A three-dimensional tidal problem

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(Manuscript received August 18, 1974; in final form July 7, 1975)

ABSTRACT

The investigation is of a linear three-dimensional model of the tides in an ocean covering a rotating spherical earth. It differs from the classical problem in that the oscillations are governed essentially by a partial differential equation whose type depends on the period of the oscillation, whereas the classical problem is resolved into the solution of an ordinary differential equation. Results of the two theories are compared at points where this is possible.

Introduction

The usual method used in investigating the motion of a relatively thin layer of fluid of large horizontal extent, is to assume that the vertical equation of motion may be replaced by the hydrostatic equation and the vertical velocity determined by means of the continuity equation. It was demonstrated by Solberg (1936) that this approximation is not essential to the solving of the problem of the free symmetrical oscillations of a homogeneous fluid covering the surface of the rotating earth. The results obtained by Solberg differ, only slightly, from those obtained by Hough (1897) who solved a similar problem but made the hydrostatic approximation.

Formulation

Solberg worked with a nonorthogonal coordinate system and made an allowance for the spheroidal shape of the mean earth and ocean surfaces. He considered it a contradiction to retain the dynamical effect of the earth's rotation on the tides but to neglect its effect on the shape of the mean surface of the ocean.

The present problem is investigated using an orthogonal coordinate system, spherical polars, thus avoiding the considerable complication which is necessarily involved in setting up the problem in a nonorthogonal system. The spheroidal shape of the earth has not been considered and, as shown by J. Proudman (1942), the coriolis forces may be retained and the centrifugal forces neglected without any mathematical inconsistency being introduced into the problem. Using this coordinate system the problem is far simpler in detail but eventually takes an exactly similar form to that which arises from the method of Solberg.

The spherical polar coordinates (R, θ, ψ) are taken with the pole at the centre of the earth. R measures the distance radially from the centre, θ measures the angular distance from the North Pole, and ψ is the longitude measured positive to the east.

In this coordinate system the linearised equations of motion take the form

$$\frac{\partial u}{\partial t} - 2\omega v \cos\theta = -\frac{1}{R}\frac{\partial \Phi}{\partial \theta} \tag{1}$$

$$\frac{\partial v}{\partial t} + 2\omega u \cos\theta + 2\omega w \sin\theta = -\frac{1}{R\sin\theta}\frac{\partial \Phi}{\partial \psi} \tag{2}$$

$$\frac{\partial w}{\partial t} - 2\omega v \sin\theta = -\frac{\partial \Phi}{\partial R} \tag{3}$$

where u, v, and w represent respectively the components of the perturbed velocity in the directions of θ, ψ and R increasing, p is the perturbation in the pressure, Ψ is the perturbation in the gravitational potential which is due to the tides themselves, $\Phi = p/\varrho + \Psi$, ω is the angular velocity of the earths rotation and ϱ the density of the water.

The equation of continuity has the form

$$\frac{\partial}{\partial\theta}[u \sin\theta] + \frac{\partial v}{\partial\psi} + \frac{\sin\theta}{R}\frac{\partial}{\partial R}[R^2 w] = 0 \quad (4)$$

In order that these equations should represent oscillatory states, solutions are sought having the following form

$$[u, v, w, \Phi] = \exp(im\psi - i\sigma t)[A, B, C, D] \quad (5)$$

where A, B, C and D are now functions of θ and R only. With this substitution, eqs. (1), (2), (3) and (4) take the form

$$-iA\sigma - 2\omega B\cos\theta = -\frac{1}{R}\frac{\partial D}{\partial\theta} \quad (6)$$

$$-iB\sigma + 2\omega A\cos\theta + 2\omega C\sin\theta = -\frac{im}{R\sin\theta}D \quad (7)$$

$$-iC\sigma - 2\omega B\sin\theta = -\frac{\partial D}{\partial R} \quad (8)$$

$$\frac{\partial}{\partial\theta}(A\sin\theta) + imB + \frac{\sin\theta}{R}\frac{\partial}{\partial R}(R^2 C) = 0 \quad (9)$$

Solving eqs. (6), (7) and (8) for A, B, and C in terms of D and its partial derivatives gives

$$A = \frac{i}{2\omega\lambda(\lambda^2-1)}\left[-\frac{\lambda^2 - \sin^2\theta}{R}\frac{\partial D}{\partial\theta} + \frac{m\lambda D\cot\theta}{R} - \sin\theta\cos\theta\frac{\partial D}{\partial R}\right] \quad (10)$$

$$B = \frac{1}{2\omega(\lambda^2-1)}\left[-\frac{\cos\theta}{R}\frac{\partial D}{\partial\theta} + \frac{\lambda m D}{R\sin\theta} - \sin\theta\frac{\partial D}{\partial R}\right] \quad (11)$$

$$C = \frac{i}{2\omega\lambda(\lambda^2-1)}\left[-(\lambda^2 - \cos^2\theta)\frac{\partial D}{\partial R} - \frac{\sin\theta\cos\theta}{R}\frac{\partial D}{\partial\theta} + \frac{\lambda m D}{R}\right] \quad (12)$$

where $\lambda = \dfrac{\sigma}{2\omega}$

Substituting these values of A, B and C into eq. (9), the equation of continuity, yields the following partial differential equation for D.

$$(\lambda^2 - \sin^2\theta)\frac{\partial^2 D}{\partial\theta^2} + (\lambda^2 - \cos^2\theta)R^2\frac{\partial^2 D}{\partial R^2} + 2\sin\theta\cos\theta R\frac{\partial^2 D}{\partial R\partial\theta} + \cot\theta(\lambda^2 - 2\sin^2\theta)\frac{\partial D}{\partial\theta} + R(2\lambda^2 - \sin^2\theta)\frac{\partial D}{\partial R} - \frac{\lambda^2 m^2}{\sin^2\theta}D = 0 \quad (13)$$

which is of the elliptic, parabolic, or hyperbolic type according to whether $\lambda^2 > 1$, $\lambda^2 = 1$ or $\lambda^2 < 1$ respectively.

Boundary conditions

The conditions to be satisfied at the boundaries of the ocean are taken to be that:

(i) there is no motion perpendicular to the bed of the ocean,
(ii) the surface of the ocean is a surface of constant pressure.

As a spherical earth is being considered, the first condition is equivalent to the vanishing of the vertical velocity of the water at the bed of the ocean which is taken to be at $R = a - h$, where h is the depth of the ocean

i.e. $w = 0$ at $R = a - h$

or, on using eq. (5), $C = 0$ at $R = a - h$ (14)

The surface of the ocean is taken to be at $R = a$, hence the second condition, when expressed mathematically is

$$\frac{Dp}{Dt} = 0 \quad \text{at} \quad R = a.$$

By applying a perturbation analysis and neglecting second order products of small quantities, consistent with earlier assumptions, this becomes

$$\frac{\partial p}{\partial t} - wg\varrho = 0 \quad \text{at} \quad R = a$$

or

$$\frac{\partial\Phi}{\partial t} - wg - \frac{\partial\Psi}{\partial t} = 0 \quad \text{at} \quad R = a$$

Expanding Φ_s and w_s in a series of surface harmonics in the form

$$(w_s, \Phi_s) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (a_{m,n}, G_{m,n})\, e^{im\psi} P_n^m(\cos\theta),$$

where the suffix s denotes the surface value of a variable, it can be shown, subject to the same linearization approximations, that

$$\left(\frac{\partial \Psi}{\partial t}\right)_s = -\sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{4\pi\gamma\varrho a}{2n+1} a_{m,n}\, e^{im\psi} P_n^m(\cos\theta)$$

where γ is the gravitational constant.

Then using the fact that, to a good degree of approximation, $g = (4\pi/3)\gamma a \varrho_s$, where ϱ_s is the mean density of the earth's solid core, the boundary condition at $R = a$ becomes

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{\partial G_{m,n}}{\partial t} - g_n a_{m,n}\right) e^{im\psi} P_n^m(\cos\theta) = 0,$$

where

$$g_n = g\left[1 - \frac{3}{2n+1}\varrho/\varrho_s\right].$$

When considering only the type of oscillation associated with any particular value of m, this may be written as

$$\sum_{n=m}^{\infty} \left[\frac{\partial G_{m,n}}{\partial t} - g_n a_{m\,n}\right] P_n^m(\cos\theta) = 0 \quad \text{for all } m,$$

or on using the orthogonality of the Legendre functions

$$\frac{\partial G_{m,n}}{\partial t} - g_n a_{m,n} = 0 \quad \text{at} \quad R = a \tag{15}$$

for all m and $n \geq m$.

The analytical solution

The solution of the problem follows very closely the method used by Solberg (1936) in the case when only the symmetrical ($m = 0$) oscillations were investigated. As in Solberg's investigation only the case of $\lambda^2 > 1$ is considered so that the equation to be solved is of the elliptic type.

Since this equation is homogeneous in R it will have solutions of the form $D = (R/a)^n D_n(\theta)$. Making this substitution in equation (13) yields an ordinary differential equation for $D_n(\theta)$, namely

$$(\lambda^2 - \sin^2\theta)\frac{d^2 D_n}{d\theta^2} + [2(n-1)\sin\theta\cos\theta + \lambda^2\cot\theta] \times \frac{dD_n}{d\theta} + \left[\lambda^2 n(n+1) - n - n(n-2)\cos^2\theta - \frac{m^2\lambda^2}{\sin^2\theta}\right] D_n = 0$$

which on making the transformation $v = \lambda\cos\theta/\sqrt{\lambda^2 - \sin^2\theta}$ of the independent variable may be written in the form

$$(1 - v^2)\frac{d^2}{dv^2}[(\lambda^2 - v^2)^{n/2} D_n] - 2v\frac{d}{dv}[(\lambda^2 - v^2)^{n/2} D_n] + \left[n(n+1) - \frac{m^2}{1 - v^2}\right](\lambda^2 - v^2)^{n/2} D_n = 0$$

from which it is apparent that the function $(\lambda^2 - v^2)^{n/2} D_n$ may be written

$$(\lambda^2 - v^2)^{n/2} D_n = d_n^m P_n^m(v) + K_n^m Q_n^m(v)$$

where P_n^m and Q_n^m are linearly independent solutions of the associated Legendre differential equation and where d_n^m and K_n^m are constants.

$P_n^m(v)$ has a singularity at $v = \infty$, and $Q_n^m(v)$ has singularities at $v = -1$, $+1$ and ∞. Now at the North Pole ($\theta = 0$) and the South Pole ($\theta = \pi$), $v = +1$ and -1 respectively, thus if $K_n^m \neq 0$ then D_n has singularities which do not exist in the physical problem, hence it is necessary that K_n^m is chosen to be zero.

In view of the assumption $(u, v, w, \Phi) = \exp(im\psi - i\sigma t)[A, B, C, D]$ it is clear that m must be chosen to be an integer in order that u, v, w and Φ are continuous round a latitude circle. Further it may be demonstrated, by expressing $P_n^m(v)$ in terms of the hypergeometric function, that $P_n^m(v)$ has a singularity at $v = -1$ if n is not an integer, and is zero at $v = -1$ if n is an integer. Thus it is clear that n can only take integer values.

A solution of eq. (13), pertinent to the problem under consideration is therefore

$$D_n = \sum_{n=-\infty}^{\infty} d_n^m \left(\frac{R}{a}\right)^n (\lambda^2 - v^2)^{-n/2} P_n^m(v).$$

Using the fact that $P^m_{-n-1}(v) = P^m_n(v)$ and that $P^m_n(v) = 0$ if $m < n$ this may be written in the form

$$D = \sum_{n=m}^{\infty} \left[d_n^m \left(\frac{R}{a}\right)^n (\lambda^2 - v^2)^{-n/2} + \bar{d}_n^m \left(\frac{R}{a}\right)^{-n-1} (\lambda^2 - v^2)^{(n+1)/2} \right] P_n^m(v)$$

where $\bar{d}_n^m = d^m_{-n-1}$

This solution, when expressed in this form, is totally unsuited to the satisfying of the boundary conditions which require the solution to be expressed in a series of surface harmonics at the top and bottom of the ocean. It is therefore necessary to express the solution in a series of Legendre functions of argument $\cos\theta$.

Substituting for v gives

$$(\lambda^2 - v^2)^{-n/2} P_n^m(v) = \left[\frac{\lambda^2 - \sin^2\theta}{\lambda^2(\lambda^2-1)}\right]^{n/2} P_n^m \left[\frac{\lambda\cos\theta}{\sqrt{\lambda^2 - \sin^2\theta}}\right]$$

and

$$(\lambda^2 - v^2)^{(n+1)/2} P_n^m(v) = \left[\frac{\lambda^2(\lambda^2-1)}{\lambda^2 - \sin^2\theta}\right]^{(n+1)/2} P_n^m \left[\frac{\lambda\cos\theta}{\sqrt{\lambda^2 - \sin^2\theta}}\right].$$

The factors $[\lambda^2(\lambda^2-1)]^{-n/2}$, and $[\lambda^2(\lambda^2-1)]^{(n+1)/2}$ may be absorbed into d_n^m and $\bar{d}_n^m$ respectively and it can then be shown that

$$(\lambda^2 - 1 + u^2)^{n/2} P_n^m \left[\frac{\lambda u}{\sqrt{\lambda^2 - 1 + u^2}}\right] \equiv \sum_{r=m}^{n} C(n,r,m)\, P_r^m(u)\, P_r^m(\lambda)$$

and

$$(\lambda^2 - 1 + u^2)^{-(n+1)/2} P_n^m \left[\frac{\lambda u}{\sqrt{\lambda^2 - 1 + u^2}}\right] \equiv \sum_{r=n}^{\infty} \bar{C}(n,r,m)\, P_r^m(u)\, Q_r^m(\lambda)$$

where $u = \cos\theta$,

$$C(n,r,m) \begin{cases} = \dfrac{(n+m)!\,(2r+1)\,(r-m)! \times (2n+1)\,(2n-1)\ldots(n+r+3)}{(2n+1)\,(2n-1)\ldots 5\times 3\times 1 \times (r+m)!\,(n-r)\ldots 4\times 2} & \text{if } n-r \text{ is even} \\ = 0 \quad \text{if } n-r \text{ is odd} \end{cases}$$

and

$$\bar{C}(n,r,m) \begin{cases} = \dfrac{(-1)^{(r-n)/2+m}(r-m)!\,(2r+1) \times (2n-1)\ldots 5\times 3\times 1 \times (2n+1)\,(2n+3)\ldots(r+n-1)}{(n-m)!\,(r+m)!\,2\times 4\times 6\ldots(r-n)} & \text{if } r-n \text{ is even} \\ = 0 \quad \text{if } r-n \text{ is odd}^1 \end{cases}$$

The constants $C(n, r, m)$ all contain the factor $\dfrac{(m+n)!}{(2n+1)(2n-1)\ldots 5\times 3\times 1}$ for any given n and m, this factor may therefore be taken outside the summation and absorbed into the constant d_n^m. Similarly the constants $\bar{C}(n, r, m)$ all contain the factor $\dfrac{(2n-1)(2n-3)\ldots 5\times 3\times 1}{(n-m)!}$ which may be absorbed into $\bar{d}_n^m$. Using these results in the solution for D, D may be written in the form

$$D = \sum_{n=m}^{\infty} \left[d_n^m \left(\frac{R}{a}\right)^n \sum_{r=0}^{X} C(n, n-2r, m)\, P^m_{n-2r}(\cos\theta) \times P^m_{n-2r}(\lambda) + \bar{d}_n^m \left(\frac{R}{a}\right)^{-1-n} \times \sum_{r=0}^{\infty} \bar{C}(n, n+2r, m)\, P^m_{n+2r}(\cos\theta)\, Q^m_{n+2r}(\lambda)\right]$$

$$\text{where } X \begin{cases} = \dfrac{n-m}{2} & \text{if } n-m \text{ is even} \\ = \dfrac{n-m-1}{2} & \text{if } n-m \text{ is odd} \end{cases}$$

This expression for D still does not present it in a series of surface harmonics at the top and bottom of the ocean although this may now be obtained merely by rearranging the series to the following form

$$D = \sum_{n=m}^{\infty} P_n^m(\cos\theta) \left[P_n^m(\lambda) \sum_{r=0}^{\infty} C(n+2r, n-2r, m) \times d^m_{n+2r} \left(\frac{R}{a}\right)^{n+2r} + Q_n^m(\lambda) \times \sum_{r=0}^{X} \bar{C}(n-2r, n+2r, m)\, \bar{d}^m_{n-2r} \left(\frac{R}{a}\right)^{-n+2r-1} \right] \qquad (16)$$

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[1] These results do not appear to be in the literature, hence an ad hoc derivation of the constants is given in the appendix.

The upper boundary condition involves C which is expressed in terms of D in eq. (12). Hence substituting the above solution for D in this expression, by using the recurrence relations

$$(n-m+1)\,P^m_{n+1}(\cos\theta) - (2n+1)\cos\theta\,P^m_n(\cos\theta) + (n+m)\,P^m_{n-1}(\cos\theta) = 0$$

$$\sin\theta\frac{d}{d\theta}P^m_n(\cos\theta) = (n-m+1)\,P^m_{n+1}(\cos\theta) - (n+1)\cos\theta\,P^m_n(\cos\theta),$$

and the equivalent relations for $P^m_n(\lambda)$ and for $Q^m_n(\lambda)$, C may be expressed in the following way

$$C = \frac{i}{2\omega(\lambda^2-1)\,a}\sum_{n=m}^{\infty} P^m_n(\cos\theta) \times \left\{\left[(1-\lambda^2)\frac{d}{d\lambda}P^m_n(\lambda) + mP^m_n(\lambda)\right]\sum_{r=0}^{\infty} G + \left[(1-\lambda^2)\frac{d}{d\lambda}Q^m_n(\lambda) + mQ^m_n(\lambda)\right]\sum_{r=0}^{X} H\right\}$$

where $G = C(n+2r, n-2r, m)\, d^m_{n+2r}\left(\frac{R}{a}\right)^{n+2r-1}$

$$H = \bar{C}(n-2r, n+2r, m)\,\bar{d}^m_{n-2r}\left(\frac{R}{a}\right)^{-n+2r-2}$$

C and D have thus been obtained in a form which presents them as a series of surface harmonics at the boundaries of the ocean.

The first boundary condition (14) that is $C = 0$ on $R = a - h$ gives

$$\sum_{n=m}^{\infty} P^m_n(\cos\theta)\left[I(n,m,\lambda) \times \sum_{r=0}^{\infty} C(n+2r, n-2r, m)\, d^m_{n+2r}\, x^{n+2r-1} + J(n,m,\lambda)\sum_{r=0}^{X}\bar{C}(n-2r, n+2r, m) \times \bar{d}^m_{n-2r} x^{-n+2r-2}\right] = 0,$$

where $I(n,m,\lambda) = (1-\lambda^2)\dfrac{d}{d\lambda}P^m_n(\lambda) + mP^m_n(\lambda)$

$$J(n,m,\lambda) = (1-\lambda^2)\frac{d}{d\lambda}Q^m_n(\lambda) + mQ^m_n(\lambda)$$

and $x = 1 - h/a$.

Using the orthogonality of the $P^m_n(\cos\theta)$ this becomes an infinite set of linear equations namely

$$\sum_{r=0}^{\infty} C(n+2r, n-2r, m)\, d^m_{n+2r}\, x^{n+2r-1} + Y(n,m,\lambda)\sum_{r=0}^{X}\bar{C}(n-2r, n+2r, m) \times x^{-n+2r-2}\,\bar{d}^m_{n-2r} = 0 \tag{17}$$

for all m, λ satisfying $\lambda^2 > 1$, and $n \geqslant m$,

where $Y(n, m, \lambda) = J(n, m, \lambda)/I(n, m, \lambda)$

In a similar way the second boundary condition (15) yields the following set of linear equations

$$\sum_{r=0}^{\infty} C(n+2r, n-2r, m)\, d^m_{n+2r} + Z(n,m,\lambda)\sum_{r=0}^{X}\bar{C}(n-2r, n+2r, m)\,\bar{d}^m_{n-2r} = 0 \tag{18}$$

for all m, λ and $n \geqslant m$, where

$$Z(n,m,\lambda) = \frac{4a\omega^2\lambda(\lambda^2-1)\,Q^m_n(\lambda) + g_n\left[(1-\lambda^2)\dfrac{d}{d\lambda}Q^m_n(\lambda) + mQ^m_n(\lambda)\right]}{4a\omega^2\lambda(\lambda^2-1)\,P^m_n(\lambda) + g_n\left[(1-\lambda^2)\dfrac{d}{d\lambda}P^m_n(\lambda) + mP^m_n(\lambda)\right]}$$

For any given values of m and λ, the infinite sets of homogeneous linear eqs. (17) and (18) in the d^m_n and $\bar{d}^m_n$ are consistent with each other only when the determinant of the coefficients is zero. As the coefficients are functions of h the determinant will be zero for only certain discrete values of h, the eigen depths. In general the determinant will be of infinite order and have an infinity of eigen depths, but only the fundamental and second modes of oscillation are investigated.

Having selected a particular value for m it becomes apparent that there are two possible types of oscillation which can take place. One type is associated with 'even' coefficients and the other type with 'odd' coefficients. It is clear from equations (17) and (18) that if n is chosen to be even the only coefficients to occur are of the type d_{2s}^m and $\bar{d}_{2s}^m$ where s is an integer and if n is odd the only coefficients will be of the type d_{2s+1}^m and $\bar{d}_{2s+1}^m$.

Solberg, in solving the symmetrical problem, considered only the oscillations associated with the even coefficients, but in the present work both types of oscillations will be considered. In order to illustrate the procedure, the case $m = 0$ with n even is considered in detail.

Putting $n = 0$, the first and second boundary conditions give respectively

$$x^{-1}d_0^0 + \frac{5}{2}xd_2^0 + \frac{9\times 7}{4\times 2}x^3d_4^0 + \frac{13\times 11\times 9}{6\times 4\times 2}x^5d_6^0 + \frac{17\times 15\times 13\times 11}{8\times 6\times 4\times 2}x^7d_8^0 + \dots + Y(0,0,\lambda)\,x^{-2}\bar{d}_0^0 = 0 \tag{19}$$

and

$$d_0^0 + \frac{5}{2}d_2^0 + \frac{9\times 7}{4\times 2}d_4^0 + \frac{13\times 11\times 9}{6\times 4\times 2}d_6^0 + \frac{17\times 15\times 13\times 11}{8\times 6\times 4\times 2}d_8^0 + \dots + Z(0,0,\lambda)\,\bar{d}_0^0 = 0. \tag{20}$$

Now in eq. (19) since $\frac{d}{d\lambda}P_0^0(\lambda) = 0$, $\bar{d}_0^0 = 0$. Hence from eq. (20)

$$d_0^0 + \frac{5}{2}d_2^0 + \frac{9\times 7}{4\times 2}d_4^0 + \frac{13\times 11\times 9}{6\times 4\times 2}d_6^0 + \dots = 0$$

which may be considered as merely an expression for d_0^0 in terms of d_2^0, d_4^0, etc. When $n = 2,4,6$ and 8 the first boundary condition gives respectively

$$xd_2^0 + \frac{9}{2}x^3d_4^0 + \frac{13\times 11}{4\times 2}x^5d_6^0 + \frac{17\times 15\times 13}{6\times 4\times 2}x^7d_8^0 + \dots + Y(2,0,\lambda)\,[x^{-4}\bar{d}_2^0] = 0 \tag{21}$$

$$x^3d_4^0 + \frac{13}{2}x^5d_6^0 + \frac{17\times 15}{4\times 2}x^7d_8^0 + \dots + Y(4,0,\lambda)\left[-\frac{5}{2}x^{-4}\bar{d}_2^0 + x^{-6}\bar{d}_4^0\right] = 0 \tag{22}$$

$$x^5d_6^0 + \frac{17}{2}x^7d_8^0 + \dots + Y(6,0,\lambda) \times \left[\frac{7\times 5}{4\times 2}x^{-4}\bar{d}_2^0 - \frac{9}{2}x^{-6}\bar{d}_4^0 + x^{-8}\bar{d}_6^0\right] = 0 \tag{23}$$

$$x^7d_8^0 + \dots + Y(8,0,\lambda)\left[-\frac{9\times 7\times 5}{6\times 4\times 2}x^{-4}\bar{d}_2^0 + \frac{11\times 9}{4\times 2}x^{-6}\bar{d}_4^0 - \frac{13}{2}x^{-8}\bar{d}_6^0 + x^{-10}\bar{d}_8^0\right] = 0 \tag{24}$$

and the second boundary condition gives respectively

$$d_2^0 + \frac{9}{2}d_4^0 + \frac{13\times 11}{4\times 2}d_6^0 + \frac{17\times 15\times 13}{4\times 6\times 2}d_8^0 + \dots + Z(2,0,\lambda)\,[\bar{d}_2^0] = 0 \tag{25}$$

$$d_4^0 + \frac{13}{2}d_6^0 + \frac{17\times 15}{4\times 2}d_8^0 + \dots + Z(4,0,\lambda)\left[-\frac{5}{2}\bar{d}_2^0 + \bar{d}_4^0\right] = 0 \tag{26}$$

$$d_6^0 + \frac{17}{2}d_8^0 + \dots + Z(6,0,\lambda)\left[\frac{7\times 5}{4\times 2}\bar{d}_2^0 - \frac{9}{2}\bar{d}_4^0 + \bar{d}_6^0\right] = 0 \tag{27}$$

$$d_8^0 + \dots + Z(8,0,\lambda)\left[-\frac{9\times 7\times 5}{6\times 4\times 2}\bar{d}_2^0 + \frac{11\times 9}{4\times 2}\bar{d}_4^0 - \frac{13}{2}\bar{d}_6^0 + \bar{d}_8^0\right] = 0 \tag{28}$$

As a first approximation it is assumed that d_4^0, d_6^0, etc. are all negligibly small compared with d_2^0. With this assumption eqs. (21) and (25) become a pair of homogeneous equations in d_2^0 and $\bar{d}_2^0$. The condition that these two equations are consistent is

$$\begin{vmatrix} x & Y(2,0,\lambda)\,x^{-4} \\ 1 & Z(2,0,\lambda) \end{vmatrix} = 0$$

i.e. $x^5 = Y(2, 0, \lambda)/Z(2, 0, \lambda)$.

This leads to a first approximation, h_1, to the depth h in which free oscillations of this type and period can exist.

Table 1. *Tabulation of the first, second, and third order approximations to the depth associated with the fundamental mode of oscillation when n is even and m = 0*

λ	h_1 km	h_2 km	h_3 km
1.1	11.46	13.27	13.28
1.3	19.99	21.03	21.04
1.5	29.47	30.14	30.15
1.7	40.14	40.65	40.66
2.0	58.52	58.87	58.88

The second approximation is found by retaining d_2^0 and d_4^0 and neglecting d_6^0, d_8^0 etc., the corresponding consistency condition being;

$$\begin{vmatrix} x & \frac{9}{2}x^3 & Y(2,0,\lambda)\,x^{-4} & 0 \\ 0 & x^3 & -\frac{5}{2}\,Y(4,0,\lambda)\,x^{-4} & Y(4,0,\lambda)\,x^{-6} \\ 1 & \frac{9}{2} & Z(2,0,\lambda) & 0 \\ 0 & 1 & -\frac{5}{2}Z(4,0,\lambda) & Z(4,0,\lambda) \end{vmatrix} = 0$$

which gives a second approximation, h_2, to h, the depth associated with the fundamental mode of oscillation, and a first approximation to the second mode. The next approximation, retaining an extra term in the infinite series, leads to a sixth order determinant. Solving this equation gives a third approximation to the fundamental mode, a second approximation to the second mode, and a first approximation to the third mode of oscillation. Significant differences are apparent between the first and second order approximations to the fundamental mode, but the differences between the second and third order approximations are of order 0.1 %. Hence in all cases the third order approximation, h_3, is assumed to be a good approximation to the depth associated with the fundamental mode of oscillation. The various approximations obtained for the fundamental and second modes of oscillation are listed in Tables 1 and 2 respectively.

Table 2. *Tabulation of the first and second order approximations to the depth associated with the second mode of oscillation when n is even and m = 0*

λ	h_1 km	h_2 km
1.1	2.56	3.29
1.3	5.15	5.58
1.5	7.93	8.22
1.7	11.02	11.23
2.0	16.27	16.43

Table 3. *Tabulation of the third order approximations to the depth in km associated with the fundamental mode of oscillation for various values of n and m*

λ	n even $m = 1$	n even $m = 2$	n odd $m = 2$	n even $m = 3$	n odd $m = 3$
−2.0	53.38	52.11	26.77	15.97	26.76
−1.7	35.89	35.70	18.60	11.01	18.60
−1.4	21.21	22.19	11.91	6.88	11.88
−1.1	9.38	11.64	6.55	3.62	6.57
1.1	18.39	24.34	9.15	5.46	11.16
1.3	26.69	33.37	13.16	7.82	15.38
1.5	36.25	43.76	17.85	10.56	20.25
1.7	47.43	55.51	23.19	13.69	25.77
2.0	66.55	75.70	32.42	19.11	35.28

The above procedure is repeated for $m = 1$, 2 and 3 and also for the cases in which n takes odd values. In all cases except two, the approximation to the fundamental mode is taken to the third order and the values obtained for h_3 are given in Table 3. The exceptions are the cases in which n takes odd values and $m = 0$ and 1, when the associated depths are unrealistically large, even for application to the atmosphere, so it was considered unnecessary labour to compute the higher approximations. Further, some of the second modes of oscillation are calculated for positive values of λ only and these results are shown in Table 4.

Table 4. *Tabulation of the second order approximations to the depth in km associated with the second mode of oscillation for various values of n and m*

λ	n even $m = 1$	n even $m = 2$	n odd $m = 2$
1.1	3.90	4.64	2.78
1.3	6.18	6.93	4.27
1.5	8.83	9.63	6.03
1.7	11.86	12.70	8.03
2.0	17.14	18.02	11.50

Table 5. *Comparison of the fundamental mode of oscillation, as given by the hydrostatic and the present theories, for various values of m and λ when n takes even values*

m	2λ	$h/a \times 10^3$ (Hough)	$h/a \times 10^3$ (Present)
0	± 2.4318	2.7682	2.77005
1	$+2.1611$	2.7682	2.77846
1	-2.6288	2.7682	2.76026
2	-2.5535	2.7682	2.75327

Throughout the foregoing work the physical constants are given the values $a = 6\,371$ km, $g = 9.81$ m s^{-2}, $\varrho/\varrho_s = 0.181$, and $\omega = 7.292 \times 10^{-5}$ rad s^{-1}. In order to facilitate a close comparison between the results obtained by Hough (1897, 1898) (who solved the problem using the assumption that the vertical accelerations may be neglected) and the present work, a further set of results is calculated using exactly the information used by him, namely $g/(4a\omega^2) = 72.25$, and $\varrho/\varrho_s = 0.18093$.

Unfortunately as a result of the restriction $\lambda^2 > 1$, the number of points at which a comparison can be made is limited,[1] nevertheless Tables 5 and 6 are of some interest. In this section of the work greater care is taken to ensure that the final numbers obtained are a true representation of the theory and are not marred by the approximation to the infinite set of linear equations which are obtained at the end of the theory. To this end the 6×6, 8×8 and 10×10 determinental approximations are evaluated to eight significant figures. It is noted that for the fundamental mode the 6×6 and 8×8 determinants give results which agree to at least six significant figures and the 8×8 and 10×10 determinants agree to at least eight significant figures. In the case of the second mode of oscillation, when values of λ close to unity are excluded (say $\lambda > 1.2$), the approximations obtained from the 6×6, 8×8 and 10×10 determinants agree to at least three and five figures respectively. The agreement falls to at least two and four figures respectively when $\lambda = 1.1$.

Further calculations are made for $\lambda = 1.01$ to investigate the behaviour of the solutions as the differential equation tends towards the 'change over' point of $\lambda^2 = 1$. Nothing striking appears to happen as far as the numerical values are concerned, the points obtained continue to lie on a smooth extension of the previously smooth curves, although the convergence obtained is slower. The agreement between the results from the 6×6 and 8×8 and the 8×8 and 10×10 determinants is to at least five and eight signifi-

Table 6. *Comparison of the second mode of ascillation, as given by the hydrostatic and the present theories, for various values of m and λ when n takes even values*

m	2λ	$h/a \times 10$ (Hough)	$h/a \times 10$ (Present)
0	2.3972	6.9204	6.92086
1	2.2907	6.9204	6.92776
2	2.1547	6.9204	6.93301
0	3.0794	13.8408	13.8401
1	2.9961	13.8408	13.8592
2	2.8856	13.8408	13.8736
0	4.1208	27.682	27.6894
1	4.0506	27.682	27.7271
2	3.9592	27.682	27.7700

[1] A more complete comparison of the two theories could probably be accomplished by comparing the comprehensive work of Longuet-Higgins (1968) with the present work. Even so, as no allowance is made by Longuet-Higgins for the change in the gravitational potential produced by the tides themselves and as the comparison with Hough's results gives a reasonable indication of the differences in the theories, the above exercise has not been attempted.

Table 7. *Tabulation of the percentage difference between the hydrostatic and the present results for the fundamental mode when the depth is approximately 17.6 km*

λ	$m = 0$	$m = 1$	$m = 2$
Positive	0.07	0.37	
Negative	0.07	-0.29	-0.64

Table 8. *Tabulation of the percentage difference between the hydrostatic and the present results for the second mode, λ taking only positive values*

h km (approximately)	$m = 0$	$m = 1$	$m = 2$
4.4	-0.01	0.11	0.18
8.8	0.01	0.13	0.24
17.6	0.02	0.16	0.32

cant figures respectively for the fundamental mode and to at least one and three figures respectively for the second mode of oscillation. In all of the results in which λ^2 is close to unity, the convergence of the approximations improves as m increases.

Conclusion

In studying the results of the calculations performed for comparison with the work of Hough (1897, 1898), Tables 7 and 8, which list the percentage difference between the theories, give a clearer indication of the trends which appear to exist. Firstly there is a confirmation of the 'prediction' by Hough (1897) that the agreement between the two theories would be less good as the ratio of the depth to the horizontal extent of the ocean measured by h/a, becomes larger. Secondly, on the evidence of the rather limited figures available, it would appear that the hydrostatic approximation deteriorates as m increases. This could be another manifestation of the first observation in that as m increases the horizontal extent of the oscillation is in effect diminished.

Appendix

(i) Assume that the function

$$(\lambda^2-1+u^2)^{n/2}P_n^m\left[\frac{\lambda u}{\sqrt{\lambda^2-1+u^2}}\right] \text{ can be}$$

expanded as a series of the associated Legendre functions in the following way

$$(\lambda^2-1+u^2)^{n/2}P_n^m\left[\frac{\lambda u}{\sqrt{\lambda^2-1+u^2}}\right] \equiv \sum_{r=m} C(n,r,m)\,P_r^m(u)\,P_r^m(\lambda) \tag{29}$$

Throughout the work the definition of $P_n^m(z)$ is assumed to be

$$P_n^m(z)=(z^2-1)^{m/2}\frac{d^m}{dz^m}P_n(z) \tag{30}$$

$$P_n^m\left[\frac{\lambda u}{\sqrt{\lambda^2-1+u^2}}\right] = \left[\frac{(\lambda^2-1)(u^2-1)}{\lambda^2-1+u^2}\right]^{m/2}\frac{d^m}{d\xi^m}P_n\left[\frac{\lambda u}{\sqrt{\lambda^2-1+u^2}}\right]$$

where $\xi=\dfrac{\lambda u}{\sqrt{\lambda^2-1+u^2}}$

Substituting into the identity (29) gives

$$(\lambda^2-1+u^2)^{(n-m)/2}\frac{d^m}{d\xi^m}P_n\left[\frac{\lambda u}{\sqrt{\lambda^2-1+u^2}}\right] \equiv \sum_{r=m} C(n,r,m)\frac{d^m}{du^m}P_r(u)\frac{d^m}{d\lambda^m}P_r(\lambda)$$

As this is an identity and true for all λ and u, let $u\to 1$, then

$$\lambda^{n-m}\left[\frac{d^m}{d\xi^m}P_n\left[\frac{\lambda u}{\sqrt{\lambda^2-1+u^2}}\right]\right]\Bigg|_{u=1} \equiv \sum_{r=m} C(n,r,m)\left[\frac{d^m}{du^m}P_r(u)\right]\Bigg|_{u=1}\frac{d^m}{d\lambda^m}P_r(\lambda)$$

i.e.

$$\lambda^{n-m}\frac{n!}{2^m m!\,(n-m)!}\,\frac{(n+m)!}{n!} \equiv \sum_{r=m}^{n} C(n,r,m)\frac{d^m}{d\lambda^m}P_r(\lambda)\frac{(r+m)!}{(r-m)!}\,\frac{1}{2^m m!} \tag{31}$$

(Magnus & Oberhettinger, 1943)

Now considering $z^n\equiv \Sigma A_r P_r(z)$.

Multiplying both sides by $P_r(z)\,dz$, integrating from $z=-1$ to $+1$, and assuming uniform convergence it is seen that

$$A_r=\frac{2r+1}{2}\int_{-1}^{+1} z^n P_r(z)\,dz,$$

$$\int_{-1}^{+1} z^n P_r(z)\,dz \begin{cases} =0 & \text{if } r>n \\ & \text{or if } r \text{ is such that } n-r \text{ is odd} \\ =\dfrac{n!\,(2n+1)\,(2n-1)\dots(n+r+3)}{(2n+1)\,(2n-1)\dots5\times3\times1 \times(n-r)\,(n-r-2)\dots6\times4\times2} & \text{if } r \text{ is such that } n-r \text{ is even.}\end{cases}$$

Hence $\lambda^n\equiv \sum_{r=p}^{n} A_r P_r(\lambda)$, differentiate m times with respect to λ

$$\frac{n!}{(n-m)!}\lambda^{n-m} \equiv \sum_{r=m}^{n} A_r \frac{d^m}{d\lambda^m} P_r(\lambda),$$

comparing this with (31) it is seen that

$$C(n,r,m) = \frac{(n+m)!\,(2r+1)\cdot(r-m)! \times (2n+1)\cdot(2n-1)\ldots(n+r+3)}{(2n+1)\,(2n-1)\ldots 5\times 3\times 1 \times (r+m)!\,(n-r)\ldots 4\times 2}$$

if r is such that $n-r=2s$, s an integer,

$$C(n,r,m) = 0 \quad \text{if} \quad n-r = 2s+1 \quad \text{or if} \quad r > n.$$

(ii) Assume

$$(\lambda^2 - 1 + u^2)^{(-n-1)/2} P_n^m\left[\frac{\lambda u}{\sqrt{\lambda^2-1+u^2}}\right] \equiv \sum C(n,r,m,\lambda)\, P_r^m(u).$$

Using the definition (30) from part (i)

$$(\lambda^2-1+u^2)^{-(n+m+1)/2}(\lambda^2-1)^{m/2} \times \frac{d^m}{d\xi^m} P_n\left[\frac{\lambda u}{\sqrt{\lambda^2-1+u^2}}\right] \equiv \frac{d^m}{du^m}\sum C(n,r,m,\lambda)\, P_r(u)$$

this is an identity true for all values of u, let $u \to 1$, then

$$\frac{(\lambda^2-1)^{m/2}(n+m)!}{\lambda^{n+m+1}2^m m!\,(n-m)!} \equiv \sum C(n,r,m,\lambda)\frac{(r+m)!}{2^m m!\,(r-m)!}. \tag{32}$$

Now it may be shown that (Whittaker & Watson, 1927)

$$\frac{1}{\lambda - x} \equiv \sum_{r=0}^{\infty} (2r+1)\, P_r(x)\, Q_r(\lambda),$$

differentiating m times with respect to λ, n times with respect to x and multiplying by $(\lambda^2-1)^{m/2}$ it is seen that

$$(\lambda^2-1)^{m/2}\frac{(-1)^m(m+n)!}{(\lambda-x)^{n+m+1}} \equiv \sum_{r=n}^{\infty}(2r+1)\frac{d^n}{dx^n}P_r(x)\, Q_r^m(\lambda).$$

Let $x \to 0$

$$(\lambda^2-1)^{m/2}\frac{(-1)^m(m+n)!}{\lambda^{n+m+1}} \equiv \sum_{r=n}^{\infty}(2r+1)\left[\frac{d^n}{dx^n}P_r(x)\right]\Bigg|_{x=0} Q_r^m(\lambda) \tag{33}$$

$$\text{But}\quad \left[\frac{d^n}{dx^n}P_r(x)\right]\Bigg|_{x=0} \begin{cases} = 0 & \text{when } r-n \text{ is odd or negative} \\ = \dfrac{(2r)!\,(-1)^{(r-n)/2}}{2^r r!\,2\times 4\ldots(r-n)\,(2r-1) \times (2r-3)\ldots(r+n+1)} & \text{when } r-n \text{ is even} \end{cases}$$

(Magnus & Oberhettinger, 1943)

Thus comparing (32) and (33) it can be seen that

$$C(n,r,m,\lambda) \begin{cases} = 0 & \text{if } r-n \text{ is odd or negative} \\ = Q_r^m(\lambda)\dfrac{(-1)^{(r-n)/2+m}(r-m)! \times (2r+1)\,(2n+1)\ldots 5\times 3\times 1 \times (2n+3)\,(2n+5)\ldots(r+n-1)}{(n-m)!\,(r+m)!\,2\times 4\times 6\ldots(r-n)} & \text{if } r-n \text{ is even} \end{cases}$$

Put $C(n, r, m, \lambda) = \bar{C}(n, r, m) Q_r^m(\lambda)$.

Acknowledgement

The author is very much indebted to Professor T. V. Davies for his considerable help and encouragement during the course of this work.

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ТРЕХМЕРНАЯ ЗАДАЧА ПРИЛИВОВ

Исследуется линейная трехмерная модель приливов в океане, покрывающем вращающуюся сферическую землю. Задача отличается от классической тем, что колебания описываются дифференциальным уравнением в частных производных, тип которого зависит от периода колебания, в то время как классическая задача сводится к решению обыкновенного дифференциального уравнения. Результаты этих двух теорий сравниваются там, где это возможно.