Euclidean space (x_1, x_2, x_3). A *thermodynamic process* in a fluid body is a collection of eight time-dependent fields over the body that are compatible with the law of balance (conservation) of momentum and the law of balance of energy. The eight fields are:

1. the *velocity* $\mathbf{v} = \mathbf{v}(\mathbf{X}, t)$,
2. the symmetric Cauchy *stress tensor* $\mathbf{T} = \mathbf{T}(\mathbf{X}, t) = \{T_{ij}(\mathbf{X}, t)\}$,
3. the specific *body force* $\mathbf{b} = \mathbf{b}(\mathbf{X}, t)$ exerted on the fluid body at $\mathbf{X}$ by other bodies which do not intersect the fluid body,
4. the specific *internal energy* $e = e(\mathbf{X}, t)$,
5. the *heat flux* vector $\mathbf{q} = \mathbf{q}(\mathbf{X}, t)$,
6. the *net radiation flux* vector $\mathbf{R} = \mathbf{R}(\mathbf{X}, t)$,
7. the specific *entropy* $s = s(\mathbf{X}, t)$,
8. the *absolute temperature* $T = T(\mathbf{X}, t) > 0$.

The *laws* of *balance of momentum* and *balance of energy* are

$$\frac{d}{dt}\int \varrho \mathbf{v} dV = \int \varrho \mathbf{b} dV + \int \mathbf{n}\mathbf{T} d\sigma \qquad (1)$$

[1] The thermodynamics of linearly viscous fluids presented in this section follows Coleman & Greenberg (1967).

$$\frac{d}{dt}\int \varrho(e+v^2/2)\,dV = \int \varrho \mathbf{v}\cdot\mathbf{b}\,dV + \int [\mathbf{v}\cdot\mathbf{T}\mathbf{n} - \mathbf{n}\cdot(\mathbf{q}+\mathbf{R})]\,d\sigma \tag{2}$$

where $v = |\mathbf{v}|$, ϱ is the density at a point $\mathbf{X}$, $\mathbf{n}$ is the exterior unit normal at the boundary, dV an element of volume, and $d\sigma$ an element of area.

Constitutive assumptions which restrict the processes that are admissible in a fluid body define a *fluid material*. *Response functions* specify the relationships between the time-dependent fields and are denoted by an overbar. A material is *materially homogeneous* if its response functions do not vary with $\mathbf{X}$.

The constitutive assumptions themselves are restricted by the *second law of thermodynamics*: For every admissible process in a body consisting of a given material, the *Clausius-Duhem inequality*

$$\frac{d}{dt}\int \varrho s\,dV \geqslant -\int \frac{\nabla\cdot(\mathbf{q}+\mathbf{R})}{T}\,dV \tag{3}$$

must hold at all times t and all parts of the body.

One fluid material will be considered here. A *linearly viscous fluid* is defined by the following constitutive equations that hold at each material point $\mathbf{X}$ and at all times t,

$$\begin{aligned} &s = \bar{s}(\alpha, e),\quad T = \bar{T}(\alpha, e)\\ &T_{ij} = -p\delta_{ij} + \Delta_{ij},\quad p = \bar{p}(\alpha, e),\\ &\Delta_{ij} = 2\lambda_1 e_{ij} + \lambda_2 e_{kk}\delta_{ij}\\ &\lambda_1 = \bar{\lambda}_1(\alpha, e),\quad \lambda_2 = \bar{\lambda}_2(\alpha, e)\\ &\mathbf{q} = -\varkappa\nabla T,\quad \varkappa = \bar{\varkappa}(\alpha, e) \end{aligned} \tag{4}$$

where α is the specific volume, p the thermodynamic pressure, e_{ij} the rate of strain tensor, and $\varkappa$ the coefficient of heat conduction.

These constitutive equations are compatible with the second law if and only if

$$T = [\bar{s}_e(\alpha, e)]^{-1},\quad p = T[\bar{s}_\alpha(\alpha, e)] \tag{5}$$

$$\bar{\varkappa}(\alpha, e) \geqslant 0,\ \bar{\lambda}_1(\alpha, e) \geqslant 0,\ \bar{\lambda}_2(\alpha, e) + \tfrac{2}{3}\bar{\lambda}_1(\alpha, e) \geqslant 0 \tag{6}$$

for all α and e. The subscripts denote partial differentiation.

It is assumed that the response functions defining the material are sufficiently smooth with respect to the independent thermodynamic variables to permit the arguments below, as in the applications of the implicit function theorem in Section 2.2. To simplify notation the overbars denoting response functions will be omitted henceforth.

Before proceeding, it is necessary to review some properties of concave entropy functions. The function s is strictly concave if and only if for every pair (α_2, e_2) distinct from (α_1, e_1) it is true that

$$s(\alpha_2, e_2) < s(\alpha_1, e_1) + (\alpha_2 - \alpha_1)s_\alpha(\alpha_1, e_1) + (e_2 - e_1)s_e(\alpha_1, e_1) \tag{7}$$

Furthermore, if s is strictly concave for a linearly viscous fluid then it can be shown that

$$T_e > 0 \tag{8}$$

Definition 2.1.1. *Linearly viscous fluids whose entropy functions are strictly concave are referred to here as* equilibrium regular fluids.

Equilibrium regular fluid bodies in which the body force is a consequence of a steady, positive, monotonic potential field,

$$\mathbf{b} = -\nabla\Phi,\ \Phi(\mathbf{X}) > 0,\ \nabla\Phi \neq 0 \tag{9}$$

will be considered in the sequel.

Definition 2.1.2. *A* fluid system *consists of an equilibrium regular fluid of fixed mass M in which* (9) *holds and either*

1. *the fluid is bounded by fixed solid walls, on which $\mathbf{n}\cdot\mathbf{v} = 0$, and a condition on the tangential velocity is specified, or*
2. *the fluid is bounded by fixed solid walls except in a direction normal to surfaces of constant Φ towards increasing Φ, the integrals of mass, entropy, and kinetic, potential and internal energies are finite, and $\nabla\Phi\cdot p\mathbf{v} \to 0$ as $\Phi \to \infty$.*

Conditions can be placed on a fluid system and its interactions with the external environment to ensure that entropy is non-decreasing and that the sum of kinetic, internal, and potential energies is conserved.

Definition 2.1.3. *A system is in* isolation *if*

$$\int [\mathbf{v}\cdot\mathbf{T}\mathbf{n} - \mathbf{n}\cdot(\mathbf{q}+\mathbf{R})]\,d\sigma = 0 \tag{10}$$

and

$$\int \left[\frac{\nabla \cdot \mathbf{R}}{T} + \nabla \cdot \left(\frac{\mathbf{q}}{T} \right) \right] dV = 0 \tag{11}$$

A system is in energetic isolation *if only holds.* (10)

With the definitions

$$\textit{system internal energy, } I = \int \varrho e dV, \tag{12}$$

$$\textit{system potential energy, } P = \int \varrho \Phi dV, \tag{13}$$

$$\textit{system kinetic energy, } K = \int \varrho \frac{v^2}{2} dV, \tag{14}$$

$$\textit{total system energy, } E = I + P + K \tag{15}$$

$$\textit{system entropy, } S = \int \varrho s dV, \tag{16}$$

$$\textit{system mass, } M = \int \varrho dV, \tag{17}$$

a system in isolation has constant E and non-decreasing S with time, for from (2), (3), and (4) it follows that in isolation we have the global statements

$$\frac{d}{dt} \int \varrho (e + v^2/2 + \Phi) dV = 0, \tag{18}$$

$$\frac{d}{dt} \int \varrho s dV \geqslant \int \frac{1}{\varkappa} \left(\frac{|\mathbf{q}|}{T} \right)^2 dV \geqslant 0. \tag{19}$$

Definitions 2.1.1–2.1.3 define the fluids and the environments for which the theorems in Section 2.2 are valid. To conclude the present section, the meaning of Gibbs' concept of stability will be stated.

Definition 2.1.4. *A configuration of a fluid system is* thermodynamically stable in Gibbs sense *if every other configuration of the system with the same system mass M and total system energy E has smaller system entropy S.*

2.2. *Entropic energy and the dynamical stability of systems in isolation*

The Gibbs stable state of fluid systems may be calculated explicitly and then used to develop a relationship between system entropy and system kinetic energy and to prove global stability theorems for systems in isolation. Specifically, the difference $S_0 - S$, where the subscript zero refers to an entropy maximizing configuration of the system, will be determined and shown to be strictly positive for any deviation of the fluid from the maximizing state.

First, necessary conditions for a unique entropy maximizing configuration will be determined by variational techniques used by Gibbs (1875) and (in a recent version) by Dutton (1973). To begin, the functional

$$S = \int \varrho s dV + \gamma_1 \left(M - \int \varrho dV \right) + \gamma_2 \left(E - \int \varrho [e + \Phi + v^2/2] dV \right) \tag{1}$$

is formed where γ_1 and γ_2 are Lagrange multipliers. Necessary conditions are sought for maximizing the functional under the constraints that M and E are constant.

With the substitutions

$$\varrho = \varrho_0 + \varepsilon \delta \varrho, \; e = e_0 + \varepsilon \delta e, \; \mathbf{v} = \mathbf{v}_0 + \varepsilon \delta \mathbf{v} \tag{2}$$

where the subscript zero denotes the maximizing state, the functional gives the Euler equations,

$$\delta v: \; -\gamma_2 \varrho_0 v_0 = 0 \tag{3}$$

$$\delta e: \frac{\varrho_0}{T(\alpha_0, e_0)} - \gamma_2 \varrho_0 = 0 \tag{4}$$

and

$$\delta \varrho: s(\alpha_0, e_0) - \frac{p(\alpha_0, e_0)}{\varrho_0 T(\alpha_0, e_0)} - \gamma_1 - \gamma_2 [e_0 + \Phi + v_0^2/2] = 0 \tag{5}$$

Eqs. (3) and (4) demand that the maximizing state be motionless and isothermal,

$$\mathbf{v}_0 = 0, \; T_0 = \gamma_2^{-1} \tag{6}$$

where the subscript zero will now be appended to response functions to indicate their value in the maximizing state. Substitution of (6) into (5) gives

$$\gamma_1 = s_0 - \frac{1}{T_0} [p_0 \alpha_0 + e_0 + \Phi] \tag{7}$$

which, by application of the gradient operator to both sides of (7) and substitution of (2.1.5), can be shown to be equivalent to the hydrostatic condition

$$-\alpha_0 \nabla p_0 = \nabla \Phi \tag{8}$$

Thus the maximizing state must be motionless, isothermal, and hydrostatic. This state will be shown to be Gibbs stable in Theorem 2.2.

The next task is to find the conditions on a fluid system with total system energy E and system mass M that ensure that the maximizing state is uniquely determined under the constraints

$$M_0 = M \tag{9}$$

$$E_0 = I_0 + P_0 = E = I + P + K \tag{10}$$

Eq. (10) follows from (6).

An orthogonal coordinate system (x_1', x_2', η) is defined such that

$$\mathbf{i}_3' = \frac{\nabla \Phi}{|\nabla \Phi|} \tag{11}$$

By definition $\Phi = \Phi(\eta)$, and therefore (8) becomes

$$-\alpha_0 \frac{\partial p_0}{\partial x_i'} = \frac{\partial \Phi}{\partial x_i'} = \frac{\partial \Phi}{\partial \eta} \delta_{i3} \tag{12}$$

giving $p_0 = p_0(\eta)$.

With $T_0 = T(\alpha_0, e_0)$, the function T^* is defined for T_0 constant by

$$T^* = T(\alpha, e) - T_0 \tag{13}$$

Therefore by the implicit function theorem and the fact that s is strictly concave ($T_e > 0$), there exists a function f_1 such that

$$e_0 = f_1(\alpha_0; T_0) \tag{14}$$

It then follows that

$$p_0 = p(\alpha_0, e_0) = p(\alpha_0, f_1(\alpha_0; T_0)) = f_2(\alpha_0; T_0) \tag{15}$$

so that

$$-\alpha_0 \frac{\partial p_0}{\partial x_i'} = -\alpha_0 \frac{\partial f_2}{\partial \alpha_0} \frac{\partial \alpha_0}{\partial x_i'} = \frac{\partial \Phi}{\partial \eta} \delta_{i3} \tag{16}$$

and thus $\alpha_0 = \alpha_0(\eta)$.

Eq. (8) now reduces to

$$\frac{\partial \Phi}{\partial \eta} = f_3(\alpha_0; T_0) \frac{\partial \alpha_0}{\partial \eta} \left(f_3(\alpha_0; T_0) = -\alpha_0 \frac{\partial f_2}{\partial \alpha_0} \right) \tag{17}$$

a first-order, nonlinear and separable, ordinary differential equation in α_0 and η (T_0 is treated as a parameter). With α^* the value $\alpha_0(\eta^*)$ specified at some η^* in the fluid domain, a unique solution

$$\alpha_0 = f_4(\eta; \alpha^*, T_0) \tag{18}$$

exists satisfying $\alpha^* = f_4(\eta^*)$.

Eqs. (9) and (10) now become

$$M = \int \frac{1}{\alpha_0} dV = \int \frac{1}{f_4} dV' = \mathcal{M}(\alpha^*, T_0) \tag{19}$$

$$E = \int \frac{1}{\alpha_0} [e_0 + \Phi] dV = \int \frac{1}{f_4} [f_1(f_4; T_0) + \Phi] dV' = \mathcal{E}(\alpha^*, T_0) \tag{20}$$

where

$$dV' = \left| \frac{\partial(x_1, x_2, x_3)}{\partial(x_1', x_2', \eta)} \right| d\eta dx_1' dx_2' \tag{21}$$

Theorem 2.1. *The entropy maximizing configuration of a fluid system that satisfies* (6) *and* (7) *is uniquely determined by the values of the system mass* M *and total system energy* E *if and only if*

$$\left| \frac{\partial(\mathcal{M}, \mathcal{E})}{\partial(x, y)} \right|_{\substack{x=\alpha^* \\ y=T_0}} \neq 0 \tag{22}$$

Proof. Forming the functions

$$M^* = \mathcal{M}(x, y) - M$$

$$E^* = \mathcal{E}(x, y) - E \tag{23}$$

and applying the implicit function theorem gives

$$\alpha^* = h_1(M, E)$$

$$T_0 = h_2(M, E) \tag{24}$$

if and only if

$$\left| \frac{\partial(M^*, E^*)}{\partial(x, y)} \right|_{\substack{x=\alpha^* \\ y=T_0}} \neq 0 \tag{25}$$

Because M and E are fixed in (23), (25) reduces to (22).

The usual model atmosphere forms a fluid system that satisfies (22) and was studied by Dutton (1973). Such an atmosphere consists of an ideal gas confined to a planetary body by a gravitational field,

$$\Phi = gz \tag{26}$$

in which g is a constant (by assumption) gravitational acceleration and z is the height above the surface of the planetary body. For convenience $\eta^* = z^*$ is taken to be zero. The mass and energy as functions of α^* and T_0 are

$$M = \mathcal{M}(\alpha^*, T_0) = \frac{RAT_0}{g\alpha^*}$$

$$E = \mathcal{E}(\alpha^*, T_0) = \frac{c_p RAT_0^2}{g\alpha^*} = c_p \mathcal{M}(\alpha^*, T_0) T_0 \tag{27}$$

where R is the ideal gas constant for the mixture making up the atmosphere, c_p is the specific heat at constant pressure, and A is the surface area of the planetary body.

With $\mathcal{M}$ and $\mathcal{E}$ as functions of the independent variables α^* and T_0, the Jacobian in (22) becomes

$$\left|\frac{\partial(\mathcal{M}, \mathcal{E})}{\partial(x, y)}\right|_{\substack{x=\alpha^* \\ y=T_0}} = \frac{c_p}{\alpha^*}[\mathcal{M}(\alpha^*, T_0)]^2 \tag{28}$$

which is non-zero for all finite values of $\alpha^* > 0$ and $T_0 > 0$.

If a fluid system does satisfy the condition (22), as in the example above, then the quantities $\mathcal{M}$ and $\mathcal{E}$ are independent functions of α^* and T_0. One material to which the present theory cannot be applied and for which (22) cannot be satisfied is a Boussinesq fluid, whose equation of state specifies temperature as a linear function of specific volume only. This fluid, and others with degenerate equations of state, do not realistically represent the physics of geophysical fluids, except as approximations in the analysis of limited situations where some thermodynamic phenomena play minor roles. In most cases, use of a more accurate equation of state will eliminate the difficulty.

The only degenerate systems that need specific mention are those that do not satisfy (22) and those where pressure is independent of internal energy, a situation that proves detrimental to the study of convective origins in Chapter 3.

Definition 2.2.1. *A* geophysical fluid system *is a fluid system in which* (22) *and* $p_e \neq 0$ *are both satisfied.*

From Theorem 2.1 it is clear that in all geophysical fluid systems M and E uniquely determine a motionless, hydrostatic, and isothermal state.

Definition 2.2.2. *The unique configuration of a geophysical fluid system that satisfies* (6) *and* (7) *is called the* associated equilibrium state.

Necessary conditions for a fluid system to have a unique entropy maximizing configuration are embodied in (6) and (7) and (22), but these conditions do not yet ensure that the associated equilibrium state is in fact thermodynamically stable in the Gibbs sense. The next theorem demonstrates this property of the associated equilibrium state through representation and analysis of the entropy difference, $S_0 - S$.

Theorem 2.2. *In a geophysical fluid system*

$$N = T_0(S_0 - S) = K + T_0\Sigma \tag{29}$$

where

$$\Sigma = -\int \varrho\left[(s - s_0) - (\alpha - \alpha_0)\frac{p_0}{T_0} - (e - e_0)\frac{1}{T_0}\right] dV \tag{30}$$

Furthermore the associated equilibrium state is thermodynamically stable in Gibbs sense.

Proof. The proof begins with a decomposition used by Dutton (1973),

$$S_0 - S = \int (\varrho_0 s_0 - \varrho s)\, dV = \int \varrho(s_0 - s)\, dV + \int s_0(\varrho_0 - \varrho)\, dV. \tag{31}$$

Substitution of (7) in the second integral on the right gives

$$\int s_0(\varrho_0 - \varrho)\, dV = \int \left[\gamma_1 + \frac{1}{T_0}(p_0\alpha_0 + e_0 + \Phi)\right] \times (\varrho_0 - \varrho)\, dV \tag{32}$$

and then application of the definitions in Section 2.1 and associated equilibrium state properties yields

$$\int s_0(\varrho_0-\varrho)\,dV = \int \varrho\left[(\alpha-\alpha_0)\frac{p_0}{T_0}-\frac{e_0}{T_0}\right]dV + \frac{1}{T_0}(I_0+P_0-P). \tag{33}$$

Adding and subtracting I/T_0 on the right of (33) and application of (10) produces

$$\int s_0(\varrho_0-\varrho)\,dV = \int \varrho\left[(\alpha-\alpha_0)\frac{p_0}{T_0} + (e-e_0)\frac{1}{T_0}\right]dV + \frac{K}{T_0} \tag{34}$$

Substitution of (34) into (31) and multiplication by T_0 completes the proof of the first part of the theorem.

With the use of (2.1.5) Equation (30) becomes

$$\Sigma = -\int \varrho[(s(\alpha,e)-s(\alpha_0,e_0))-(\alpha-\alpha_0)s_\alpha(\alpha_0,e_0) - (e-e_0)s_e(\alpha_0,e_0)]\,dV \tag{35}$$

The inequality (2.1.7) applied to the integrand of (35) shows that the expression is negative for any deviation of the fluid system from the associated equilibrium state. Therefore Σ is positive for any deviation from the associated equilibrium state and the assertion of Gibbs stability is proven.

Definition 2.2.3. *The energy form* $N = K + T_0\Sigma$ *is called the* entropic energy.

Application of Taylor's formula with remainder to the integrand of (35) gives the alternate form

$$\Sigma = -\frac{1}{2}\int \varrho[(\alpha-\alpha_0)^2 s_{\alpha\alpha}(\hat{\alpha},\hat{e}) + 2(\alpha-\alpha_0)(e-e_0)s_{\alpha e}(\hat{\alpha},\hat{e}) + (e-e_0)^2 s_{ee}(\hat{\alpha},\hat{e})]\,dV$$
$$(\hat{\alpha}=\alpha_0+\hat{r}(\alpha-\alpha_0), \quad \hat{e}=e_0+\hat{r}(e-e_0), \quad 0<\hat{r}<1) \tag{36}$$

The cross-product term in (36) can be eliminated through replacement of $e-e_0$ by means of the variable

$$\theta = s_{ee}(\hat{\alpha},\hat{e})(e-e_0)+s_{\alpha e}(\hat{\alpha},\hat{e})(\alpha-\alpha_0) \tag{37}$$

which vanishes when both $\alpha=\alpha_0$ and $e=e_0$. This operation yields

$$\Sigma = -\frac{1}{2}\int \varrho\left[\frac{\theta^2}{s_{ee}(\hat{\alpha},\hat{e})} + \left(s_{\alpha\alpha}(\hat{\alpha},\hat{e})-\frac{s_{\alpha e}^2(\hat{\alpha},\hat{e})}{s_{ee}(\hat{\alpha},\hat{e})}\right)(\alpha-\alpha_0)^2\right]dV \tag{38}$$

Substitution of (38) into (29) produces

$$N = \frac{1}{2}\int \varrho\left\{v^2 - T_0\left[\frac{\theta^2}{s_{ee}(\hat{\alpha},\hat{e})} + \left(s_{\alpha\alpha}(\hat{\alpha},\hat{e})-\frac{s_{\alpha e}^2(\hat{\alpha},\hat{e})}{s_{ee}(\hat{\alpha},\hat{e})}\right)(\alpha-\alpha_0)^2\right]\right\}dV \tag{39}$$

That the factors involving second derivatives of s are negative is equivalent to the concavity of s. Thus entropic energy is a measure, which resembles an $\mathcal{L}_2$-norm, of the closeness of a geophysical fluid system to its associated equilibrium state. Consequently, closeness of T_0S to T_0S_0 implies closeness, in the sense of (39), of the fluid system to its associated equilibrium state.

To establish dynamical stability of fluid systems in isolation it only remains to be shown that N is bounded by its value at the time isolation is imposed and that it is non-increasing for all succeeding times.

Theorem 2.3. *Let a geophysical fluid system be in isolation for all time* $t \geqslant \tau$. *Then the deviations from the associated equilibrium state are for all time* $t \geqslant \tau$ *restricted by*

$$N(t) = K(t) + T_0\Sigma(t) \leqslant T_0[S_0 - S(\tau)] \tag{40}$$

Furthermore the entropic energy is nonincreasing for all time $t \geqslant \tau$.

Proof. The first part of the theorem follows from the fact that S is non-decreasing and M and E are constant in a fluid system in isolation so that S_0 and T_0 are constant for all time $t \geqslant \tau$. Therefore

$$T_0[S_0 - S(\tau)] \geqslant T_0[S_0 - S(t)] = K(t) + T_0\Sigma(t) = N(t) \tag{41}$$

Differentiation of (29) with respect to time gives, under the present conditions, the result

$$\dot{N}(t) = \dot{K}(t) + T_0\dot{\Sigma}(t) = -T_0\dot{S}(t) \leqslant 0 \tag{42}$$

because $\dot{S}(t) \geqslant 0$ for a system in isolation, proving the second part of the theorem.

These results show that the exact relationship between system entropy and system kinetic energy embodied in (29) is crucial to this work. First of all, it is the key to the simple proof of Theorem 2.3, which asserts that geophysical fluid systems in isolation are globally stable and that their future global behavior is determined and known at the onset of permanent isolation. But of equal significance is the insight (29) provides to convection in systems not in isolation. One immediate consequence of (29) is that destruction of entropy implies the production of entropic energy in fluid systems that are energetically isolated, a fact utilized later in Theorem 3.3, in which the maintenance and intensification of the global circulation for such systems is linked to the detailed arrangement of entropy production and destruction processes.

2.3. *A geophysical fluid system—an ideal liquid in a tank*

This section presents an example of the application of the theory to a geophysical fluid system other than a model planetary atmosphere. The fluid system to be considered is an ideal liquid, in a tank with a rigid contact lid, under the influence of a linear gravitational potential field. The tank may be rectangular, as in Bénard convection experiments, or cylindrical or annular, as in rotating convection experiments.

The spatial distribution of the fluid can be conveniently described by either a Cartesian or a polar coordinate system, stationary or rotating, whose origin is at the center of the tank bottom and whose vertical direction is toward increasing linear gravitational potential. With the coordinates (x_1', x_2', z), the potential is written

$$\Phi = gz \tag{1}$$

where g is a constant gravitational acceleration. Tanks with vertical sidewalls (constant cross-sectional area A), and with their tops at $z = z_T$ and bottoms at $z = 0$, are considered here.

The fluid itself is a linearly viscous fluid whose equations of state are those given by Dutton & Fichtl (1969) for an ideal liquid,

$$\alpha = \alpha_1[1 + \varepsilon_1(T - T_1) - \varepsilon_2(p - p_1)] \tag{2}$$

$$e = e_1 + c_p(T - T_1) + p_1(\alpha_1 - \alpha) + \frac{\alpha_1\varepsilon_2}{2}(p - p_1)^2 - \alpha_1\varepsilon_1(p - p_1)T \tag{3}$$

$$s = s_1 + c_p \ln\left(\frac{T}{T_1}\right) - \alpha_1\varepsilon_1(p - p_1) \tag{4}$$

$$\varepsilon_1 = \left[\frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial T}\right)_p\right]_1 > 0, \quad \varepsilon_2 = -\left[\frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial p}\right)_T\right]_1 > 0 \tag{5}$$

$$c_V = c_p(p_1, T_1) - \alpha_1\varepsilon_1^2 T/\varepsilon_2 > 0 \tag{6}$$

where α_1, e_1, p_1, T_1, s_1 are constant reference values of the thermodynamic variables, c_p and c_V are the specific heats at constant pressure and volume respectively, and ε_1, ε_2, and c_p are constants for a given reference state. Eqs. (2)–(6) apply only if the deviations of the thermodynamic variables from the reference state are small and temperature always satisfies (6). For water, we have the values $T_1 \sim 3 \times 10^2$°K, $p_1 \sim 10^6$ dynes cm^{-2}, $\alpha_1 \sim 1$ cm^3 g^{-1}, $c_p \sim 4.2 \times 10^7$ erg g^{-1}, $\varepsilon_1 \sim 2.5 \times 10^{-4}$(°K)$^{-1}$, $\varepsilon_2 \sim 5 \times 10^{-11}$ cm^2 dynes^{-1}, and $c_p\varepsilon_2/(\varepsilon_1\alpha_1) \sim 10$. With these estimates, (6) is satisfied if $T < 4 \times 10^4$°K, which is certainly true if $|T - T_1|/T_1$ is always small.

The ideal liquid has not yet been shown to be the material constituent of a fluid system according to Definition 2.1.2, for Definition 2.1.1 requires that an equilibrium regular fluid have a strictly concave entropy function. This is true for $T = \bar{T}(\alpha, e)$ and $p = \bar{p}(\alpha, e)$ if and only if $T_e > 0$ and

$$pT_\alpha - p_\alpha T > \frac{T_\alpha^2}{T_e} > 0 \tag{7}$$

Implicit differentiation of (2) with respect to e and α separately produces

$$p_e = \frac{\varepsilon_1}{\varepsilon_2}T_e \tag{8}$$

and

$$p_\alpha = \frac{1}{\varepsilon_2}\left(\varepsilon_1 T_\alpha - \frac{1}{\alpha_1}\right) \tag{9}$$

Likewise implicit differentiation of (3) with respect to e and α, and substitution of (8) and (9) into the appropriate expression yield

$$T_e = \frac{\varepsilon_2}{\varepsilon_2 c_p - \alpha_1 \varepsilon_1^2 T} \tag{10}$$

and

$$T_\alpha = \frac{\varepsilon_2 p - \varepsilon_1 T}{\varepsilon_2 c_p - \alpha_1 \varepsilon_1^2 T} \tag{11}$$

Replacement of the denominator in (10) by $\varepsilon_2 c_V$ and utilization of the inequality (6) reveal that

$$T_e = \frac{1}{c_V} > 0 \tag{12}$$

Formation of the left side of (7) with the aid of (11) and (9), then manipulation of the resulting expressions and subsequent substitution of (10) and (11), produce by virtue of (12) the result

$$pT_\alpha - p_\alpha T = \frac{T_\alpha^2}{T_e} + \frac{T}{\varepsilon_2 \alpha_1} > \frac{T_\alpha^2}{T_e} > 0 \tag{13}$$

Thus an ideal liquid is an equilibrium regular fluid, and consequently the fluid contained in a tank with a contact lid comprises a fluid system according to Definition 2.1.2.

It is now permissible to proceed to the task of determining the relevant parameters, α^* and T_0, of the associated equilibrium state. After differentiation of (2) with respect to z, the differential equation describing the variation of α_0 with height is obtained through use of the properties $(2.2.6)_2$ and (2.2.8) of the associated equilibrium state,

$$\alpha_0 \frac{\partial \alpha_0}{\partial z} = g \alpha_1 \varepsilon_2 \tag{14}$$

Integration from zero to z gives

$$\alpha_0(z) = (2g\varepsilon_2 \alpha_1 z + \alpha^{*2})^{1/2} \tag{15}$$

where $\alpha_0(0) = \alpha^*$.

Eq. (2.2.19) can be written for this system as

$$M = A \int_0^{z_T} (2g\varepsilon_2 \alpha_1 z + \alpha^{*2})^{-1/2} dz$$

$$= \frac{A}{g\varepsilon_2 \alpha_1} (\alpha_0(z_T) - \alpha^*) = \mathcal{M}(\alpha^*) \tag{16}$$

where $\alpha_0(z_T) = (2g\varepsilon_2 \alpha_1 z_T + \alpha^{*2})^{1/2}$. Combination of (15) and (16) now give

$$\alpha^* = \frac{z_T A}{M} - \frac{g\varepsilon_2 \alpha_1 M}{2A} \tag{17}$$

so that α^* is completely determined by system constants and is not dependent on E.

The derivation of an expression for T_0 is somewhat more complicated, and most of the details will not be given. To begin, (2.2.20) is rewritten for this system as,

$$E = A \int_0^{z_T} \frac{1}{\alpha_0} (e_0 + gz) dz \tag{18}$$

where α_0 is given by (15) and (17) and $e_0(z; \alpha^*, T_0)$ is given by the elimination of p_0 from (3) with the use of (2) and substitution of $\alpha_0(z; \alpha^*)$. After integration and extensive manipulation of the result, (18) becomes

$$E = M[e_\iota + C(M) + f(T_0)] = \mathcal{E}(\alpha^*, T_0),$$

$$C(M) = \frac{1}{\varepsilon_2} \left(\frac{Az_T}{M} - \alpha_1 \right) (\varepsilon_1 T_1 - \varepsilon_2 p_1 - 1) - \frac{\alpha_1}{2\varepsilon_2}$$

$$+ \frac{Az_T}{M\varepsilon_2} \frac{\alpha_0(z_T)}{\alpha_1} - \frac{A}{6gM\varepsilon_2^2 \alpha_1^2} (\alpha_0^3(z_T) - \alpha^{*3}),$$

$$f(T_0) = -\frac{\alpha_1 \varepsilon_1^2}{2\varepsilon_2} \left[\left(T_0^2 - \frac{2\varepsilon_2 c_p T_0}{\alpha_1 \varepsilon_1^2} \right) \right.$$

$$\left. - \left(T_1^2 - \frac{2\varepsilon_2 c_p T_1}{\alpha_1 \varepsilon_1^2} \right) \right] \tag{19}$$

Before proceeding it should be shown that (19) will uniquely determine T_0 in this system. The fluid and tank make up a geophysical system only if (2.2.22) is satisfied, for (8) and (12) ensure that $p_e \neq 0$. Because $\partial \mathcal{M}/\partial T_0 = 0$ it must be shown that

$$\frac{\partial \mathcal{M}}{\partial \alpha^*} \frac{\partial \mathcal{E}}{\partial T_0} \neq 0 \tag{20}$$

Now, from (6) and (19)

$$\frac{\partial \mathcal{E}}{\partial T_0} = -M \frac{\alpha_1 \varepsilon_1^2}{\varepsilon_2} \left(T_0 - \frac{\varepsilon_2 c_p}{\alpha_1 \varepsilon_1^2} \right)$$

$$= \left(c_p - \frac{\alpha_1 \varepsilon_1^2 T_0}{\varepsilon_2} \right) M = c_{V_0} M > 0 \tag{21}$$

and from (16)

$$\frac{\partial \mathcal{M}}{\partial \alpha^*} = \frac{A}{g\varepsilon_2\alpha_1}\left(\frac{\alpha^*}{\alpha_0(z_T)} - 1\right) < 0 \tag{22}$$

Condition (20) is therefore satisfied and (19) must produce a unique T_0 consistent with the physical constraints of the system.

From (19)

$$T_0 = \frac{\varepsilon_2 c_p}{\alpha_1 \varepsilon_1^2}\left\{1 \pm \left[1 - \frac{2\alpha_1\varepsilon_1^2}{\varepsilon_2 c_p^2}\left(\frac{E}{M} - e_1 - C(M) + T_1\left(c_p - \frac{\alpha_1\varepsilon_1^2 T_1}{2\varepsilon_2}\right)\right)\right]^{1/2}\right\} \tag{23}$$

Instead of resorting to complicated analytical techniques to determine which sign is correct in (23), a simple scale analysis of the coefficient outside of the brackets will immediately reveal the answer. Estimates for water applied to a tank with $A = 10^2$ cm^2, $z_T = 10$ cm, and $M = 10^3$ g, give

$$T_0 \sim 40\,000°\text{K}\{1 \pm \sqrt{B}\}, \quad B \sim 1 \tag{24}$$

where B denotes the quantity within the square brackets in (23). Because $|T_0 - T_1|/T_1$ will be small and $T_1 \sim 3 \times 10^2$°K it is obvious from (24) that the minus sign in (23) gives the correct form of T_0. This completes the specification of the associated equilibrium state; the eqs. (15), (17), (23), $(19)_2$, and $(19)_3$ along with (2)–(6) completely describe the distribution of the thermodynamic variables in the equilibrium state.

The form of the quantity $T_0\Sigma$ can be determined by the elimination of p and p_0 from (3) and (4) through the use of (2) for the natural and equilibrium states respectively, and then substituting the results of these operations into (2.2.30). The result of all these manipulations is

$$T_0\Sigma = -\int \varrho\left[c_p T_0 \ln\frac{T}{T_0} - (T - T_0) \times \left(c_p - \frac{\alpha_1\varepsilon_1^2}{2\varepsilon_2}(T - T_0)\right)\right] dV + \frac{1}{2\varepsilon_2\alpha_1}\int \frac{(\alpha - \alpha_0)^2}{\alpha} dV \tag{25}$$

This can be rewritten as

$$T_0\Sigma = -\int \varrho\left[c_p T_0 \ln\frac{T}{T_0} - (T - T_0) \times \left(c_p - \frac{\alpha_1\varepsilon_1^2}{2\varepsilon_2}(T - T_0)\right)\right] dV + \int \varrho g z dV + \frac{1}{2\varepsilon_2\alpha_1}\int \alpha dV + \frac{\alpha^{*2}M}{2\varepsilon_2\alpha_1} - \frac{A}{3g\varepsilon_2^2\alpha_1^2}(\alpha_0^3(z_T) - \alpha^{*3}) \tag{26}$$

The entropic energy can now be readily computed with knowledge of the distribution of $\mathbf{v}$, α, and T in the tank. Computation of M and E allow computation of α^*, $\alpha_0(z_T)$, and T_0 and ultimately $T_0\Sigma$ and N.

2.4. *Simple-solution fluid systems*

The theory of entropic energy and dynamical stability of systems in isolation can be extended to fluids that are made up of two chemical constituents, a solvent and a solute, whose proportional mix can vary in space but whose respective total masses are constant in the system.

This extension makes the theory of entropic energy available to oceanographic studies, and to other studies of the thermodynamics of solutions. With additional sophistication, the ideas presented in this section may find application to the investigation of certain cloud equilibria and other facets of moist-air meteorology.

In the interests of brevity, only new material and modifications of that in Sections 2.1 and 2.2 is presented. Notational conventions and smoothness assumptions remain the same. The thermodynamics of solutions was adapted from Fofonoff (1962) and Eckart (1962).

To the list of time-dependent fields in Section 2.1 the following are added:

9. the *net solute flux vector* $\mathbf{f}_\chi = \mathbf{f}_\chi(\mathbf{X}, t)$,
10. the *chemical potential* $\mu = \mu(\mathbf{X}, t)$.

Eq. (2.1.1) remains the same but the *law of balance of energy* and the *Clausius-Duhem inequality* take the forms

$$\frac{d}{dt}\int \varrho(e + v^2/2)\,dV = \int \varrho \mathbf{v}\cdot\mathbf{b}\,dV + \int [\mathbf{v}\cdot\mathbf{T}\mathbf{n} - \mathbf{n}\cdot(\mathbf{q} + \mathbf{R} + \mu\mathbf{f}_\chi)]\,d\sigma, \tag{1}$$

$$\frac{d}{dt}\int \varrho s\,dV \geqslant -\int \frac{\nabla\cdot(\mathbf{q} + \mathbf{R} + \mu\mathbf{f}_\chi)}{T}\,dV$$

where ϱ is the solution density.

The constitutive equations for a *simple-solution linearly-viscous fluid* are

$$s = \bar{s}(\alpha, e, \chi),\ T = \bar{T}(\alpha, e, \chi),\ \mu = \bar{\mu}(\alpha, e, \chi)$$
$$T_{ij} = -p\delta_{ij} + \Delta_{ij}$$
$$p = \bar{p}(\alpha, e, \chi),\ \Delta_{ij} = 2\lambda_1 e_{ij} + \lambda_2 e_{kk}\delta_{ij}$$
$$\lambda_1 = \bar{\lambda}_1(\alpha, e, \chi),\ \lambda_2 = \bar{\lambda}_2(\alpha, e, \chi)$$
$$\mathbf{q} = -\varkappa\nabla T,\ \varkappa = \bar{\varkappa}(\alpha, e, \chi)$$
$$\mathbf{f}_\chi = -\varkappa_1\nabla\mu,\ \varkappa_1 = \bar{\varkappa}_1(\alpha, e, \chi) \tag{3}$$

where χ is the *salinity* at a point $\mathbf{X}$, the ratio of the mass of solute to the mass of solution, and $\varkappa_1$ is the coefficient of solute diffusion.

The constitutive assumptions (3) are compatible with the second law if and only if

$$T = \bar{s}_e^{-1},\ p = \bar{T}s_\alpha,\ \mu = -\bar{T}\bar{s}_\chi \tag{4}$$

$$\varkappa \geqslant 0,\ \varkappa_1 \geqslant 0,\ \lambda_1 \geqslant 0,\ \lambda_2 + \tfrac{2}{3}\lambda_1 \geqslant 0 \tag{5}$$

for all α, e, and χ.

For strict concavity of entropy in simple-solution fluids it is necessary and sufficient that for every triple (α_2, e_2, χ_2) distinct from (α_1, e_1, χ_1)

$$\begin{aligned} s(\alpha_2, e_2, \chi_2) < s(\alpha_1, e_1, \chi_1) &+ (\alpha_2 - \alpha_1)s_\alpha(\alpha_1, e_1, \chi_1) \\ &+ (e_2 - e_1)s_e(\alpha_1, e_1, \chi_1) \\ &+ (\chi_2 - \chi_1)s_\chi(\alpha_1, e_1, \chi_1) \end{aligned} \tag{6}$$

Moreover it can be shown that (6) implies that

$$T_e\mu_\chi - T_\chi\mu_e > 0 \tag{7}$$

Definition 2.4.1. *Simple-solution fluids referred to above with strictly concave entropy functions are called* simple-solution equilibrium fluids.

Definition 2.4.2. *A* simple-solution fluid system *is made up of a simple-solution equilibrium fluid of fixed total mass M (containing fixed solute mass M_χ), in which* (2.1.9) *holds and either alternative* 1 *or* 2 *in Definition* 2.1.2 *is true.*

Definition 2.4.3. *A simple solution system is in* energetic isolation *if*

$$\int [\mathbf{v}\cdot\mathbf{T}\mathbf{n} - \mathbf{n}\cdot(\mathbf{q} + \mathbf{R} + \mu\mathbf{f}_\chi)]\,d\sigma = 0 \tag{8}$$

and is in isolation *if* (8) *holds and*

$$\int \left[\frac{\nabla\cdot\mathbf{R} + \mu\nabla\cdot\mathbf{f}_\chi}{T} + \nabla\cdot(\mathbf{q}/T)\right] dV = 0 \tag{9}$$

To the list of definitions (2.1.12)–(2.1.17) must be added

$$\textit{system solute mass, } M_\chi = \int \varrho\chi\,dV \tag{10}$$

Eq. (2.1.18) still holds, but (2.1.19) must be replaced by

$$\frac{d}{dt}\int \varrho s\,dV \geqslant \int \left[\frac{1}{\varkappa}\left(\frac{|\mathbf{q}|}{T}\right)^2 + \frac{1}{\varkappa_1}\frac{|\mathbf{f}_\chi|^2}{T}\right] dV \geqslant 0 \tag{11}$$

To conclude the preliminary work, Definition 2.1.4 is modified to read:

Definition 2.4.4. *A configuration of a simple-solution fluid system is* thermodynamically stable in Gibbs sense *if every other configuration of the system with the same system mass M, system solute mass M_χ, and total system energy E, has smaller system entropy S.*

The determination of the properties of the associated equilibrium state and the proof of its uniqueness for a given, M, M_χ, and E follow the same procedures used in Section 2.2. The functional

$$\begin{aligned} S = &\int \varrho s\,dV + \gamma_1\left(M - \int \varrho\,dV\right) + \gamma_2 \\ &\times \left(E - \int \varrho\left[e + \Phi + \frac{v^2}{2}\right] dV\right) + \gamma_3\left(M_\chi - \int \varrho\chi\,dV\right) \end{aligned} \tag{12}$$

is formed, where the γ's are Lagrange multipliers. The substitutions

$$\varrho = \varrho_0 + \varepsilon\delta\varrho,\ e = e_0 = \varepsilon\delta e,\ \chi = \chi_0 + \varepsilon\delta\chi,\ \mathbf{v} = \mathbf{v}_0 + \varepsilon\delta\mathbf{v}$$

and use of (4) gives the Euler equations for the maximizing state

$$\delta v\colon\ -\gamma_2\varrho_0 v_0 = 0 \tag{14}$$

$$\delta e\colon\ \frac{\varrho_0}{T_0} - \gamma_2\varrho_0 = 0 \tag{15}$$

$$\delta s\colon\ \frac{-\varrho_0\mu_0}{T_0} - \gamma_3\varrho_0 = 0 \tag{16}$$

$$\delta\varrho\colon\ s_0 - \gamma_1 - \frac{p_0}{\varrho_0 T_0} - \gamma_3\chi_0 - \gamma_2\left[e_0 + \Phi + \frac{v^2}{2}\right] = 0 \tag{17}$$

Eqs. (14)–(16) require the associated equilibrium state to be motionless and of constant temperature and chemical potential,

$$v_0 = 0,\ T_0 = \gamma_2^{-1},\ \mu_0 = -\gamma_3\gamma_2^{-1} \tag{18}$$

Combination of (14)–(17) yields

$$\gamma_1 = s_0 - \frac{1}{T_0}[p_0\alpha_0 - \mu_0\chi_0 + e_0 + \Phi] \tag{19}$$

As before, application of the gradient operator to (19), observation that $\nabla T_0 = \nabla\mu_0 = 0$, and use of (4) yield the hydrostatic condition (2.2.8) as a final condition on the associated equilibrium state.

The proof of the uniqueness of the state (18) and (2.2.8) follows from arguments similar to those used in Section 2.2.

The definitions

$$M = \mathcal{M}(\alpha^*, T_0, \mu_0) \tag{20}$$

$$E = \mathcal{E}(\alpha^*, T_0, \mu_0) \tag{21}$$

$$M_\chi = \mathcal{M}_\chi(\alpha^*, T_0, \mu_0) \tag{22}$$

allow us to state the following theorem.

Theorem 2.4. *The entropy maximizing configuration of a simple-solution fluid system that satisfies* (18) *and* (2.2.8) *is uniquely determined by the values of the system mass M, system solute mass M_χ, and total system energy E, if and only if*

$$\left|\frac{\partial(\mathcal{M}, \mathcal{M}_\chi, \mathcal{E})}{\partial(x, y, z)}\right|_{\substack{x=\alpha^* \\ y=T_0 \\ z=\mu_0}} \neq 0 \tag{23}$$

The proof proceeds as before; the details are omitted.

It is now appropriate to formally define the associated equilibrium state for the present case.

Definition 2.4.5. *If a simple-solution fluid system satisfies* (23), *then the unique configuration that satisfies* (18) *and* (2.2.8) *is called the* associated equilibrium state *of the simple-solution fluid system.*

The only tasks remaining in this section are the proofs of the simple-solution analogs of Theorems 2.2 and 2.3. The procedures are virtually the same as before and only differences are noted.

Theorem 2.5. *If a simple-solution fluid system satisfies* (23), *then*

$$N = T_0(S_0 - S) = K + T_0\Sigma \tag{24}$$

where

$$\Sigma = -\int \varrho\left[(s - s_0) - (\alpha - \alpha_0)\frac{p_0}{T_0} - (e - e_0)\frac{1}{T_0} + (\chi - \chi_0)\frac{\mu_0}{T_0}\right] dV \tag{25}$$

Furthermore, the associated equilibrium state is thermodynamically stable in Gibbs sense.

Proof. After using the decomposition (2.2.31), (19) is employed to eliminate s_0 in the integral on the right. This procedure produces only two terms in addition to the terms shown in (2.2.32). Thus, except for these new terms, the proof proceeds identically to that in Theorem 2.2.

The additional terms are

$$\begin{aligned} -\int \frac{1}{T_0}(\varrho_0 - \varrho)\mu_0\chi_0 dV &= -\int \frac{1}{T_0}(\varrho_0\mu_0\chi_0 - \varrho\mu_0\chi_0) dV \\ &= -\int \frac{1}{T_0}[\varrho\mu_0(\chi - \chi_0) + \mu_0(\varrho_0\chi_0 - \varrho\chi)] dV \\ &= -\int \varrho\left[(\chi - \chi_0)\frac{\mu_0}{T_0}\right] dV \end{aligned} \tag{26}$$

The second term on the right in the second line vanishes because μ_0/T_0 is constant and $M_{\chi_0} = M_\chi$. Addition of the last line of (26) to (2.2.30) completes the proof of the first part of the theorem, and application of (4) and the inequality (6) to (25) proves the second part.

Finally, the theorem establishing the dynamical stability of simple-solution fluid systems in isolation is stated without proof, for the method is identical to that used to prove Theorem 2.3 except for the additional observation that M_χ is constant.

Theorem 2.6. *Let a simple-solution fluid system satisfy* (23) *and be in isolation for all time $t \geq \tau$. Then the deviations from the associated equilibrium state are for all time $t \geq \tau$ restricted by*

$$N(t) = K(t) + T_0\Sigma(t) \leq T_0[S_0 - S(\tau)] \tag{27}$$

Furthermore the entropic energy for all time $t \geq \tau$ is nonincreasing.

3.0. Origin and maintenance of convection in geophysical fluid systems

Consideration of the implications of the concept of entropic energy for the global circulation of geophysical systems has been deferred so far, except for the discussion of the dynamical stability of systems in isolation. The emphasis now shifts to the study of (2.2.29), and to the information it contains about convection. Several of the questions posed in Chapter 1 are satisfactorily answered, while the answers to others are suggested. The theorems and ideas developed below and in Chapter 2 lend support to a hope that the global circulation of geophysical fluid systems can be studied profitably from an entropic viewpoint.

3.1. *Material and integral rates of change*

The equations governing the changes associated with a material point **X** of an equilibrium regular fluid body are

$$\varrho \dot{v}_i = \varrho b_i + \frac{\partial}{\partial x_j} T_{ij}, \quad b_i = -\frac{\partial \Phi}{\partial x_i}, \quad T_{ij} = -p\delta_{ij} + \Delta_{ij} \tag{1}$$

$$\varrho \dot{e} = -\frac{\partial}{\partial x_i}(R_i + q_i) + T_{ij}\frac{\partial v_i}{\partial x_j}, \quad q_i = -\varkappa \frac{\partial T}{\partial x_i} \tag{2}$$

$$\dot{\alpha} = \alpha \frac{\partial v_i}{\partial x_i} \tag{3}$$

$$p = \bar{p}(\alpha, e), \quad T = \bar{T}(\alpha, e) \tag{4}$$

in which the dot indicates the material derivative $d(\ )/dt$, and the summation convention is applied to repeated indices. By virtue of the fact that fluid systems are material volumes (1)–(4) are compatible with the laws of balance of momentum and balance of energy ((2.1.1) and (2.1.2)).

The rate of change of total system energy can be written

$$\dot{E} = \int \nabla \cdot (-\mathbf{R} + \varkappa \nabla T)\, dV + \int n_j v_i \Delta_{ij}\, d\sigma \tag{5}$$

It is clear from Definition 2.1.2 that the energy flux $n_j p v_j$ vanishes on all fluid system boundaries.

The sign of the last integral in (5) is now determined for rigid boundaries because the result will prove useful in Section 3.2. The analysis begins with the expansion of the integrand with the aid of $(2.1.4)_2$, resulting in

$$n_j v_i \Delta_{ij} = \lambda_1 n_j \left(\frac{1}{2}\frac{\partial v_i^2}{\partial x_j} + v_i \frac{\partial v_j}{\partial x_i}\right) + \lambda_2 n_j v_j \frac{\partial v_i}{\partial x_i} \tag{6}$$

The assumption is now made that a rigid boundary is made up of m small planes. The velocity at a particular point P_m on each plane can be decomposed as

$$\mathbf{v} = v_n \mathbf{n} + v_\tau \boldsymbol{\tau} \tag{7}$$

where $\mathbf{n}$ is the exterior unit normal to the plane, and $\boldsymbol{\tau}$ is a unit vector pointing in the direction of the projection of $\mathbf{v}$ onto the plane (if $\mathbf{n} \times \mathbf{v} \equiv 0$ on a plane, $\boldsymbol{\tau}$ can be taken to be any unit vector parallel to the plane). With the definition of a second tangential unit vector $\boldsymbol{\tau}'$, normal to $\boldsymbol{\tau}$ and $\mathbf{n}$, the velocity at any other point on the plane can be expressed as components of the orthonormal system $(\mathbf{n}, \boldsymbol{\tau}, \boldsymbol{\tau}')$,

$$\mathbf{v} = v_n \mathbf{n} + v_\tau \boldsymbol{\tau} + v_{\tau'} \boldsymbol{\tau}' \tag{8}$$

At the point P_m where $\boldsymbol{\tau}$ is defined, (6) becomes

$$(n_j v_i \Delta_{ij})_{P_m} = \left(\lambda_1 + \frac{\lambda_2}{2}\right)\frac{\partial v_n^2}{\partial n} + \lambda_1 \left(v_\tau \left[\frac{\partial v_\tau}{\partial n} + \frac{\partial v_n}{\partial \tau}\right] + v_\tau' \left[\frac{\partial v_{\tau'}}{\partial n} + \frac{\partial v_n}{\partial \tau'}\right]\right) + \lambda_2 v_n \left(\frac{\partial v_\tau}{\partial \tau} + \frac{\partial v_{\tau'}}{\partial \tau'}\right) \tag{9}$$

where (n, τ, τ') are the new coordinates. Definition 2.1.2 requires that v_n vanish at rigid boundaries, and so

$$v_n = \frac{\partial v_n}{\partial \tau} = \frac{\partial v_n}{\partial \tau'} = 0 \tag{10}$$

Thus, because velocities at the boundary and their spatial derivatives with respect to directions parallel to the boundary must be finite, the fourth and the sixth through eighth terms on the right in (9) vanish and the equation becomes

$$(n_j v_i \Delta_{ij})_{P_m} = \left(\lambda_1 + \frac{\lambda_2}{2}\right)\frac{\partial v_n^2}{\partial n} + \lambda_1 \left(v_\tau \frac{\partial v_\tau}{\partial n} + v_{\tau'} \frac{\partial v_{\tau'}}{\partial n}\right) \tag{11}$$

Only the the point P_m need be considered be-

cause the last integral in (5) over rigid boundaries can be approximated by

$$\int n_j v_i \Delta_{ij} d\sigma \cong \sum_m (n_j v_i \Delta_{ij})_{P_m} \Delta_m \sigma \tag{12}$$

where $\Delta_m \sigma$ is the area of the mth plane, and where the sum is taken over all the planes approximating the rigid boundaries. The sum in (12) converges to the value of the integral as m goes to infinity.

Two types of fixed solid boundary conditions are commonly encountered:

1. *Non-slip*, $v_n = v_\tau = v_{\tau'} = 0$,

2. *Free-slip*, $v_n = \dfrac{\partial v_\tau}{\partial n} = \dfrac{\partial v_{\tau'}}{\partial n} = 0$.

The free-slip condition is used on boundaries in numerical models whose grid size greatly exceeds the depth of the viscous boundary layer.

For a non-slip boundary,

$$(n_j v_i \Delta_{ij})_{P_m} \overset{\text{non-slip}}{=\!=} \left(\lambda_1 + \frac{\lambda_2}{2}\right) \frac{\partial v_n^2}{\partial n} + \frac{\lambda_1}{2}\left(\frac{\partial v_\tau^2}{\partial n} + \frac{\partial v_{\tau'}^2}{\partial n}\right) \leqslant 0 \tag{13}$$

because the speed is zero at the boundary and its square cannot decrease away from the boundary. The free-slip boundary conditions require that the last two terms in (11) vanish and thus

$$(n_j v_i \Delta_{ij})_{P_m} \overset{\text{free-slip}}{=\!=} \left(\lambda_1 + \frac{\lambda_2}{2}\right) \frac{\partial v_n^2}{\partial n} \leqslant 0 \tag{14}$$

by the same reasoning.

Thus on free-slip or non-slip fixed solid boundaries

$$\int n_j v_i \Delta_{ij} d\sigma = \lim_{m \to \infty} \sum_m (n_j v_i \Delta_{ij})_{P_m} \Delta_m \sigma \leqslant 0 \tag{15}$$

Lemma 3.1. *If the fixed solid boundaries of bounded fluid systems are of either the non-slip or (and) free-slip types, then*

$$\int n_j v_i \Delta_{ij} d\sigma \leqslant 0 \tag{16}$$

The relationship (16) *also holds for unbounded fluid systems if, in addition, the viscous integral in* (5) *vanishes at the free surface at infinite* Φ, *so that*

$$\int n_j v_i \Delta_{ij} d\sigma_\infty \leqslant 0 \tag{17}$$

Proof. With (15) and (17), (16) is true for all cases, completing the proof.

The second integral whose rate of change is of interest is the system entropy. Its time derivative can be expressed exactly by the first law of thermodynamics for a material point **X** of an equilibrium regular fluid body. Use of (1)–(3), (2.1.5), and the fact that fluid systems are material volumes produces

$$\frac{d}{dt}\int \varrho s dV = \dot{S} = \int \frac{\nabla \cdot (-\mathbf{R} + \varkappa \nabla T)}{T} dV + \int \frac{\Delta_{ij}}{T} \frac{\partial v_i}{\partial x_j} dV \tag{18}$$

Eq. (18) satisfies the Clausius-Duhem inequality (2.1.3), since it is known[1] that the integrand of the last integral is non-negative when the inequalities (2.1.6) are true.

To complete the preparatory work for the analyses in the next section, another form of the rate of change of entropic energy will be derived. Temporal differentiation of (2.2.29) gives

$$\dot{N} = \dot{T}_0(S_0 - S) + T_0(\dot{S}_0 - \dot{S}) \tag{19}$$

Because system mass is constant in fluid systems, the first term in (19) becomes

$$\dot{T}_0(S_0 - S) = \frac{\partial T_0}{\partial E}(S_0 - S)\dot{E} = \Gamma \dot{E} \tag{20}$$

where $\Gamma = \Gamma(t)$ is defined as the coefficient of $\dot{E}$ in (20). The entropy difference $S_0 - S$ is non-negative, and examination of (2.2.27), and (2.3.23) with the minus sign, reveals the positivity of $\partial T_0/\partial E$ for both the ideal gas atmosphere and the ideal liquid in a tank respectively, thus the factor Γ is non-negative for both examples of geophysical fluid systems presented in Chapter 2.

The rate $\dot{S}_0$ in (19) can be expressed in terms of other system rates upon the observation that all changes of thermodynamic variables in the associated equilibrium state are the result of total system energy changes. Thus, with

[1] For one method of proof, see Dutton (1973).

use of the transport theorem for material volumes,

$$\dot{S}_0 = \int \frac{\partial}{\partial t}(\varrho_0 s_0)\,dV + \int \mathbf{n}\cdot\varrho_0 s_0 \mathbf{v}\,d\sigma \tag{21}$$

The integrand of the boundary integral in (21) vanishes identically at all boundaries: at fixed solid boundaries, the normal component of velocity vanishes, and in unbounded systems, $\varrho_0 s_0 \mathbf{v}$ vanishes at infinity by virtue of the finiteness of the integrals of mass, energy, and entropy. Expansion of the remaining integrand in (21), making use of (2.1.5), (2.2.7), the chain rule, and the local invariance of Φ, gives

$$\dot{S}_0 = \int \frac{p_0}{T_0}\frac{\partial}{\partial t}(\alpha_0\varrho_0)\,dV + \gamma_1\int \frac{\partial \varrho_0}{\partial t}\,dV + \frac{1}{T_0}\int \frac{\partial}{\partial t}(\varrho_0 e_0 + \varrho_0\Phi)\,dV \tag{22}$$

The first integral in (22) is trivially zero, and the second integral vanishes after reuse of the transport theorem and from the fact that mass is conserved in geophysical systems. Application of the transport theorem to the remaining term and (2.2.10) now produce

$$\dot{S}_0 = \frac{1}{T_0}\frac{d}{dt}\int \varrho_0(e_0+\Phi)\,dV = \frac{\dot{E}}{T_0} \tag{23}$$

Substitution of (20) and (23) into (19) gives finally

$$\dot{N} = (1+\Gamma)\dot{E} - T_0\dot{S} \tag{24}$$

This compact form of the rate of change of entropic energy is employed frequently in the proof and discussion of the theorems given in Section 3.2, but it can also be used as a starting point for analysis of the entropic energy budgets of natural systems when sufficient data is available.

3.2. *Motion and the rate of change of entropic energy*

Before proceeding to the analysis below, a careful distinction must be made between fluid systems in which motions or accelerations are or are not present.

Definition 3.2.1. *A fluid system is*
1. *in* motion *if* $\mathbf{v} \not\equiv 0$,
2. *in* weak motion *if either* $\mathbf{v} \not\equiv 0$ *or* $\dot{\mathbf{v}} \not\equiv 0$,
3. *at* rest *if both* $\mathbf{v} \equiv 0$ *and* $\dot{\mathbf{v}} \equiv 0$,
4. *at* permanent rest *if at rest with* $\dot{e} \equiv 0$.

It is also convenient to establish terminology for geophysical systems in which the entropic energy is chaging.

Definition 3.2.2. *A geophysical fluid system is*
1. evolving *on* $[t_1, t_2]$ *if* $\dot{N} \neq 0$ *at all times* t, $t_1 \leqslant t \leqslant t_2$,
2. globally unstable, stable, *or* neutral *if* $\dot{N} \gtrless 0$ *respectively.*

Finally, the theorems proved in this section contain constraints on the sum of the divergences of radiation and heat flux.

Definition 3.2.3. *The sum*

$$H = -\nabla\cdot\mathbf{R} + \nabla\cdot\varkappa\nabla T \tag{1}$$

at a point in the fluid domain is called the local heating.

With these definitions, the results of Section 3.1, and ideas taken from a theorem concerning motion in ideal gas systems by Jeffreys (1925), it can be shown immediately that motions or accelerations are direct consequences of evolution in N ($\dot{N} \neq 0$).

Theorem 3.1. *Suppose the local heating vanishes somewhere in the domain of a geophysical fluid system. If* $\dot{N} \neq 0$, *then the fluid system must be in at least weak motion.*

Proof. Suppose $\mathbf{v} \equiv \dot{\mathbf{v}} \equiv 0$, then (3.1.1)–(3.1.3) become

$$\nabla p = -\varrho\nabla\Phi \tag{2}$$

$$\frac{\partial \alpha}{\partial t} = 0 \tag{3}$$

and with (1),

$$\varrho\frac{\partial e}{\partial t} = H \tag{4}$$

The local derivative of (2), after expansion of $\partial p/\partial t$, use of (3) on each side of the equality, and employment of the temporal invariance of Φ, can be written

$$\frac{\partial}{\partial t}\nabla p = \nabla\left(p_e\frac{\partial e}{\partial t}\right) = 0 \tag{5}$$

Thus,

$$p_e \frac{\partial e}{\partial t} = \frac{p_e}{\varrho} H \tag{6}$$

can only be a function of time. Because H is zero at least at one (spatial) point in the fluid domain, H is identically zero, for $p_e \neq 0$ in geophysical fluid systems. Therefore in the domain

$$H \equiv 0 \tag{7}$$

and

$$\frac{\partial e}{\partial t} = \dot{e} \equiv 0 \tag{8}$$

Use of the fact that $\mathbf{v}$ vanishes identically and the result (7) in (3.1.5) and (3.1.18) give $\dot{E} = \dot{S} = 0$. Subsequently, from (3.1.24), $\dot{N} = 0$. Under the conditions of the theorem, $\mathbf{v} \equiv \dot{\mathbf{v}} \equiv 0$ implies $\dot{N} = 0$, and the contrapositive statement establishes the theorem.

The proof of the theorem by Jeffreys (1925) is contained in the course of the proof of Theorem 3.1. For an ideal gas (6) becomes

$$\varrho c_V \frac{\partial T}{\partial t} = H \tag{9}$$

and thus local heating can only be a function of time. Therefore, $\mathbf{v} \equiv \dot{\mathbf{v}} \equiv 0$ in an ideal gas system implies that the heating rate (9) does not vary in space. Subsequently any spatial variation in local heating guarantees the presence of at least weak motion in an ideal gas system, and this conclusion is essentially the content of Jeffreys' theorem.

An extension of Jeffreys' results to the atmosphere culminates in an atmospheric version of Theorem 3.1 discussed in Dutton (1973). Because the integrals of mass and energy are required to be finite in that case, $\varrho\, \partial T/\partial t$ vanishes at infinity. Thus, the local heating condition is satisfied naturally in this specific case, whereas the condition must be imposed when studying more general geophysical fluid systems.

The first part of the following corollary to Theorem 3.1 is a version of the theorems of Jeffreys and Dutton for geophysical fluid systems.

Corollary 3.1.1. *Suppose the local heating vanishes somewhere in the domain of a geophysical fluid system. If either*

1. *the local heating does not vanish everywhere in the domain of the fluid system, or*
2. *the fluid system is not in permanent rest, or*
3. *the fluid system is not in isolation,*

then the fluid system is at least in weak motion.

The proof of all parts of Corollary 3.1.1 is contained contrapositively in the proof of Theorem 3.1.

The next corollary is of interest because it reveals the stringent conditions that must be satisfied for a geophysical fluid system to be in its associated equilibrium state.

Corollary 3.1.2. *If a geophysical fluid system is in its associated equilibrium state and if the divergence of radiation flux vanishes somewhere in the domain of the fluid system, then*

1. $-\nabla \cdot \mathbf{R}$ *vanishes identically,*
2. *the fluid system is in permanent rest, and*
3. *the fluid system is in isolation.*

Proof. The associated equilibrium state is motionless and hydrostatic. As a consequence eq. (3.1.1) shows that $\dot{\mathbf{v}} \equiv 0$ in the fluid domain, and the fluid is at least at rest. In addition $\nabla \cdot \varkappa \nabla T_0 \equiv 0$, and so local heating vanishes somewhere in the fluid domain, for $-\nabla \cdot \mathbf{R}$ vanishes somewhere. The hypotheses for the contrapositive of Corollary 3.1.1 are then satisfied. This completes the proof.

With the relatively unrestrictive condition that somewhere in a geophysical fluid system the gain of heat by conduction must equal the divergence of radiant energy flux, Theorem 3.1 states that any change in the entropic energy must be at least accompanied by some accelerations in the fluid. However, it does not guarantee the presence of motion. Theorem 3.2 does make this guarantee for evolving geophysical fluid systems.

Theorem 3.2. *Suppose local heating vanishes somewhere in the domain of a geophysical fluid system at every time t, $t_1 < t < t_2$. If the system is evolving on $[t_1, t_2]$ then it must be in motion on at least part of $[t_1, t_2]$.*

Proof. Let $\mathbf{v} \equiv 0$ on $[t_1, t_2]$, then $\dot{\mathbf{v}} \equiv 0$ on (t_1, t_2). But $\dot{N} \neq 0$ at every time on $[t_1, t_2]$ because the system is evolving on the closed interval, and from Theorem 3.1 the fluid system must be at

least in weak motion at every time on (t_1, t_2). Thus $\dot{\mathbf{v}} \not\equiv 0$ at every time on (t_1, t_2), a contradiction, consequently $\mathbf{v} \not\equiv 0$ on at least part of $[t_1, t_2]$.

Theorems 3.1 and 3.2 establish definite links between entropic energy change and the origins of convection. They state that motions or accelerations are necessary conditions for changes in entropic energy. In other words, motions and accelerations must accompany increases or decreases in entropic energy. However, both theorems require that local heating vanish somewhere in the domain of a geophysical fluid system, and this condition must be satisfied in a system before the theorems can be utilized.

In geophysical fluid systems, $\dot{N}$ and $\dot{K}$ are usually of the same sign, so that a qualitative assessment of the sign of $\dot{N}$ is often sufficient to determine changes in circulation intensity ($\dot{K}$). The calculations in Dutton (1973) reveal that this is the case for seasonal variations in the atmosphere—increases in N are accompanied by increases in K from summer to winter and vice versa. With this example and the likelihood that in many evolving systems changes in circulation intensity follow the direction of evolution, it is desirable to determine under what conditions a geophysical system is globally unstable, neutral, or stable. The rates of change of system total energy and system entropy that determine $\dot{N}$ through (3.1.24) have a complicated dependence on phenomena occurring at the boundaries of the system and on the internal distribution of the velocity and the thermodynamic variables. Hence, in order to proceed, certain simplifying assumptions that are generally justified in geophysical fluid systems are made.

First, because real geophysical systems driven by a reasonably steady external energy source, such as the earth–atmosphere system driven by the sun, are observed to be in approximate energetic equilibrium, energetic isolation is imposed.

Next, it is assumed that either the local heating is both positive and negative somewhere in the fluid, or that $H < 0$ somewhere in the fluid, that fixed solid boundaries are of either the free-slip or non-slip, and that (3.1.17) holds if the system is unbounded. The second alternative of $H < 0$ and the boundary conditions combined with energetic isolation also implies that $H > 0$ somewhere in the fluid. These assumptions also ensure that the local heating vanishes somewhere in the fluid system.

The integral of H can now be decomposed as

$$\int H dV = \int H^+ dV - \int H^- dV \tag{10}$$

where

$$H^+ = \begin{Bmatrix} H \text{ if } H \geqslant 0 \\ 0 \text{ if } H < 0 \end{Bmatrix}, \quad H^- = \begin{Bmatrix} 0 \text{ if } H \geqslant 0 \\ H \text{ if } H < 0 \end{Bmatrix} \tag{11}$$

Utilization of this decomposition then permits the definitions

$$T^{\pm} = \frac{\int H^{\pm} dV}{\int (H^{\pm}/T) dV} \tag{12}$$

Lastly, the viscous integral in (3.1.5) is denoted by $D(E)$, and the irreversible production of entropy by viscosity in (3.1.18) by $\dot{S}_i$.

Theorem 3.3. *Suppose a geophysical fluid system is in energetic isolation and the local heating is both positive and negative somewhere in the domain of the fluid system. Then,* $\dot{N} \gtreqless 0$ *respectively if and only if*

$$T^- \lesseqgtr T^+ \left\{ \int H^- dV \left[\int H^+ dV + T^+ \dot{S}_i \right]^{-1} \right\} \tag{13}$$

Further, suppose, instead of the requirement that local heating be positive somewhere, the system is bounded by free-slip or (and) non-slip rigid boundaries and if unbounded satisfies (3.1.17). *If either* $D(E) \neq 0$ *or* $\dot{S}_i \neq 0$, *then for the fluid system to be globally unstable or neutral it is necessary that*

$$T^- < T^+ \tag{14}$$

Proof. For global instability, stability, and neutrality respectively, (3.1.24) and energetic isolation demand that

$$\dot{N} = -T_0 \dot{S} \gtreqless 0 \tag{15}$$

Thus, with (3.1.18), the decomposition (10)–(11), and the definitions (12), the conditions become

$$\dot{S} = \frac{1}{T^+} \int H^+ dV - \frac{1}{T^-} \int H^- dV + \dot{S}_i \lesseqgtr 0 \tag{16}$$

Because all of the integrals and coefficients in (16) are non-negative, (13) is obtained by transposition of the second term of the inequality and cross multiplication, proving the first part of the theorem.

The boundary conditions specified in the second part of the theorem give by virtue of Lemma 3.1

$$D(E) \leqslant 0 \tag{17}$$

Further, (17) and energetic isolation imply that $\int H^+ dV \geqslant \int H^- dV$ respectively. Thus, because $\dot{S}_i$ is never negative,

$$\dot{N} \geqslant 0 \Rightarrow T^- \leqslant T^+ \left\{ \int H^- dV \left[\int H^+ dV + T^+ \dot{S}_i \right]^{-1} \right\} < T^+ \tag{18}$$

for if either $\dot{S}_i \neq 0$ or $D(E) \neq 0$ the factor multiplying T^+ in (13) must be less than one. This completes the proof of both parts of Theorem 3.3[2].

In essence the theorem just proved states that changes in intensity of the global circulation in energetically isolated systems (in the sense that $\dot{N}$ is a measure of intensity change) depend on the relative magnitudes of the temperatures at which heating and cooling is taking place in the fluid. However, the theorem reveals no way by which the detailed arrangement of heating and cooling and the distribution of convective circulations can be deduced from $\dot{K}$ or $\dot{N}$. This problem is a result of the integral nature of Theorem 3.3; statements are made in terms of global characteristics of geophysical fluid systems and contain no specific information about what must be taking place locally. Budgets of N, S, E, K, and $T_0\Sigma$ also fail to answer crucial questions about patterns of convective response, despite their value in revealing the relative importance of entropy and energy production to changes in entropic energy and in showing how these changes are shared with kinetic and static energies.

It has been shown above that, in a geophysical fluid system, motions and accelerations must accompany the disturbance of an entropic energy equilibrium when local heating vanishes somewhere in the fluid domain. In situations where local heating is non-zero everywhere, motion occurs after certain minimal forcing is applied to the system. Presumably, in both cases the motion is initiated to aid the fluid system in reaching a new equilibrium, and perhaps the expedient accomplishment of this end governs the choice of a particular pattern of convection from several possible alternatives, including no convection. The efficiency of a particular response must depend both on the system's goal and on the nature and extent of the disturbance. It seems reasonable, in light of the emphasis on $\dot{N}$ in Theorems 3.1–3.3, that a mode of response is an attempt by the fluid system to offset the production of entropic energy brought about by external forcing.

These ideas could be tested by computation of the relevant integrals and their rates of change at each time step in numerical models of systems that reproduce mode changes. The models developed by Williams (1967 a, 1967 b, 1968, 1971) for rotating annuli lend themselves to this undertaking, thus providing part of the motivation for the preliminary derivations shown in Section 2.3 for ideal liquids in tanks. In Williams' models a temperature difference is set up between the outer and inner cylinders of a rotating annulus by means of hot and cold reservoirs outside the walls. The fluid contained in the annulus is initially at uniform temperature, in solid rotation, and in equilibrium. With certain imposed rotational rates and temperature differences between the walls, the convective pattern is observed to evolve from the solid rotation mode (no convection) to an axi-symmetric circulation similar to a direct thermal cell, and then to a regular wave pattern, finally achieving an entropic energy balance. In analogous laboratory experiments, some initial conditions produce the same sequence plus an additional transformation from the regular wave pattern to a more chaotic circulation characterized by eddies.

Convective modes in these experiments vary from simple to complicated circulations, and the patterns upon reaching an entropic energy steady state are dependent upon the imposed initial conditions. Comparative analyses of the energy, entropy, and entropic energy integrals before, during, and after mode transformations may reveal the role of the budget of entropic energy in a system's choice of a convective response.

[2] Theorems given in Dutton (1973) for the planetary atmosphere are special cases of Theorems 3.1–3.3.

4.0. Concluding remarks

The contribution of this article to the study of the global circulation of geophysical fluid systems is the addition of entropic energy to the list of integral quantities available to study the circulation. The new concept, embodied in (2.2.29), has an apparent applicability to a wide class of fluid systems and gives a direct coupling of the mechanistic and kinematic aspects of kinetic energy to the thermodynamic and predictive aspects of entropy. From the concept of entropic energy, definite conclusions about the dynamical stability of systems in isolation and the origins and maintenance of convection follow naturally.

Among the direct uses of the concept of entropic energy at its current stage of sophistication is its application to both the atmosphere and to laboratory experiments in which enough data is available for the accurate estimation of all the relevant and desired quantities. Budgets of the energy forms, entropy, and entropic energy could then be computed, and the degree of interaction and mutual dependence of these integrals can be studied under a variety of situations.

However, the application of the present work that is at once the most feasible and would be of great interest is the analysis of patterns of convective response in numerical models, especially those of rotating or Bénard convection. The application to general circulation models of the atmosphere at this time is less promising. The complicated interrelationships between long-wave radiation and the properties of the atmosphere and the lower boundary are not yet well understood and imperfectly parameterized in models, whereas approximate radiative equilibrium can be assumed with small error in most models of laboratory experiments. Nonetheless, insight gained from the analysis of convection models may not only provide insight to the real systems simulated but to other fluid systems as well.

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ЭНТРОПИЙНАЯ ЭНЕРГИЯ ГЕОФИЗИЧЕСКИХ ЖИДКИХ СИСТЕМ

Главная цель работы состоит в создании интегральной теории для широкого класса жидких систем, которая не только учитывает концепции баланса массы, импульса и энергии, но также энтропии и второе начало термодинамики. Развитая теория дает явное соотношение между интегралами кинетической энергии и энтропии в терминах новой формы энергии — энтропийной энергии. Жидкости с линейной вязкостью, чьи энтропийные функции строго вогнуты, определяются как равновесные регулярные жидкости. Жидкая система состоит из равновесной регулярной жидкости с фиксированной массой M, чья массовая сила обусловлена стационарным положительным монотонным потенциальным полем и чьи границы удовлетворяют специфическим условием. Геофизические жидкости представляют собою невырожденный ансамбль жидкостей.

Для геофизической жидкой системы существует единственное состояние при данной массе и общей энергии E, называемое состоянием равновесия, в котором отсутствуют движения и которое гидростатично и изотермично (теорема 2.1). Произведение температуры T_0 этого состояния и разности энтропий $S_0 - S$ этого и естественного состояния равно энтропийной энергии N, т. е. сумме кинетической энергии K и статической энергии $T_0\Sigma$. Энтропийная энергия является положительной мерой (напоминающей норму $\mathcal{L}_2$) отклонения естественного состояния от соответствующего равновесного состояния, и таким образом, равновесное состояние термодинамически устойчиво в смысле Гиббса (теорема 2.2). Если геофизическая жидкая система изолирована ($\dot{E} = 0$, $\dot{S} \geqslant 0$), ее энтропийная энергия ограничена по величине, а скорость ее изменения никогда не становится положительной (теорема 2.3). Когда система энергетически изолирована ($\dot{E} = 0$), скорость изменения интенсивности циркуляции, измеряемая величиной $\dot{N}$, контролируется только скоростью увеличения энтропии, и необходимые условия для роста или поддержания интенсивности в определенных физически реальных системах могут быть сформулированы только в терминах осредненной температуры в областях нагревания или охлаждения (теорема 3.3). Наконец, предположим, что сумма конвергенции радиации и потоков тепла исчезает где-либо в области, занятой геофизической жидкой системой. Тогда в состоянии энергетической или полной изоляции, или вообще без изоляции, в системе должны в любой момент присутствовать ускорения или движения, если в этот момент скорость изменения энтропийной энергии не равна нулю (теорема 3.1.). Более того, система должна находиться в движении в течение части интервала времени, если ее энтропийная энергия постоянна в течение этого интервала (теорема 3.2).