Zonal and eddy forms of the available potential energy equations in pressure coordinates

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ABSTRACT

An approximation to the zonal and eddy forms of the available potential energy equations is developed from the exact equations in pressure coordinates. These equations preserve the form and symmetry of the exact equations to a large extent and reduce without approximation to the exact equations when the fields are purely zonal. The expressions for zonal and eddy available potential energy are applied to long-term atmospheric circulation statistics for the globe for the December–February season. The values of A_Z and A_E thus calculated differ somewhat from those obtained using Lorenz's approximate equations.

1. Introduction

Since its elaboration by Lorenz (1955*a*), the concept of available potential energy has been widely and fruitfully used in general circulation studies. Lorenz developed the "exact" form of the equations in isentropic coordinates and he also developed an approximation to these equations in pressure coordinates. These equations, applicable to the entire atmosphere, have also been modified for use with data from limited regions by the inclusion of appropriate boundary terms (e.g. Muench, 1965).

The exact equations in isentropic coordinates were derived and discussed by Dutton & Johnson (1967) and were applied to data without recourse to the pressure coordinates. Smith (1969) developed the exact equations in pressure coordinates and applied them to data from a portion of the Northern Hemisphere.

Most studies involving observational data have made use of Lorenz's approximate equations in pressure coordinates and have, following his original paper, decomposed them further into zonal and eddy forms. In studies using the exact equations, in either isentropic or isobaric coordinates, this decomposition has not been used. It is desirable to have a form of the available potential energy equations which includes both zonal and eddy forms of energy, which is suitable for use with observational data on pressure surfaces, and which preserves the form of the exact equations in pressure coordinates as far as possible.

2. Available potential energy

Following Smith (1969) the equation for the available potential energy in pressure coordinates may be written as

$$A = C_p \int \left(\frac{p^\varkappa - p_r^\varkappa}{p_s^\varkappa} \right) \theta \, dm \tag{1}$$

where the integration is over the entire mass of the atmosphere, p_s is the standard pressure of 1 000 mb, $\varkappa = R/C_p$ and the remaining symbols have their usual meteorological meanings.

The reference pressure p_r is defined as

$$p_r(\theta, t) = \frac{1}{\sigma} \iint_{\theta_T}^{\theta} \frac{\partial p}{\partial \theta} d\theta \, d\sigma \tag{2}$$

where σ is the area of the earth and θ_T is the potential temperature at the top of the atmosphere.

It follows from Poisson's equation applied to the reference state that

$$\theta = T_r \left(\frac{p_r}{p_s} \right)^\varkappa \tag{3}$$

and that eq. (1) may be written in any of the following ways

$$A = \int C_p M\theta \, dm = \int C_p NT \, dm$$

$$= \int C_p(T - T_r) \, dm \tag{4}$$

where $M = (p^\varkappa - p_r^\varkappa)/p_s^\varkappa$, $N = 1 - (p_r/p)^\varkappa$. N has been called the efficiency factor by Lorenz (1955*b*) where it appears in the generation term for available potential energy. Perhaps the third form of the equation expresses most clearly the definition of available potential energy as the difference between the total potential energy and that of the reference state possessing the minimum total potential energy.

Making use of Smith's result that $dp_r/dt = 0$, and the thermodynamic equation in isobaric coordinates, one obtains

$$\frac{d}{dt} C_p M\theta = \frac{d}{dt} C_p NT = \frac{d}{dt} C_p(T - T_r)$$

$$= \omega\alpha + \left(\frac{\theta}{T}\right) MQ = \omega\alpha + NQ \tag{5}$$

whence, integrating over the mass of the atmosphere,

$$\frac{\partial A}{\partial t} = G - C \tag{6}$$

where

$$G = \int \left(\frac{\theta}{T}\right) MQ \, dm = \int NQ \, dm \tag{7}$$

is the rate of generation of available potential energy and

$$C = -\int \omega\alpha \, dm \tag{8}$$

is the rate of conversion of available potential energy to kinetic energy.

The efficiency factor N represents the effectiveness of diabatic processes in the production (or destruction) of available potential energy. In this formulation the efficiency factor also appears in the specification of the available potential energy where it represents the relative contribution of the local temperature to the available potential energy.

3. Zonal and eddy components

The expressions (4), (7), (8) may be evaluated in pressure coordinates without approximation. The corresponding terms in Lorenz's approximate equations in pressure coordinates are

$$A \simeq \int \tfrac{1}{2} C_p \gamma_L T''^2 dm \tag{9}$$

$$G \simeq \int \gamma_L Q'' T'' dm \tag{10}$$

where $T'' = T - \tilde{T}$, $Q'' = Q - \tilde{Q}$. The tilde represents an average of the quantity over a pressure surface and the double prime the deviation therefrom. The stability factor γ_L is given by

$$\gamma_L = \left(\frac{\theta}{T}\right)\left(\frac{-\varkappa}{p}\left(\frac{\partial\tilde{\theta}}{\partial p}\right)^{-1}\right) = \frac{\Gamma_d}{\tilde{T}(\Gamma_d - \tilde{\Gamma})} \tag{11}$$

where $\Gamma_d = g/C_p$ is the dry adiabatic lapse rate and $\Gamma = -\partial T/\partial z$.

When T is decomposed into its zonally averaged and eddy parts in eq. (9)

$$[T''^2] = [T]''^2 + [T^{*2}]$$

The zonal and eddy forms of the available potential energy are identified with the terms dependent on the zonal average and the deviation therefrom to give

$$A_Z = \int \tfrac{1}{2} C_p \gamma_L [T]''^2 dm \tag{12}$$

$$A_E = \int \tfrac{1}{2} C_p \gamma_L [T^{*2}] \, dm \tag{13}$$

Similarly

$$G_Z = \int \gamma_L [Q]''[T]'' dm \tag{14}$$

$$G_E = \int \gamma_L [Q^* T^*] \, dm \tag{15}$$

Conversion terms between the zonal and eddy forms of available potential energy and between available potential and kinetic energy arise in this decomposition. The equations for A_z and A_E become (Lorenz, 1967)

$$\frac{\partial A_z}{\partial t} = G_Z - C_Z - C_A \tag{16}$$

$$\frac{\partial A_E}{\partial t} = G_E - C_E + C_A \tag{17}$$

where

$$C_Z = -\int [\omega][\alpha]\, dm \tag{18}$$

$$C_E = -\int [\omega^*\alpha^*]\, dm \tag{19}$$

$$C_A = -\int C_p\left(\frac{\theta}{T}\right)\left\{\frac{[v^*T^*]''}{a}\frac{\partial}{\partial\varphi} + [\omega^*T^*]''\frac{\partial}{\partial p}\right\} \times \left\{\frac{T}{\theta}\gamma_L[T]''\right\} \tag{20}$$

Here square brackets denote a zonal average and asterisks the deviation from the zonal average. Thus

$$[T] = \frac{1}{2\pi}\int_0^{2\pi} T\, d\lambda$$

$$T^* = T - [T]$$

and

$$T'' = T - \tilde{T} = [T]'' + T^*$$

The averages and the deviations are calculated on constant pressure surfaces.

If the fields of T and Q were purely zonal, eqs. (4) and (7) would represent exact expressions for the zonal available potential energy and its generation while eqs. (12) and (14) would still be approximate. It is desirable to obtain approximate expressions for the zonal and eddy available potential energy equations in pressure coordinates which reduce to the exact equations in the zonal case and which retain the form and the symmetry of the exact equations.

4. Derivation of the zonal and eddy equations

Consider the expression for the available potential energy as given in eq. (4). The reference pressure, p_r, is a function of θ and hence of p and T. The potential temperature is decomposed into its zonal and eddy components, $\theta = [\theta] + \theta^*$ and $p_r^\varkappa$ and M are expanded in the Taylor series:

$$p_r^\varkappa = p_r^\varkappa\Big|_0 + \varkappa p_r^{\varkappa-1}\frac{\partial p_r}{\partial\theta}\Big|_0 \theta^* + \left\{\varkappa(\varkappa-1)\, p_r^{\varkappa-2}\left(\frac{\partial p_r}{\partial\theta}\right)^2 + \varkappa p_r^{\varkappa-1}\frac{\partial^2 p_r}{\partial\theta^2}\right\}\Big|_0 \frac{\theta^2}{2} + \dots \tag{21}$$

where

$$\frac{\partial p_r}{\partial\theta} = \frac{-p_r\Gamma_d}{\varkappa\theta(\Gamma_d - \Gamma_r)}$$

$$\frac{\partial^2 p_r}{\partial\theta^2} = \frac{\partial p_r}{\partial\theta}\left(\frac{1}{p_r}\frac{\partial p_r}{\partial\theta} - \frac{1}{\theta} + \frac{\partial\Gamma_r/\partial\theta}{\Gamma_d - \Gamma_r}\right) \tag{22}$$

and $\Gamma_r = -\partial T_r/\partial z$, the lapse rate of temperature for the reference state. The zero subscript implies the expression is evaluated at $\theta = [\theta]$.

Ignoring the small term involving $\partial\Gamma_r/\partial\theta$ in the expression for $\partial^2 p_r/\partial\theta^2$, and substituting eq. (21) in M, one obtains, neglecting higher order terms,

$$[M([\theta]+\theta^*)] = \left[\left(\frac{p}{p_s}\right)^\varkappa - \left(\frac{p_r}{p_s}\right)^\varkappa\Big|_0\left\{1 + \left(\frac{\varkappa}{p_r}\frac{\partial p_r}{\partial\theta}\right)^2\Big|_0 \frac{[\theta^{*2}]}{2}\right\}\right][\theta] - \left(\frac{p_r}{p_s}\right)^\varkappa\Big|_0\left(\frac{\varkappa}{p_r}\frac{\partial p_r}{\partial\theta}\right)\Big|_0 \frac{[\theta^{*2}]}{2}$$

The term multiplying $[\theta]$ is identified with the zonal available potential energy and the remaining term with the eddy available potential energy. Since the second term in the cursive brackets is small compared to one,

$$\left(\frac{\varkappa}{p_r}\frac{\partial p_r}{\partial\theta}\right)^2\Big|_0 \frac{[\theta^{2*}]}{2} \sim \frac{[\theta^{*2}]}{[\theta]^2} \ll 1$$

it may be neglected to good approximation.

The zonal and eddy forms of the available potential energy in this approximation then become

$$A_Z \simeq \int C_p[M][\theta]\,dm = \int C_p[N][T]\,dm \tag{23}$$

$$A_E \simeq \int \tfrac{1}{2} C_p \gamma [T^{*2}]\,dm \tag{24}$$

where

$$\gamma = \left(\frac{\theta}{T}\right)\left(\frac{p_r}{p}\right)^{\varkappa}\Bigg|_0 \left(\frac{-\varkappa}{p_r}\frac{\partial p_r}{\partial \theta}\right)\Bigg|_0$$

Square brackets about M and N are used for notational convenience. The terms are defined as:

$$[M] = (p^{\varkappa} - p_r^{\varkappa}|_0)/p_s^{\varkappa}$$

$$[N] = 1 - (p_r|_0/p)^{\varkappa}$$

Eq. (23) resembles the exact eq. (4) for available potential energy to which it reduces when the fields are purely zonal. Eq. (24) for A_E resembles Lorenz's eq. (14) with γ_L replaced by a slightly different expression, although one of the same form. The new stability factor γ will vary with latitude while γ_L does not.

A similar evaluation gives

$$\begin{aligned}[M([Q]+Q^*)] = &\left[\left(\frac{p}{p_s}\right)^{\varkappa} - \left(\frac{p_r}{p_s}\right)^{\varkappa}\Bigg|_0 \right.\\ &\times \left.\left\{1 + \left(\left(\frac{\varkappa}{p_r}\frac{\partial p_r}{\partial \theta}\right)^2\Bigg|_0 - \left(\frac{\varkappa}{\theta p_r}\frac{\partial p_r}{\partial \theta}\right)\Bigg|_0\right)\frac{[\theta^{*2}]}{2}\right\}\right]\\ &\times [Q] - \left(\frac{p_r}{p_s}\right)^{\varkappa}\Bigg|_0 \left(\frac{\varkappa}{p_r}\frac{\partial p_s}{\partial \theta}\right)\Bigg|_0 [\theta^* Q^*]\end{aligned}$$

so that the zonal and eddy generation terms are approximated as

$$G_Z \simeq \int \left(\frac{\theta}{T}\right)[M][Q]\,dm = \int [N][Q]\,dm \tag{25}$$

$$G_E \simeq \int \gamma [Q^* T^*]\,dm \tag{26}$$

The remaining terms in the zonal and eddy available potential energy eqs. (16) and (17) may be obtained as follows.

The equation

$$\frac{d[M]}{dt} = \left(\frac{p}{p_s}\right)^{\varkappa}\frac{\varkappa\omega}{p}.$$

follows from the fact that $dp_r/dt = 0$.

Multiplying this equation by $C_p[\theta]$ and the thermodynamic equation by $[M]$, combining the two, and integrating over the mass of the atmosphere gives eq. (16) where C_Z is given by eq. (18), G_Z by eq. (25) and

$$\begin{aligned}C_A &= -\int C_p\left(\frac{[v^*\theta^*]}{a}\frac{\partial}{\partial\varphi} + [\omega^*\theta^*]\frac{\partial}{\partial p}\right)[M]\,dm\\ &= -\int C_p\left(\frac{\theta}{T}\right)\left(\frac{[v^*T^*]}{a}\frac{\partial}{\partial\varphi} + [\omega^*T^*]\frac{\partial}{\partial p}\right)\\ &\qquad\times\left(\frac{[N][T]}{[\theta]}\right)dm\\ &= -\int C_p\left(\frac{\theta}{T}\right)\left(\frac{[v^*T^*]}{a}\frac{\partial}{\partial\varphi} + [\omega^*T^*]\frac{\partial}{\partial p}\right)\\ &\qquad\times\left(\frac{[T]-T_r|_0}{[\theta]}\right)dm\end{aligned} \tag{28}$$

In this approximation a conversion of A_Z to A_E is represented by the eddy flux of heat down the gradient of what might be called the "local zonal available potential energy density" (inversely weighted by the potential temperature).

The equation for A_E may be obtained by subtraction from eq. (6) after decomposing it into zonal and eddy parts with the use of eqs. (23–26) such that

$$\frac{\partial}{\partial t}(A_Z + A_E) = -(C_Z + C_E) + (G_Z + G_E)$$

The result may also be obtained by multiplying the thermodynamic equation by $(T/\theta)^2\gamma\theta^*$, neglecting triple correlations, integrating, and making use of the relations;

$$C_p\frac{[v^*\theta^*]}{a}\frac{\partial[M]}{\partial\varphi} = C_p\left(\frac{T}{\theta}\right)^2\gamma\frac{[v^*\theta^*]}{a}\frac{\partial[\theta]}{\partial\varphi}$$

$$C_p[\omega^*\theta^*]\frac{\partial[M]}{\partial p} = C_p\left(\frac{T}{\theta}\right)^2\gamma[\omega^*\theta^*]\frac{\partial[\theta]}{\partial p} + [\omega^*\alpha^*]$$

The proposed new approximation to the zonal and eddy available potential energy equations is therefore specified by the expressions (18) and (19) for the conversions between zonal and eddy available potential energy and kinetic energy; eqs. (23–26) for the amount and rate of generation of zonal and eddy available po-

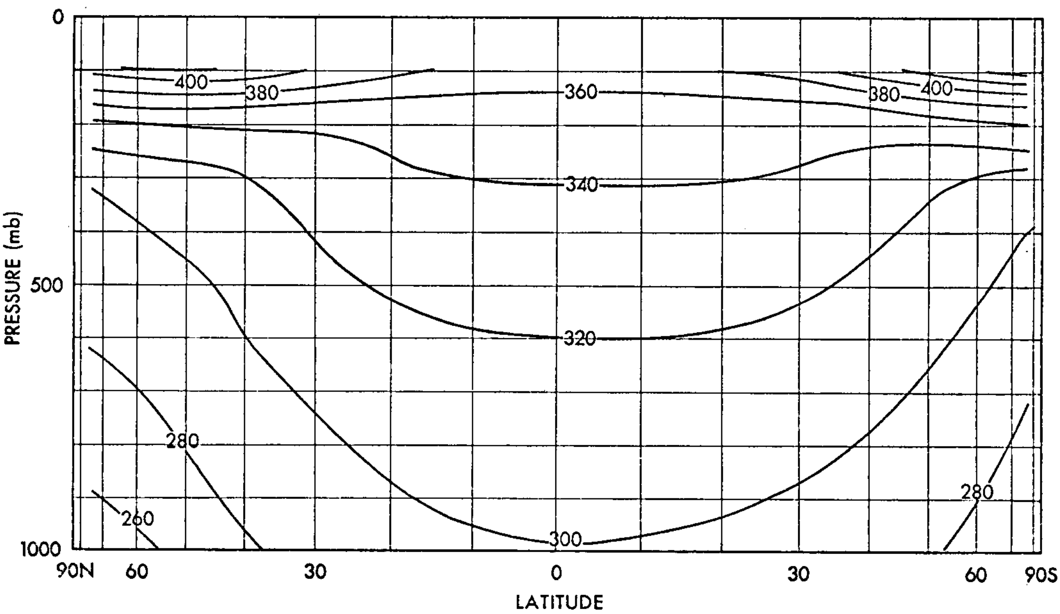

Fig. 1. The distribution of the long-term temporally and zonally averaged potential temperature $[\bar{\theta}]$ for the December–February season (units: °K).

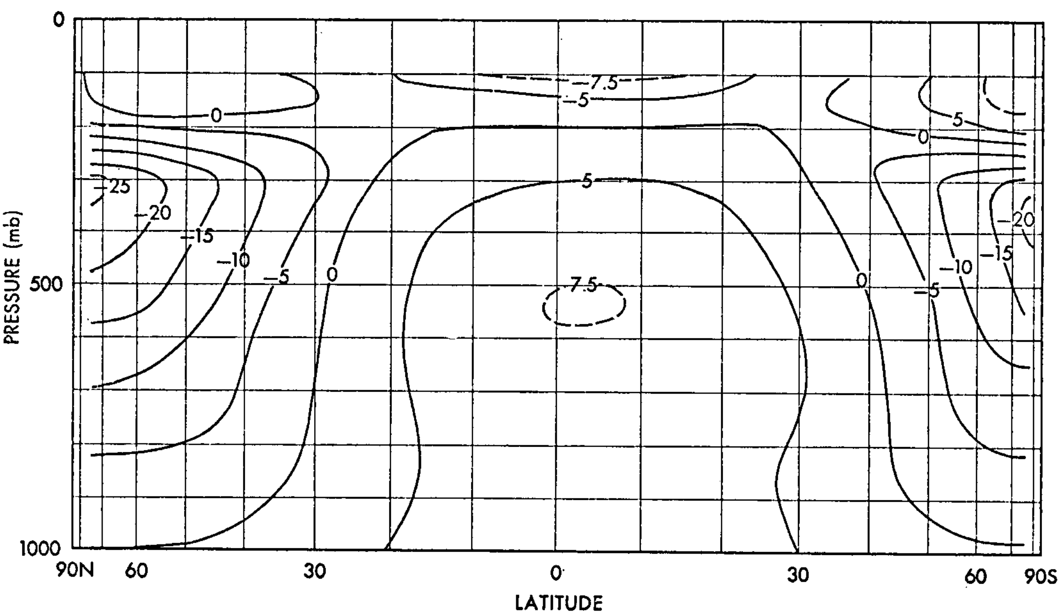

Fig. 2. The distribution of the efficiency factor $[\bar{N}]$ based on the potential temperature distribution of Fig. 1 (units: 10^{-2} deg^{-1}).

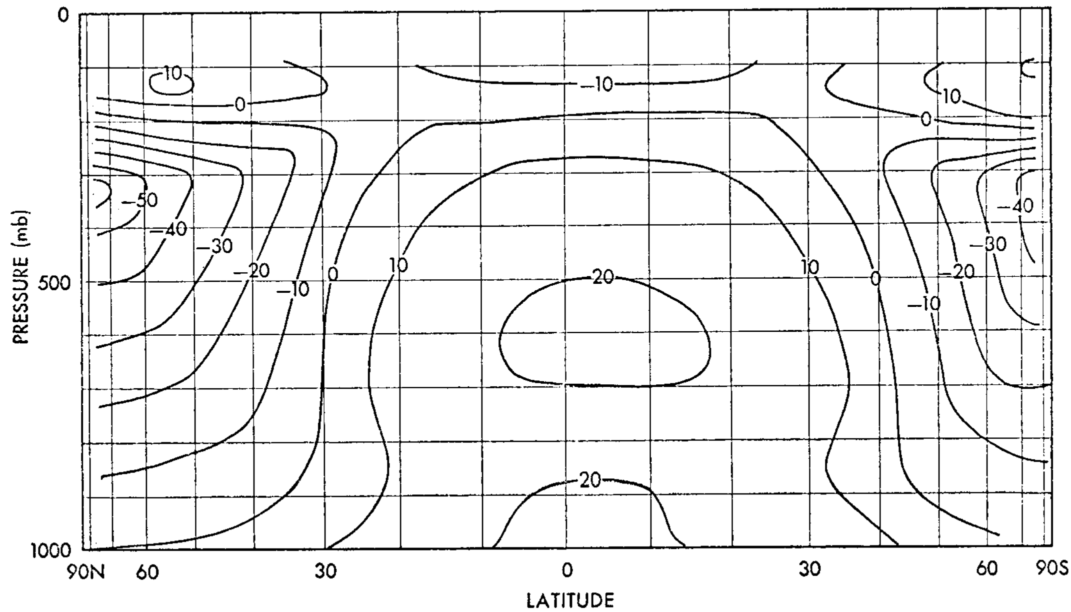

Fig. 3. The integrand of the expression for A_z using the new approximation as given in eq. (23) (units: 10^7 ergs g^{-1}).

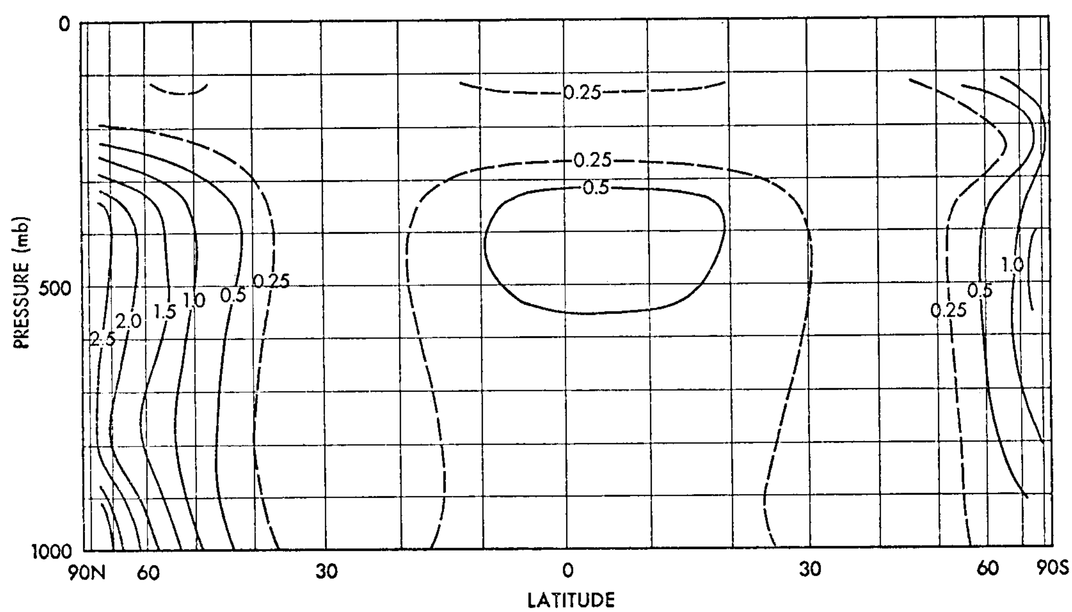

Fig. 4. The integrand of the expression for A_z using the Lorenz approximation as given in eq. (12) (units: 10^7 ergs g $^{-1}$).

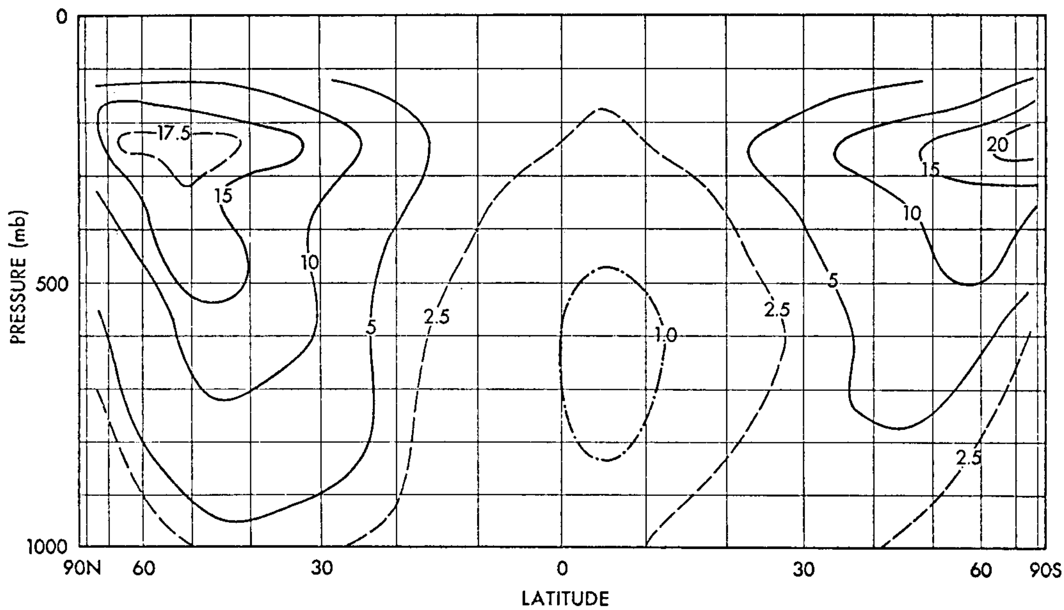

Fig. 5. The integrand of the expression for A_E using the new approximation as given in eq. (24) (units: 10^5 ergs g^{-1}).

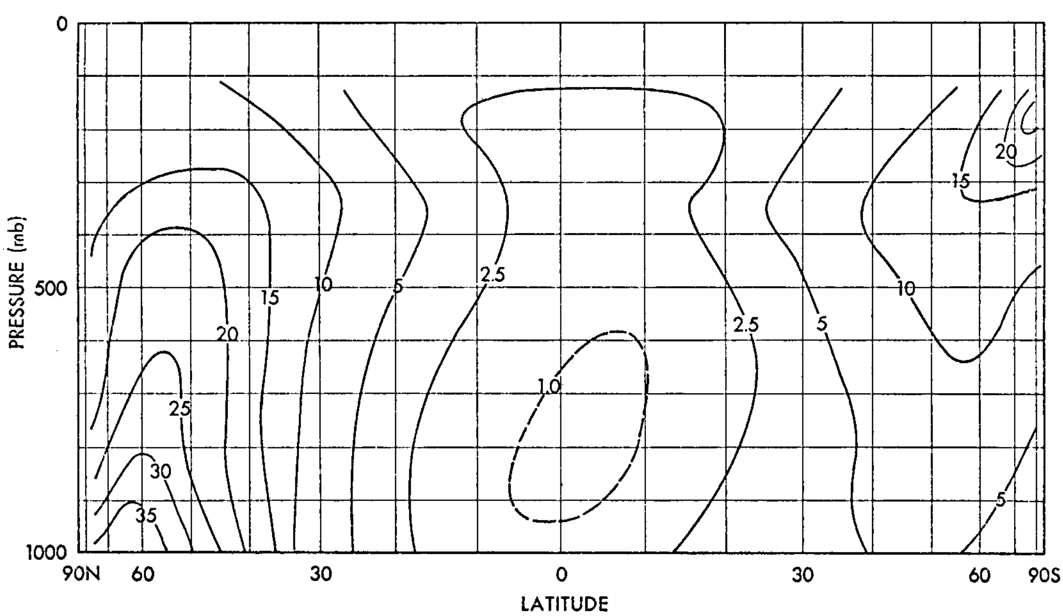

Fig. 6. The integrand of the expression for A_E using the Lorenz approximation as given in eq. (13) (units: 10^5 ergs g^{-1}).

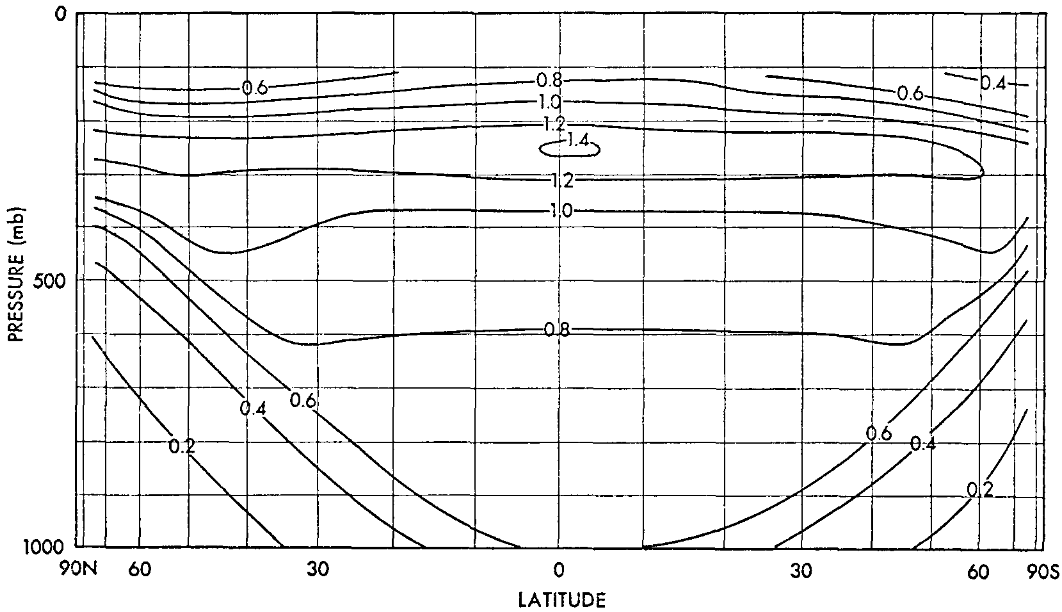

Fig. 7. The distribution of γ for December–February (units: 10^{-2} deg^{-1}).

tential energy; and eq. (28) for the conversion between zonal and eddy forms of available potential energy. The equations preserve the form of the exact equations in pressure coordinates to a large extent, retain the symmetric occurrence of the efficiency factor (or of M) in the expression for A_z and G_z and reduce without approximation to the exact equations when the fields are purely zonal.

The equations have been developed in what Oort (1964) has referred to as the "space domain", where the variables are decomposed into zonal averages and deviations therefrom. They may be simply extended to what he terms the "mixed space–time domain" or the "time domain". Equations for a limited region of the atmosphere may also be obtained (in the manner of Muench (1965) for instance) by retaining suitable boundary terms. This would of course represent a decomposition of Smith's (1969) equations into zonal and eddy forms.

5. Application to data

These equations are presently being applied to long-term general circulation statistics and the results will be presented elsewhere (Newell et al., 1974).

Dutton & Johnson (1967) have made calculations of some of the terms in the available potential energy equations using the exact form in isentropic coordinates. They used daily data from global cross-sections of temperature along the 75° W meridian of longitude for January 1958. They pointed out some difficulties with Lorenz's approximate expression for G_z and also investigated the difference between the available potential energy as calculated from the Lorenz expression and that calculated from the exact expression in isentropic coordinates.

As an illustration of the difference in results which may be obtained using different formulae for available potential energy, A_z and A_E have been calculated using longterm general circulation statistics for the December–February season (Newell et al., loc. cit.). Since these data are decomposed in the "mixed space–time domain" the appropriate expressions become

$$A_Z = \int C_p[\bar{N}]\,[\bar{T}]\,dm$$

$$A_E = \int \tfrac{1}{2} C_p \gamma [\bar{T}^{*2} + \overline{T'^2}]\,dm$$

where

Table 1. *The distribution with pressure of γ_L (in units of 10^{-2} deg^{-1}) based on the distribution of $[\bar{\theta}]$ for the December–February season*

Layer (mb)	1 000–850	850–700	700–500	500–400	400–300	300–200	200–150	150–100
γ_L	0.61	0.68	0.92	1.16	1.34	0.92	0.71	0.63

$$[\bar{N}] = 1 - (p_r([\bar{\theta}])/p)^{\varkappa}$$

An overbar represents a time average and a single prime the deviation therefrom.

Fig. 2 gives the distribution of $[\bar{N}]$ which was calculated from the distribution of $[\bar{\theta}]$ given in Fig. 1. The integrand of the expression for A_z is presented in Fig. 3. The corresponding integrand of the expression for A_z using the Lorenz approximation is given in Fig. 4. They are of course quite different in appearance since the Lorenz approximation depends on the square of a variable and cannot be negative. The expression involving the efficiency factor clearly indicates the local difference between the observed temperature and that of the reference state. The integrated values of A_z for the 1 000–100 mb region obtained from these two formulae differ considerably. The Lorenz approximation gives a value of 203×10^{26} ergs while the formulae using the efficiency factor gives a value of 100×10^{26} ergs. Since this latter result would not be an approximation if the temperature field were purely zonal, it is apparent that the use of the approximate eq. (9) for available potential energy can, with realistic temperature fields, lead to appreciable differences from values obtained with the exact equations. Dutton & Johnson, using the exact and approximate equations with daily data for the Northern Hemisphere, also found the available potential energy to be overestimated by the Lorenz equation. The discrepancy was largest in the winter season but were not as large as that found here. It is quite possible that the use of global data to calculate the reference state would have yielded larger differences in their case.

The integrand of the expression for A_E is given in Fig. 5 for the new approximation and in Fig. 6 using the Lorenz approximation. The difference in results is a consequence of the difference in γ and γ_L. The distribution of γ is shown in Fig. 7 and the corresponding values of γ_L, which do not vary in latitude, are given in Table 1. Apparently γ_L is weighted toward higher pressures and this results in larger contributions to the integrand in the more massive lower levels of the atmosphere. The integrated values are 38.1×10^{26} ergs for the Lorenz formula as compared to 26.4×10^{26} ergs for the new formulation. Estimates of both A_z and A_E are greater using the Lorenz approximations.

The generation and conversion terms may be expected to differ also. As has been mentioned, Dutton & Johnson (1967) have pointed this out for the case of generation and this has been reiterated by Johnson (1970). Results reported in Newell et al. (1970) using both the Lorenz formulation for G_z and that of eq. (25) showed that considerably different results could be obtained using the different formulae.

6. Summary

New approximations for the zonal and eddy forms of the available potential energy equations in pressure coordinates have been obtained. These approximate equations resemble the exact form of the equations to a large extent, preserve the symmetric appearance of the efficiency factor in the various terms, and reduce to the exact equations when the distribution of atmospheric variables is purely zonal.

Estimates of A_z and A_E based on long-term general circulation statistics for the December–February season using the Lorenz approximation and that derived here have been obtained and are found to differ appreciably.

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УРАВНЕНИЯ ДЛЯ ЗОНАЛЬНОЙ И ВИХРЕВОЙ ФОРМ ДОСТУПНОЙ ПОТЕНЦИАЛЬНОЙ ЭНЕРГИИ В КООРДИНАТАХ ДАВЛЕНИЯ

Исходя из точных уравнений, в координатах давления выводятся приближенные уравнения для зональной и вихревой форм доступной потенциальной энергии. Эти уравнения в большой степени сохраняют форму и симметрию точных уравнений и сводятся к ним без всяких приближений, если поток чисто зональный. Выражения для зональной и вихревой доступной потенциальной энергий применяются для вычислений этих величин на основе статистического материала по атмосферной циркуляции для сезона с декабря по февраль. Вычисленные таким образом величины A_Z и A_E несколько отличаются от аналогичных величин, полученных при использовании приближенных уравнений Лоренца.