A note on the β-plane approximation

By R. H. J. GRIMSHAW, *Mathematics Department, University of Melbourne, Parkville, Victoria 3052, Australia*

(Manuscript received July 25, 1974)

ABSTRACT

A rational approximation procedure is used to derive a β-plane approximation for fluid flow on a rotating earth, and to distinguish it from other related approximations. The β-plane equations obtained here have a constant horizontal component of the Coriolis parameter, while the vertical component varies with latitude. This feature of the equations enables the vorticity and angular momentum principles to hold in their usual form. The importance of the horizontal component of the Coriolis parameter is illustrated for internal gravity waves near a critical level.

1. Introduction

The equations governing fluid flow on a rotating earth present considerable mathematical difficulties, partly arising from the use of spherical polar co-ordinates. Consequently, many oceanographers and meteorologists use simpler equations obtained from a suitable approximation. Three approximations in common use are the f-plane, the β-plane and the shallow depth approximation (cf. Veronis, 1973). If L is a length scale of the flow, and R is the earth's radius, then the f-plane is derived from the limit $L/R \to 0$; the resulting equations are expressed in terms of Cartesian co-ordinates whose origin is located on the earth's surface; all terms associated with the earth's curvature are neglected. In the β-plane the same limit $L/R \to 0$ is used, but the curvature terms associated with the Coriolis parameter are retained. In the shallow depth approximation the vertical (radial) length scale is assumed to be much smaller than the horizontal length scale; this approximation is often used in conjunction with the f-plane, or β-plane, approximation, although it is conceptually distinct from both of them.

This note is concerned with the β-plane approximation, and so only the momentum equations need to be examined, as the Coriolis terms do not appear explicitly in the other equations. It is convenient to use Mercator coordinates λ, μ, r, where r is the radius, λ is longitude, and μ is related to the latitude ϕ by the relations

$$\operatorname{sech}\mu = \cos\phi, \tanh\mu = \sin\phi, \sinh\mu = \tan\phi. \quad (1)$$

The distance metric is given by

$$ds^2 = m^2(d\lambda^2 + d\mu^2) + dr^2 \quad (2)$$

where $m = r\cos\phi$ (3)

The velocity components are

$$u = m\frac{d\lambda}{dt}, \quad v = m\frac{d\mu}{dt}, \quad w = \frac{dr}{dt} \quad (4)$$

The equations of motion are

$$\left.\begin{aligned}
&\frac{du}{dt} + \frac{uw}{r} - \frac{uv\tan\phi}{r} + 2\Omega w\cos\phi - 2\Omega v\sin\phi = F_\lambda\\
&\frac{dv}{dt} + \frac{vw}{r} + \frac{u^2\tan\phi}{r} + 2\Omega u\sin\phi = F_\mu\\
&\frac{dw}{dt} - \frac{u^2+v^2}{r} - 2\Omega u\cos\phi = F_r
\end{aligned}\right\} \quad (5)$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{1}{m}\left(u\frac{\partial}{\partial\lambda} + v\frac{\partial}{\partial\mu}\right) + w\frac{\partial}{\partial r}$$

Here Ω is the earth's angular velocity, and F_λ, F_μ, F_r are the components of the force per unit mass (including pressure, gravitational and frictional terms). The f-plane approximation, as commonly used, replaces m by $R\cos\phi_0$, where ϕ_0 is the latitude of the point on the earth's surface selected as the origin; the metric (2) becomes a Cartesian metric, and in the equations of motion (5), the second and third terms (proportional to r^{-1}) are omitted, and ϕ is replaced by ϕ_0 in the Coriolis terms. The β-plane approximation retains ϕ in the Coriolis terms, but is otherwise identical to the f-plane approximation. However, as shown by Phillips (1966), the resulting equations do not possess an angular momentum principle (nor a conservation of vorticity principle, q.v. § 2); the difficulty is due to the horizontal component of the Coriolis parameter (the $2\Omega\cos\phi$ terms in (5)), and it is customary to omit these terms (the so-called "traditional" approximation, q.v. Eckart (1960)). The principal purpose of this note is to show that there is a rational β-plane approximation which leads to equations in which the horizontal component of the Coriolis parameter is retained, and which do possess an angular momentum principle (and a conservation of vorticity principle).

2. A rational β-plane approximation

Let x, y, z be "Cartesian" co-ordinates, defined by

$$Lx = m_0\lambda, Ly = m_0(\mu-\mu_0), Lz = R\log(r/R) \tag{6}$$

where

$$m_0 = R\cos\phi_0$$

Here ϕ_0 is the latitude of the chosen origin on the earth's surface (sech $\mu_0 = \cos\phi_0$ etc.), and L is the length scale of the flow in the vicinity of this origin; x, y, z are not true Cartesian co-ordinates, but become so in the limit $L/R\to 0$. The vertical (radial) co-ordinate z in (6) differs from the usual definition; the current choice is motivated by the observation that it reduces the occurrence of (R/r) factors in the equations of motion (5). If U is a velocity scale, and LU^{-1} is a time scale, then new velocities are defined to be

$$u' = \frac{dx}{dt'},\quad v' = \frac{dy}{dt'},\quad w' = \frac{dz}{dt'} \tag{7}$$

where

$$t = LU^{-1}t'$$

These are related to the old velocities by the relations

$$u = U\frac{m}{m_0}u',\quad v = U\frac{m}{m_0}v',\quad w = U\frac{r}{R}w' \tag{8}$$

It follows immediately that

$$\frac{d}{dt} \equiv \frac{U}{L}\frac{d}{dt'} \tag{9}$$

where

$$\frac{d}{dt'} \equiv \frac{\partial}{\partial t} + u'\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y} + w'\frac{\partial}{\partial z}.$$

F_λ etc. are scaled by a factor U^2L^{-1}, and the dimensionless angular velocity is

$$\Omega' = \frac{\Omega L}{U} \tag{10}$$

Substitution of eq. (6), (7), (8), (9) and (10) into eq. (5) lead to

$$\left.\begin{aligned}
&\frac{du'}{dt'} + \frac{L}{R}\left(2u'w' - 2u'v'\frac{\sin\phi}{\cos\phi_0}\right)\\
&\qquad + 2\Omega'w'\cos\phi_0 - 2\Omega'v'\sin\phi = \frac{m_0}{m}F'_x,\\
&\frac{dv'}{dt'} + \frac{L}{R}\left(2v'w' + (u'^2 - v'^2)\frac{\sin\phi}{\cos\phi_0}\right)\\
&\qquad + 2\Omega'u'\sin\phi = \frac{m_0}{m}F'_y,\\
&\frac{dw'}{dt} + \frac{L}{R}\left(\frac{rw'^2}{R} - (u'^2+v'^2)\frac{r\cos^2\phi}{R\cos^2\phi_0}\right)\\
&\qquad - 2\Omega'u'\cos\phi_0 = \frac{R}{r}F'_z
\end{aligned}\right\} \tag{11}$$

The f-plane approximation is now obtained by letting $L/R\to 0$, while keeping x, y, z, u', v', w' fixed.

The β-plane approximation also consists in letting $L/R\to 0$, while keeping x, y, z, u', v', w'

fixed. However, now $2\Omega' L/R$ is also kept fixed, so that the resulting equations are

$$\left.\begin{aligned} &\frac{du'}{dt'} + 2\Omega' w' \cos\phi_0 - 2\Omega' v' \sin\phi = F'_x, \\ &\frac{dv'}{dt'} + 2\Omega' u' \sin\phi = F'_y, \\ &\frac{dw'}{dt'} - 2\Omega' u' \cos\phi_0 = F'_z \end{aligned}\right\} \quad (12)$$

Here, for consistency, $\sin\phi$ must be expanded in powers of L/R, and so

$$2\Omega' \sin\phi = 2\Omega' \sin\phi_0 + \beta y + O(L/R) \quad (13)$$

where

$$\beta = 2\Omega' \cos\phi_0\, L/R$$

However, unlike the f-plane approximation, the β-plane approximation is not *self consistent*, i.e. it does not commute with other operations, such as differentiation, which may be applied to eqs. (5) and (12). For example, the derivative in the northerly direction is

$$\frac{\partial}{\partial y} \equiv \frac{L}{R} \frac{\cos\phi}{\cos\phi_0} \frac{\partial}{\partial\phi}$$

but

$$\frac{\partial}{\partial y} \lim_{L/R\to 0} (2\Omega' \cos\phi) \neq \lim_{L/R\to 0} \left(\frac{L}{R} \frac{\cos\phi}{\cos\phi_0} \frac{\partial}{\partial\phi} (2\Omega' \cos\phi)\right) \quad (14)$$

This has the consequence that equations derived from (5) by differentiation, such as the vorticity equation, Ertel's theorem etc., are not equal, in the β-plane limit, to their counterparts derived by differentiation of (12). The failure is due to the presence of non-constant metric coefficients in the vector invariant form of differential operators. This property of the β-plane approximation has been noted by Phillips (1966, 1968), who used it to justify the omission of the $2\Omega' \cos\phi_0$ terms from (12). (Phillips wrote the Coriolis terms in a differential form, and made the metric coefficients constant before differentiating.)

However, although the β-plane approximation is not self-consistent, the present β-plane equations are *rational*, in that they have been derived by a rational limiting procedure (rather than an *ad hoc* procedure of making the metric coefficients constant). Further, the present β-plane equations possess an angular momentum principle and a conservation of vorticity principle (although this is *not* the β-plane limit of the original vorticity principle). Eqs. (12) differ from the usual form of the β-plane equations in that the horizontal component of the Coriolis parameter is $2\Omega' \cos\phi_0$, rather than $2\Omega' \cos\phi$. This is a significant difference as $2\Omega' \cos\phi_0$ is a constant and does not vary with y; this has the consequence that some of the reasons for applying the "traditional" approximation, and omitting this term, no longer apply. Indeed, an additional assumption, such as the shallow depth approximation, must be invoked to omit this term (Veronis, 1963, 1968). Of course, this is often permissible, as the presence of strong stratification often ensures that the flow is approximately hydrostatic (thus enabling $2\Omega' \cos\phi_0$ to be omitted from the z-equation in (12)), and horizontal (thus enabling $2\Omega' \cos\phi_0$ to be omitted from the x-equation in (12)). Nevertheless there are situations of geophysical interest, such as equatorial regions, in which it is not permissible to omit $2\Omega' \cos\phi_0$ from eq. (12). Another example of such a situation, involving internal gravity waves, is described in § 3.

Eqs. (12) can be written, in Cartesian vector form

$$\frac{d\mathbf{v}'}{dt'} + 2\boldsymbol{\Omega}' \times \mathbf{v}' = \mathbf{F}' \quad (15)$$

where $\mathbf{v}' = (u', v', w')$ and $2\boldsymbol{\Omega}' = (0, 2\Omega' \cos\phi_0, 2\Omega' \sin\phi)$. (16)

The vorticity equation is derived by applying the vector operator "curl" (expressed in x, y, z co-ordinates) to eq. (15).

$$\frac{d}{dt'}\left\{\frac{\mathbf{w}' + 2\boldsymbol{\Omega}'}{\varrho}\right\} = \frac{(\mathbf{w}' + 2\boldsymbol{\Omega}')}{\varrho} \cdot \nabla \mathbf{v}' + \frac{1}{\varrho} \operatorname{curl} \mathbf{F}' \quad (17)$$

where $\mathbf{w}' = \operatorname{curl} \mathbf{v}'$, and ϱ is the density. This is the usual form of the vorticity equation and implies conservation of absolute vorticity, $\varrho^{-1}(\mathbf{w}' + 2\boldsymbol{\Omega}')$, whenever $\mathbf{F}'$ is irrotational. The vector identity leading to (17) is

$$\operatorname{curl}(2\boldsymbol{\Omega}' \times \mathbf{v}') = 2\boldsymbol{\Omega}' \operatorname{div} \mathbf{v}' - \mathbf{v}' \operatorname{div} 2\boldsymbol{\Omega}' + (\mathbf{v}' \cdot \nabla) 2\boldsymbol{\Omega}' - (2\boldsymbol{\Omega}' \cdot \nabla) \mathbf{v}' \quad (18)$$

The crucial result which enables (17) to follow from (18) is

$$\operatorname{div}(2\mathbf{\Omega}') = 0 \tag{19}$$

This holds because the horizontal component of $2\mathbf{\Omega}'$ is $2\Omega' \cos\phi_0$ and is a constant. In the more usual form of the β-plane equations in which the horizontal component of the Coriolis parameter is $2\Omega' \cos\phi$, (19) and hence (17) would not hold and there would be no principle for the conservation of vorticity. As remarked above, (17) is not the β-plane limit of the original vorticity equation; indeed, this limit produces (17) plus an extra term $L/R\ w' 2\mathbf{\Omega}'$ on the right-hand side. However, this is not a serious defect; the important feature of (17) is that it leads to a conservation of vorticity principle under the usual conditions.

A commonly used result in geophysical fluid dynamics is Ertel's theorem, which states that;

$$\left.\begin{aligned} &\text{if curl } \mathbf{F}' = 0, \ \frac{dS}{dt'} = 0 \\ &\text{then} \\ &\frac{d}{dt'}\left\{\frac{\mathbf{w}' + 2\mathbf{\Omega}' \cdot \nabla S}{\varrho}\right\} = 0 \end{aligned}\right\} \tag{20}$$

Ertel's theorem is an immediate consequence of the vorticity eq. (17), by virtue of the vector identity

$$\mathbf{a} \cdot \frac{d}{dt'}(\nabla S) = \mathbf{a} \cdot \nabla\left(\frac{dS}{dt'}\right) - \nabla S \cdot \{(\mathbf{a} \cdot \nabla)\mathbf{v}'\} \tag{21}$$

Note that Ertel's theorem holds in the present β-plane approximation only because the horizontal component of the Coriolis parameter is a constant.

The original eqs. (5) have an angular momentum principle (Phillips, 1966);

$$\frac{d}{dt}\{r \cos\phi(u + \Omega r \cos\phi)\} = r \cos\phi F_\lambda \tag{22}$$

The β-plane limit of this equation is, after some manipulation,

$$\frac{d}{dt'}\left\{u' \cos\phi_0 + 2\Omega' \cos^2\phi_0\left(z - \frac{R}{L}\log\frac{\cos\phi_0}{\cos\phi}\right)\right\} = \cos\phi_0 F'_x \tag{23}$$

This is also an angular momentum principle for the β-plane eqs. (12); indeed it is just the easterly component (or x-component) of the momentum eqs. (12). Also, it may be shown that the β-plane limit of Kelvin's theorem is

$$\frac{d}{dt'}\oint_C \left\{\mathbf{v}' + \left(2z - 2\frac{R}{L}\log\frac{\cos\phi_0}{\cos\phi}\right)\mathbf{\Omega}' \times \mathbf{k}\right\} \cdot d\mathbf{x} = \oint_C \mathbf{F}' \cdot d\mathbf{x} \tag{24}$$

where C is any closed circuit moving with the fluid, and $\mathbf{k}$ is a unit vector in the vertical direction. Further, it may be shown that (24) may be derived directly from (12), and that

$$\oint_C \left(2z - 2\frac{R}{L}\log\frac{\cos\phi_0}{\cos\phi}\right)\mathbf{\Omega}' \times \mathbf{k} \cdot d\mathbf{x} = \iint_S 2\mathbf{\Omega}' \cdot \mathbf{n}\, dS \tag{25}$$

where S is any surface with C as boundary. The substitution of (25) into (24) leads to the familiar form of Kelvin's theorem on a β-plane (cf. Holton, 1972, p. 63). Again, it should be noted that there would be no counterpart to (23) or (24) if $2\Omega' \cos\phi_0$ were to be replaced by $2\Omega' \cos\phi$.

3. Internal gravity waves on a rotating earth

In this section the effect of the horizontal component of the Coriolis parameter, vis-à-vis the vertical component, will be examined for the particular example of internal gravity waves. The relevant dispersion relation is (O. Phillips, 1966, p. 193)

$$\omega^{*2}(l^2 + m^2 + n^2) = (2\Omega_H m + 2\Omega_V n)^2 + N^2(l^2 + m^2) \tag{26}$$

Here (l, m, n) are the components of the wave-number vector in the east, north and vertical directions respectively, and ω^* is the intrinsic frequency (the frequency relative to the local mean velocity); $2\Omega_H$ and $2\Omega_V$ are the horizontal and vertical components of the Coriolis parameter, and N is the Brunt-Väisälä frequency. Eq. (26) is obtained from the linearised Boussinesq equations, and is to be interpreted as a "WKBJ" approximation (i.e. the wavelength is much smaller than the length scales associated with the Coriolis parameter, the background density gradient, and the mean flow). In most geophysical situations $N^2 \gg (2\Omega_V)^2$ (or $(2\Omega_H)^2$), and the terms in (26) involving the Coriolis parameter are not usually significant. However, as observed by Eckart (1960) and Phillips (1968), the Coriolis terms will dominate when l/n, $m/n \to 0$, and $\omega^* \to \pm 2\Omega_V$; in the approach to this limit, the terms in (26) involving $2\Omega_H$ play a crucial role.

As an illustration it will be assumed that N^2, $2\Omega_H$ and $2\Omega_V$ are constant, and that the mean flow, $V(z)$, is in the easterly direction and varies only with height z. The intrinsic frequency is then

$$\omega^* = \omega - lV(z) \tag{27}$$

where ω is the actual frequency; ω, l, m are constants and eqs. (26) and (27) determine n as a function of z. The wave is confined to the values of z for which

$$(l^2+m^2)(N^2-\omega^{*2})(\omega^{*2}-(2\Omega_V)^2) + (2\Omega_H)^2 m^2 \omega^{*2} \geqslant 0 \tag{28}$$

An internal gravity wave packet (identified by particular values of ω, l, m) has critical levels where $n \to \infty$, that is, at those levels z where ω^* equals $\pm 2\Omega_V$. The vertical component of the group velocity is $W = \partial\omega^*/\partial n$, where

$$(l^2+m^2+n^2)\,\omega^* W = (2\Omega_H m + 2\Omega_V n)(2\Omega_V(l^2+m^2) - 2\Omega_H mn) - N^2 n(l^2+m^2) \tag{29}$$

Consider first the case of no rotation ($2\Omega_H$ and $2\Omega_V$ both zero). As shown by Bretherton (1966), there is a single critical level where $\omega^* = 0$. From (26) and (29),

$$n = \pm\left\{\frac{(N^2-\omega^{*2})}{\omega^{*2}}(l^2+m^2)\right\}^{\frac{1}{2}},$$

$$W = -\frac{N^2 n(l^2+m^2)}{\omega^*(l^2+m^2+n^2)^2} \tag{30}$$

As $\omega^* \to 0$, and $z \to z_c(\omega^*(z_c) = 0)$, $n \to \infty$ as $|z-z_c|^{-1}$ and $W \to 0$ as $|z-z_c|^{-2}$. As the wave packet approaches the critical level, the time $t \to \infty$ as $|z-z_c|^{-1}$, and so the wave packet is effectively captured. The energy density within the wave packet grows as $|z-z_c|^{-1}$, but the total energy of the wave packet decreases as $|z-z_c|$ (Grimshaw, 1972), and there is a transfer of energy to the mean flow; the phenomenon is called critical layer absorption. Of course the linearised equations on which (26) is based will ultimately break down near the critical level; nevertheless this description will presumably remain valid while the wave amplitudes are sufficiently small (indeed, this analysis retains the same meaning when based on the nonlinear equations, as internal gravity waves are transverse waves and so are solutions of the nonlinear equations (Grimshaw, 1972)).

Next suppose that $2\Omega_V$ is not zero, but $2\Omega_H$ is zero (i.e. the "traditional" approximation). This case has been discussed by Jones (1967) without using the "WKBJ" approximation. There are now two critical levels, $\omega^* = \pm 2\Omega_V$. From (26) and (29),

$$n = \pm\left\{\frac{(N^2-\omega^{*2})}{(\omega^{*2}-(2\Omega_V)^2)}(l^2+m^2)\right\}^{\frac{1}{2}},$$

$$W = -\frac{(N^2-(2\Omega_V)^2)\,n(l^2+m^2)}{\omega^*(l^2+m^2+n^2)^2} \tag{31}$$

From (28), the waves are confined to the region where

$$(2\Omega_V)^2 \leqslant (\omega^*)^2 \leqslant N^2. \tag{32}$$

As $\omega^* \to 2\Omega_V$ (the case $\omega^* \to -2\Omega_V$ is similar, and will be omitted), and $z \to z_c$ (now $\omega^*(z_c) = 2\Omega_V$), $n \to \infty$ as $|z-z_c|^{-\frac{1}{2}}$, $W \to 0$ as $|z-z_c|^{-3/2}$, and $t \to \infty$ as $|z-z_c|^{-\frac{1}{2}}$; again the wave packet is captured at the critical level, although the rate of capture is different from that in the case of no rotation. In both cases a wave packet that is propagating towards a critical level is captured.

Finally suppose that both $2\Omega_V$ and $2\Omega_H$ are not zero. Then

$$n(\omega^{*2}-(2\Omega_V)^2) = (2\Omega_H)(2\Omega_V)m \pm \{(N^2-w^{*2})(\omega^{*2}-(2\Omega_V)^2)(l^2+m^2) + (2\Omega_H)^2 m^2 \omega^{*2}\} \tag{33}$$

As $\omega^* \to 2\Omega_V$,

$$n \sim n_1 = \frac{(2\Omega_H)_m}{\omega^*} \tag{34}$$

or

$$n \sim n_2 = -\frac{\{(N^2-(2\Omega_V)^2)(l^2+m^2)+(2\Omega_H)^2m^2\}}{2m(2\Omega_H)(2\Omega_V)} \qquad (35)$$

The first root becomes infinite as the critical level is approached, but the second root is finite. The corresponding forms for the group velocity are

$$W \sim \frac{(2\Omega_H)\,m}{n_1^2}$$

or

$$W \sim \frac{(2\Omega_H)\,m}{(l^2+m^2+n_2^2)} \qquad (37)$$

Thus the first root corresponds to capture ($W \to 0$ as $|z-z_c|^2$, and $t \to \infty$ as $|z-z_c|^{-1}$). However, the second root corresponds to transmission through the critical level. A wave packet is either captured or transmitted at a critical level according to the criterion

$$Wm(2\Omega_H)\begin{cases} <0 & \text{for capture} \\ >0 & \text{for transmission} \end{cases} \qquad (38)$$

The critical level is acting as a valve, a phenomenon recently discussed by Acheson (1973) in other physical contexts. Of course, a wave packet transmitted through a critical level will, in general, be reflected at a level where (28) becomes an equality, leading to a change in sign of W and hence ultimate capture at the critical level, albeit from the opposite direction.

The discussion has been based on the "WKBJ" approximation. An analysis which does not use this approximation, similar to that used by Booker & Bretherton (1967) for the non-rotating case, confirms the crucial role played by $2\Omega_H$. A wave approaching a critical level, which would be captured on the wave packet formalism given above, is in fact partially transmitted, but its wave energy flux is reduced by a factor

$$\exp\left\{-\frac{|2\pi m(\beta+2\Omega_H)|}{|l\beta|}\right\} \qquad (39)$$

where $\beta = V'(z_c)$.

The criterion for capture is, in place of (38),

$$Wm(\beta+2\Omega_H)<0 \qquad (40)$$

Note that, in the "WKBJ" approximation, $\beta < 2\Omega_H$, so this result agrees with (38) in this limit. However, the distinction between the case when $2\Omega_H$ is zero and that when $2\Omega_H$ is non-zero is less clear-cut, and becomes more a difference in degree, rather than in type. A detailed analysis will be contained in a forthcoming paper by the author.

REFERENCES

Acheson, D. J. 1973. Valve effect of inhomogeneities on anisotropic wave propagation. *J. Fluid Mech.* *58*, 27–37.

Booker, J. R. & Bretherton, F. P. 1967. The critical layer for internal gravity waves in a shear flow. *J. Fluid Mech.* *27*, 513–539.

Bretherton, F. P. 1966. The propagation of groups of internal gravity waves in a shear flow. *Quart. J. Roy. Met. Soc.* *92*, 466–480.

Eckart, C. 1960. *Hydrodynamics of oceans and atmospheres.* Pergamon Press, Oxford.

Grimshaw, R. 1972. Nonlinear internal gravity waves in a slowly varying medium. *J. Fluid Mech.* *54*, 193–207.

Holton, J. R. 1972. *An introduction to dynamic meteorolgy.* Academic Press, New York.

Jones, W. L. 1967. Propagation of internal gravity waves in fluids with shear flow and rotation. *J. Fluid Mech.* *30*, 439–448.

Phillips, N. A. 1966. The equations of motion for a shallow rotating atmosphere and "the traditional approximation". *J. Atmos. Sci.* *23*, 626–628.

Phillips, N. A. 1968. Reply (to Veronis, 1968). *J. Atmos. Sci.* *25*, 1155–1157.

Phillips, O. M. 1966. *The dynamics of the upper ocean.* Cambridge University Press.

Veronis, G. 1963. On the approximations involved in transforming the equations of motion from a spherical surface to the β-plane. II. Baroclinic systems. *J. Marine Res.* *21*, 199–204.

Veronis, G. 1968. Comment on Phillip's proposed simplification of the equations of motion for a shallow rotating atmosphere. *J. Atmos. Sci.* *25*, 1154–1155.

Veronis, G. 1973. Large scale ocean circulation, 2–92. *Advances in applied mechanics 13.* Academic Press, New York.

ЗАМЕЧАНИЕ О ПРИБЛИЖЕНИИ β-ПЛОСКОСТИ

Процедура рационального приближения используется для получения проближения β-плоскости для вращающейся земной поверхности и для различения его от других родственных приближений. Полученные здесь уравнения на β-плоскости имеют постоянный горизонтальный компонент параметра Кориолиса, в то время как их вертикальный компонент изменяется в зависимости от географической широты. Вследствие этой особенности уравнений принципы сохранения вихря и углового момента имеют свою обычную форму. Важность горизонтального компонента параметра Кориолиса показана для внутренних гравитационных волн в пределах критического уровня.