The influence of the "tidal stress" on the residual circulation

Application to the Southern Bight of the North Sea

By JACQUES C. J. NIHOUL and FRANCOIS C. RONDAY, *Institut de Mathématique, Université de Liège, Belgium*

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ABSTRACT

The residual current field in the sea is defined as the mean velocity field over a time sufficiently long to cancel transitory wind currents and tidal oscillations. The hydrodynamic equations governing the residual circulation are established and it is shown that, in the regions of intensive tides, the tidal motion has a cogent influence on the residual flow pattern. This effect which arises from the non linear terms is equivalent to the application of a "tidal stress" which combines with the wind stress to drive the water motion. The tidal stress is calculated in the Southern Bight of the North Sea from the results of a numerical tidal model and the residual circulation is computed. The comparison with the circulation obtained when neglecting the tidal stress shows determinant differences enforcing the theory.

Introduction

Frequently, when confronted with a complex system of partial differential equations, one investigates first the existence and characteristics of eventual steady state solutions. Hence one might find it surprising that so many of the first hydrodynamic models which were developed to study the circulation of coastal seas were concerned with wave motions, tides and storm surges and very little with stationary currents (e.g. Hansen, 1956; Brettschneider, 1967; Leenderste, 1967; Heaps, 1967, 1969).

One of the reasons for this situation is probably the insufficiency of present data on open sea boundaries. While boundary oscillations are reasonably simple to model, even with sparse experimental data, stationary currents cannot be specified without a detailed knowledge of their distribution along the frontier (Nihoul, 1973).

Another explanation may be that although steady state hydrodynamic models have been used successfully to compute wind-driven water circulation in lakes, the possible significance of steady state solutions for seas and oceans is not, at present, entirely clear. The physical meaning of steady currents is indeed far from obvious. It is generally agreed that water movements which have specially large characteristic times of variation can be approximated by time independent flows over any reasonable period and the name "residual currents" is broadly accepted to denote such slowly varying motions of the sea.

Different interpretations appear however when specialists undertake to describe residual currents mathematically, simulate them or identify pertinent experimental data. From a mathematical point of view, it is tempting to define the residual currents as the steady flow pattern which is described by the fundamental equations reduced by assuming no time dependence and, consequently, zero derivatives with respect to time.

Experimentalists however often prefer to regard the residual currents as the residuary flow obtained by subtracting from the actual fluid motion the (computed) main tidal currents (e.g. Otto, 1970).

Hydrodynamicists have a still different notion. They define the residual currents as mean currents over a time sufficiently long to cover several tidal periods and thus cancel out most of the tidal contributions.

The hydrodynamicist's point of view is apparently the more realistic.

One can indeed object to the experimentalist's interpretation that subtracting, from the observed values of the actual currents, uncertain calculated values of the tidal currents presumably worsens the experimental errors by the additional inexactitudes of the calculus. It is not clear moreover how time dependent wind currents are eliminated in the process and the stationary (or almost stationary) character of the result is not obvious.

Simple steady state solutions of the fundamental equations appear to deserve similar criticisms. Surely steady state solutions require steady state forcing and this implies some sort of long time average of at least the wind field.

Along the same lines it seems logical to regard the residual current field as the mean field over a time sufficiently long to cancel, to a large extent, transitory wind currents and tidal oscillations.

The residual currents, defined in this way, can only vary very slowly with time and it is reasonable to describe them by steady state equations. These equations however cannot be obtained directly by dropping the time derivative in the time dependent hydrodynamic equations. One must first average these equations over time. The average equations will have the same form as far as the linear terms are concerned but they will contain additional contributions from the non-linear terms.

These contributions are discussed here and it is shown that they can have a cogent influence on the residual circulation.

Depth-averaged equations of marine hydrodynamics

Almost all existing hydrodynamic models of marine circulation are depth-averaged two-dimensional models (e.g. Fortak, 1962; Hansen, 1956; Heaps, 1969; Ronday, 1972*b*).

Three-dimensional models have not been much attempted yet although recently Heaps (1972) suggested a method—basically replacing straight integration by integral transform over depth—by which in principle the depth variations could be recovered at the end.

If

$$x_3 = -h \tag{1}$$

and

$$x_3 = \zeta \tag{2}$$

are the equations of the bottom and the free surface, respectively and if x_1 and x_2 denote rectangular coordinates in a horizontal plane, one can describe the depth-averaged motion of the sea in terms of the mean velocity $\bar{\mathbf{u}}$ or the total flow-rate $\mathbf{U}$ defined by

$$\mathbf{U} = U_1\mathbf{e}_1 + U_2\mathbf{e}_2 = H\bar{\mathbf{u}} \tag{3}$$

$$U_i = \int_{-h}^{\zeta} u_i\,dx_3 \quad (i = 1, 2) \tag{4}$$

where u_i are the components of the current velocity vector[1] $\mathbf{u}$ and where H is the total depth, i.e.

$$H = h + \zeta \tag{5}$$

In terms of $\mathbf{U}$ (which will turn out to be more convenient for the study of residual currents) the basic equations can be written (e.g. Nihoul, 1975):

$$\frac{\partial H}{\partial t} + \nabla \cdot \mathbf{U} = 0 \tag{6}$$

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot (H^{-1}\mathbf{U}\mathbf{U}) + f\,\mathbf{e}_3 \wedge \mathbf{U}$$
$$= H\left[\boldsymbol{\xi} - \nabla\left(\frac{p_a}{\varrho} + g\zeta\right)\right] + a\nabla^2\mathbf{U} - \frac{D}{H^2}\mathbf{U}\|\mathbf{U}\| + \boldsymbol{\tau}_w \tag{7}$$

where f is the Coriolis parameter, $H\xi$ the external—tide-producing—force acting on a water column of unit cross section, p_a is the atmospheric pressure, a is the appropriate eddy viscosity and $\boldsymbol{\tau}_w$ is the specific wind stress (wind stress divided by the water density). The third term in the right-hand side represents the bottom friction and D is an empirical friction coefficient.

Steady state equations for the residual currents

To derive appropriate equations for the residual circulation, one now averages eqs. (6)

[1] By "current velocity vector" **u**, one means the ensemble average of the actual velocity vector **v**, cleared of the turbulent fluctuations whose dispersive effect, through non-linear terms, will be accounted for by a general dispersion term in the equations with the help of an eddy viscosity.

and (7) over some suitably long period of time (for instance, if the non steady flow is essentially due to tidal motions, the averaging time will comprise at least one and preferably several tidal periods). In agreement with the arguments presented in the introduction, the corresponding time average of $\mathbf{U}$ is regarded as representing the residual circulation. In a first approach, it is assumed independent of time.

Time averages being denoted by a subscript 0, one may write:

$$\mathbf{U} = \mathbf{U}_0 + \mathbf{U}_1 \tag{8}$$

$$H = H_0 + \zeta_1$$

$$H_0 = h + \zeta_0 \sim h \tag{9}$$

One may assume

$$(\mathbf{U}_1)_0 \sim 0 \tag{10}$$

$$(\zeta_1)_0 \sim 0 \tag{11}$$

Thus, averaging a linear term, one eliminates all contributions from $\mathbf{U}_1$ and ζ_1. This is not true however for the non-linear terms because mean products of the type $(\mathbf{U}_1\mathbf{U}_1)_0$ are not zero.

Such contributions will not be important if tidal and transitory wind currents are small. In certain regions like the North Sea however, as a consequence of intensive tidal oscillations, $\mathbf{U}_1$ can be 10 to 100 times larger than $\mathbf{U}_0$.

The tidal motion can then have a determinant influence on the residual circulation through the effect of the non-linear terms. Attention will now be restricted to this type of situation, with more specifically the case of the North Sea in mind. In the next section the analysis will be illustrated by an application to the Southern Bight.

After averaging, the surface elevation gives rise to two contributions, indeed

$$(gH\nabla\zeta)_0 = gH_0\nabla\zeta_0 + (g\zeta_1\nabla\zeta_1)_0. \tag{12}$$

The tidal elevation ζ_1 is less than H_0 but larger than ζ_0. Taking the respective horizontal length scales of variations of the residual and tidal currents into consideration, one finds that the two terms can be comparable and are indeed of the same order of magnitude in the Southern Bight (Ronday, 1972*a*, *b*).

Similarly, using estimates from Ronday's studies, one can see that the dominant contribution from the term $\nabla.(H^{-1}\mathbf{UU})$ is

$$[\nabla.(H^{-1}\mathbf{UU})]_0 \sim h^{-1}\nabla.(\mathbf{U}_1\mathbf{U}_1)_0 \tag{13}$$

Other contributions, and in particular those containing the residual flow rate $\mathbf{U}_0$, are found much smaller than the terms in (12).

By the continuity eq. (6), one can estimate that the tidal flow rate U_1 is of the order of the tidal elevation multiplied by the tidal phase velocity, i.e.

$$U_1 \simeq (gh)^{\frac{1}{2}}\zeta_1$$

Comparing the right-hand sides of eqs. (12) and (13), one finds then

$$\frac{(g\zeta_1\nabla\zeta_1)_0}{h^{-1}\nabla\cdot(\mathbf{U}_1\mathbf{U}_0)_0} \sim gh\frac{\zeta_1^2}{U_1^2} \sim 1$$

The bottom friction term is not a simple quadratic term as it contains the product of $\mathbf{U}$ by the norm $\|\mathbf{U}\|$. As a result, the dominant contribution is found to be

$$(DH^{-2}\mathbf{U}\|\mathbf{U}\|)_0 \sim Dh^{-2}\mathbf{U}_0\|\mathbf{U}_1\|_0$$

i.e. the average bottom stress is, in first approximation, *linear* in the residual flow rate; a fact that had already been noted by several authors (e.g. Groen & Groves, 1966).

The second term in the right-hand side of (12) and the term in the right-hand side of (13) are independent of $\mathbf{U}_0$. They combine in an external forcing on the residual circulation. Their effect can be visualized as that of a "tidal stress" which adds to the wind stress to produce the residual motion. One shall denote this tidal stress by $\boldsymbol{\tau}_t$, i.e.

$$\boldsymbol{\tau}_t = -[g\zeta_1\nabla\zeta_1 + \nabla\cdot(H^{-1}\mathbf{UU})]_0 \tag{14}$$

and write, in brief

$$\boldsymbol{\vartheta} = (\boldsymbol{\tau}_w)_0 + \boldsymbol{\tau}_t \tag{15}$$

Inasmuch as the average wind stress must be determined from observations or atmospheric models, the additional tidal stress $\boldsymbol{\tau}_t$ must be estimated from experimental data or tidal and transient circulation models.

In practice, the time average of the dispersion term $a\nabla^2\mathbf{U}$ is small compared with the contributions from the bottom friction, the elevation gradient and the Coriolis effect. If this term is neglected also, the averaged (steady

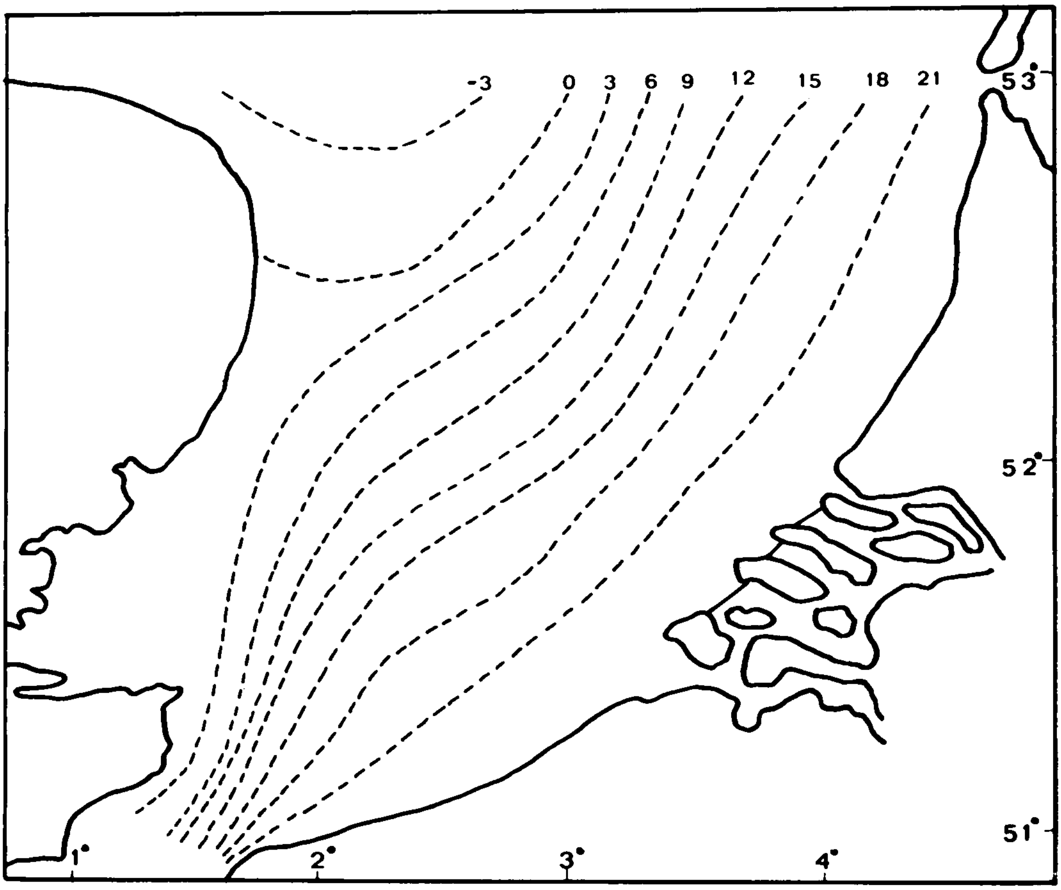

Fig. 1. Residual circulation in the Southern Bight without tidal stress. Streamlines ψ = const. (in 10^4 m^3/s). Bottom friction coefficient $K = 3\ 10^{-3}$ m/sec.

state) hydrodynamic equations can be written (taking $\xi_0 \sim 0$, in the present context):

$$f\,\mathbf{e}_3 \wedge \mathbf{U}_0 = -H_0 \nabla \left(\frac{p_a}{\varrho} + g\zeta_0\right) - KH_0^{-1}\mathbf{U}_0 + \boldsymbol{\vartheta} \quad (16)$$

where

$$K = D\|\bar{\mathbf{u}}_1\|_0 \quad (17)$$

is a new friction coefficieng.

In addition, averaging the continuity equations one obtains

$$\nabla \cdot \mathbf{U}_0 = 0 \quad (18)$$

Hence, the two components of the vector $\mathbf{U}_0$ can be derived from a stream function Ψ such that

$$U_{0,1} = -\frac{\partial \Psi}{\partial x_2} \quad (19)$$

$$U_{0,2} = \frac{\partial \Psi}{\partial x_1} \quad (20)$$

Dividing eq. (16) by H_0 and taking the curl to eliminate the surface elevation, one obtains

$$K\nabla^2\Psi - \frac{\partial \Psi}{\partial x_1}\left(f\frac{\partial H_0}{\partial x_2} + \frac{2K}{H_0}\frac{\partial H_0}{\partial x_1}\right) + \frac{\partial \Psi}{\partial x_2} \times \left(f\frac{\partial H_0}{\partial x_1} - \frac{2K}{H_0}\frac{\partial H_0}{\partial x_2}\right) = H_0\omega_3 + \frac{\partial H_0}{\partial x_2}\vartheta_1 - \frac{\partial H_0}{\partial x_1}\vartheta_2 \quad (21)$$

where

$$\omega_3 = (\nabla \wedge \boldsymbol{\vartheta})_3 \quad (22)$$

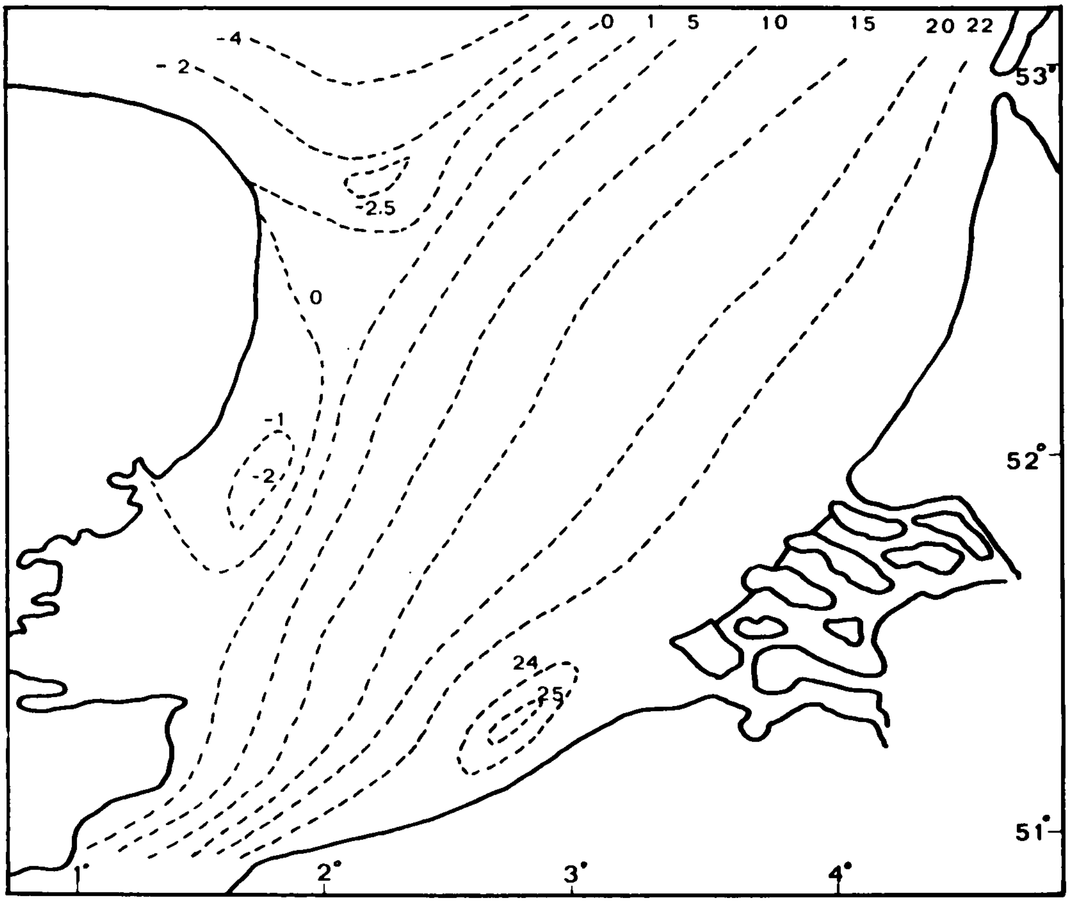

Fig. 2. Residual circulation in the Southern Bight with the tidal stress. Streamlines ψ = const. (in 10^4 m^3/s). Bottom friction coefficient $K = 3\ 10^{-3}$ m/sec.

The problem thus reduces to a boundary value problem. The solution for ψ depends on the boundary condition.

The Southern Bight is limited by coasts and by open sea boundaries: two kinds of conditions are used.

(i) *Along coasts.* One assumes that the water transport across the coast is equal to zero. The stream function ψ is then a constant along a coastal line.

(ii) *Along open sea boundaries.* The distribution of the current across the Dover Strait is approximated by a linear interpolation of ψ from zero at the British Coast to 2.38 $10^5\,m^3/sec$ at the French Coast (Cartwright, 1961). From Ramster (1965) and Otto (1970) data one determines the distribution of ψ along the northern boundary of the model (Fig. 1).

Application to the Southern Bight of the North Sea

Equations like (21) were derived by several authors previously. The essential difference is that they did not contain the effect of the tidal stress τ_t.

To illustrate the importance of this effect, it has seemed interesting to treat a practical case and eq. (21) has been applied to the computation of the residual circulation in the Southern Bight of the North Sea.

To emphasize the influence of the tidal stress, the following simple situation was considered: the average wind stress was assumed negligible. In the absence of the tidal stress, the residual flow pattern in the Southern Bight is then determined by the inflow and outflow of water at the open sea boundaries and by the bottom topography. The result of the computation is shown in Fig. 1 which is very similar to the one deduced by Ramster (1965) with Woodhead sea-bed drifters.

Fig. 2 shows the corresponding residual flow pattern taking into account the tidal stress $\boldsymbol{\tau}_t$. The tidal stress was computed numerically using the predictions of Ronday's model of tidal circulation in the North Sea (Ronday, 1972*b*). Closed streamlines appear in front of the English and Belgian coasts, figuring what might be compared to secondary flows in a suddenly divergent channel.

Due to the ellipticity of eq. (21) the spatial distribution of ψ depends on the value of K. If K is very large (10^{-2} m/sec) the stream lines are regular and the secondary flows induced by the bottom topography and by the tidal stress are reduced. If one introduces a small bottom friction coefficient (10^{-4} m/sec) the stream lines are no longer regular and one finds many local gyres.

The tidal current amplitude is of the order of 1 m/sec and the classical value of the bottom friction coefficient is 3 10^{-3}. The adopted value of K in the model is thus 3 10^{-3} m/sec.

The existence of such "secondary" circulation seems to explain several puzzling aspects of sediments transport and deposition off the Belgian coast and interesting new results in this matter will hopefully be available for future publication. The authors have now undertaken the study of the residual circulation in the whole North Sea taking into account the effect of the tidal stress computed from tidal models. The results will be reported later.

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ВЛИЯНИЕ «ПРИЛИВНОГО НАПРЯЖЕНИЯ» НА ОСТАТОЧНУЮ ЦИРКУЛЯЦИЮ. ПРИМЕНЕНИЕ К ЮЖНОЙ БУХТЕ СЕВЕРНОГО МОРЯ

Остаточное поле течения в море определяется как среднее поле скоростей за время, достаточно долгое для того, чтобы исключить переходные ветровые течения и приливные колебания. Устанавливаются гидродинамические уравнения, определяющие остаточную циркуляцию, и показывается, что в областях интенсивных приливов приливное движение имеет неоспоримое влияние на картину остаточного течения. Этот эффект, возникающий из-за нелинейных членов, эквивалентен применению «приливного напряжения», складывающегося с напряжением ветра, для приведения воды в движение. Приливное напряжение вычисляется для Южной бухты Северного моря из численной модели приливов и затем находится остаточная циркуляция. Ее сравнение с циркуляцией, полученной в пренебрежении приливным напряжением, показывает их определенную разницу, что укрепляет данную теорию.