Turbulent entrainment across stable density step structures

By E. J. WOLANSKI[1] and L. M. BRUSH, Jr, *Geography and Environmental Engineering, The Johns Hopkins University, Baltimore, Md., USA*

(Manuscript received March 5; revised version July 22, 1974)

ABSTRACT

This paper is a summary of a research on the properties of the turbulent entrainment across stable density-stratified layers, a phenomenon commonly observed in geophysical fluid dynamics. A turbulent environment is created by oscillating grids in quasi-homogeneous layers of slightly different density separated by thin sheets of fluid containing a strong density gradient. The buoyancy flux across thin stable layers is found to be a function of the Richardson and Prandtl numbers. Theoretical and experimental studies are made, leading to a determination of the order of magnitude of the four terms expressing the balance of energy in physical space. It is shown that the turbulent energy flux is of the same order of magnitude as the viscous dissipation of kinetic energy in the quasi-homogeneous layers where the slight density gradient is not altogether negligible because a small but measurable fraction of the turbulent energy flux goes into maintaining the buoyancy flux. Data taken in the stable layer show the simultaneous existence of large internal waves and patchy turbulence. Some properties of these waves as functions of the Richardson number are determined.

1. Introduction

A common observation in environmental fluid dynamics is the simultaneous existence of turbulence and stable buoyancy gradients. Emphasis is put here on the particular case of interfacial mixing, i.e. the turbulent entrainment and mixing of a fluid across relatively thin stable layers, containing a strong density stratification and separating quasi-homogeneous turbulent layers of slightly different density. This phenomenon is observed in the oceanic thermocline step structure (Woods, 1969), in temperature-stratified freshwater lakes and reservoirs (Browand, 1972, along turbidity currents of fine sediment suspensions (Middleton, 1966). It is also observed in the atmospheric boundary layer when forced convection occurs (Lumley & Panofsky, 1964), and probably also in the free atmosphere. It is well known for instance that clear-air turbulence, a major sink of energy in the atmospheric circulation (Fleagle, 1969), is not associated with gravitationally unstable layers of air, but with layers of strong gravitational stability (Buehler, et al., 1969).

The study of interfacial mixing by turbulent entrainment was initiated by Rouse & Dodu (1955), and later by Turner (1968, 1973), Thompson (1969), and others, for the situation where there was no mean velocity shear, as the turbulence was generated by oscillating grids in the quasi-homogeneous layers. Similar studies were also made for the situation where a mean velocity and mean velocity shear were present (Kato & Phillips, 1969; Long, 1970, 1971, 1972, 1973; Moore & Long, 1971; Ellison & Turner, 1960; and others). An attempt is made here to provide an answer to some of the questions arising from these earlier studies. They include, in section 3, the dependence of the entrainment rate on the coefficients of molecular diffusivity $\varkappa$ and molecular viscosity ν, and, in sections 4, 5 and 6, the balance of energy in the turbulent quasi-homogeneous layers and in the thin stable layers.

2. The experimental procedure

All the experiments described here were carried out in a tank identical to the one used

[1] Now, at the Water Pollution Control Branch, 86–88 Georges Street North, Sydney, NSW, Australia 2000.

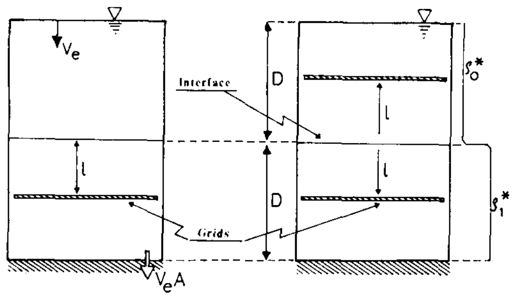

Fig. 1. A sketch of the experimental procedure.

by Turner (1968). The description of the experimental tank was given in detail by Turner. Thus only a brief sketch of the experimental procedure is necessary here. As sketched in Fig. 1, we used removable stirring grids made of 1 cm × 1 cm strips aligned into an array with 5 cm between centres. This grid was used, in case I, to generate turbulence in water of density ϱ_1^*, and of thickness D (20 cm), by oscillating the grid vertically with a fixed amplitude a (generally 1 cm), and a frequency of oscillation ω (ranging between 120 and 360 rpm). A characteristic velocity, $u_* = \omega a$, was defined. Over this layer, at a distance $l(l \gg a)$ from the stirring grid, was a quiescent homogeneous layer of slightly lower density ϱ_0^*. The difference in density, $\Delta\varrho^* = \varrho_0^* - \varrho_1^*$ was created by difference in temperature, or by dissolving salt. It was observed that the separation between the two layers took the form of a thin interface convoluted by the turbulent environment in the quasi-homogeneous lower layer. Mixing across the interface took place and tended to move the interface upwards at a velocity $\mathrm{Ve_I}$, and to incorporate, per unit time, a volume $\mathrm{Ve_I}A$ of fluid from the upper layer into the turbulent layer (A is the horizontal cross-section). This tendency was eliminated by removing fluid at a rate $\mathrm{Ve_I}A$ from the bottom layer. Thus, the mean distance l between the stirring grid and the interface, and the depth D of the turbulent layer were kept constant throughout the experiments.

In case II, sketched in Fig. 1, both quasi-homogeneous layers were stirred, thus stabilizing the interface at mid-distance between the two grids. The entrainment velocity, $\mathrm{Ve_{II}}$, was then measured by measuring the buoyancy flux, Q_0, across the interface, a function of the time rate of change of the density ϱ_1^* in the upper layer. Thus,

$$\mathrm{Ve_{II}} = -\frac{D}{\Delta\varrho^*}\frac{\partial\varrho_1^*}{\partial t} \tag{1}$$

The overall Richardson number Ri, Peclet number Pe, and Reynolds number Re, were respectively

$$\mathrm{Ri} = 4\pi^2 D\Delta\varrho/u_*^2, \quad \mathrm{Pe} = \frac{u_* D}{\varkappa}, \quad \mathrm{Re} = \frac{u_* D}{\nu}$$

where g is the gravity acceleration and $\Delta\varrho = g\Delta\varrho^*/\varrho_0^*$, is the buoyancy. The non-dimensional entrainment velocities were,

$$E_{\mathrm{I}} = \mathrm{Ve_I}/u_* \tag{2}$$

and

$$E_{\mathrm{II}} = \mathrm{Ve_{II}}/u_* \tag{3}$$

Turner (1968) observed that the entrainment velocity remains unchanged whether or not the other side of the stable layer was also turbulent, i.e., that at equal Ri,

$$E = E_{\mathrm{I}} = E_{\mathrm{II}} \tag{4}$$

This result is explained later. Turner also observed that the entrainment rate for the salinity-stratified case becomes, at large Ri, smaller than for the temperature-stratified case. For the former, he suggested the equation,

$$E \propto \mathrm{Ri}^{-1.5} \tag{5}$$

while for the latter,

$$E \propto \mathrm{Ri}^{-1} \tag{6}$$

Brush (1970) investigated the mixing across an interface where the turbulence was generated by a jet directed normal to the interface. His results also supported eqs. (5) and (6), respectively for salinity and temperature stratified fluids.

3. The parameters governing the entrainment rate

In all experiments, the Boussinesq approximation was valid because $\Delta\varrho^*/\varrho_0^*$ was much smaller than one. Then, from dimensional analysis,

$$\frac{\mathrm{Ve}}{u_*} = E = E\left(\mathrm{Ri}, \mathrm{Re}, \mathrm{Pe}, \tau, \frac{b}{a}, \frac{c}{a}, \frac{D}{a}, \frac{l}{a}\right) \quad (7)$$

where b and c are the characteristic lengths of the grid and are always constant in these experiments ($b = 5$ cm, is the distance between successive bars of a grid; $c = 1$ cm, is the dimension of the square bars of a grid), and where $\tau = \omega t$, where t is the time. When all length scales are kept constant,

$$E = E\ (\mathrm{Ri}, \mathrm{Re}, \mathrm{Pe}, \tau) \quad (8)$$

A quasi-steady state is defined by the situation where E is not an explicit function of time, i.e. if

$$\left.\frac{\partial E}{\partial \mathrm{Ri}}\right|_{\tau} \delta\mathrm{Ri} \gg \left.\frac{\partial E}{\partial \tau}\right|_{\mathrm{Ri}} \delta\tau \quad (9)$$

This was verified because, for large values of Ri ($\mathrm{Ri} > 1.0$), $\partial E/\partial\tau)_{\mathrm{Ri}}$ was the only term in eq. (9) that was too small to be measured experimentally (Wolanski, 1972).
Then,

$$E = E(\mathrm{Ri}_{(\tau)}, \mathrm{Re}, \mathrm{Pe}) \quad (10)$$

Therefore a quasi-steady state existed, whereby the Richardson number and the entrainment velocity could be assumed to remain constant throughout a given experiment even though the time rate of change of the mean density in a turbulent quasi-homogeneous layer is not vanishingly small.

It is now argued that, for the range of values of Ri, Re and Pe in this experiment, the parameters Re and Pe enter strongly in eq. (8) only as Re/Pe, or also Pr, where $\mathrm{Pr} = \nu/\varkappa$ is the Prandtl number. Then,

$$E = E(\mathrm{Ri}, \mathrm{Pr}) \quad (11)$$

In other words, one needs to prove

$$\left.\frac{\partial E}{\partial \mathrm{Re}}\right|_{\mathrm{Ri}, \mathrm{Pr}, b, l, D, a, c} \simeq 0 \quad (12)$$

Five experiments were run in the heating experiment and in an identical geometrical situation where $\Delta\varrho^*$ was varied as much as experimentally feasible (by a total factor of 10)

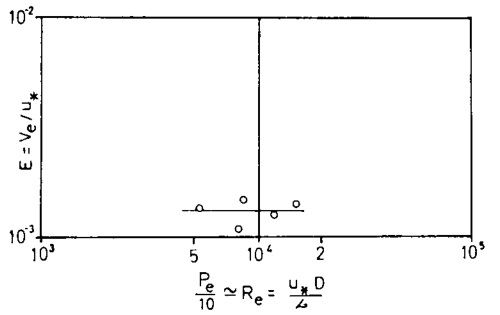

Fig. 2. Plot of the entrainment velocity against Re in the heating experiment, $\mathrm{Ri} = 1.7$.

from one experiment to the other. In order to keep Ri constant from one experiment to the other, the corresponding ω was varied (by a factor of $\sqrt{10}$). Hence, Re was varied by a factor of $\sqrt{10}$ from one experiment to the other while Ri was kept constant. The results are given in Fig. 2, from which it appears that, indeed, E does not vary strongly with Re and Pe for the range of strong stabilities investigated in the heating experiment. In other words, in these experiments, the strong variations of k had a much larger influence on the entrainment velocity by modifying the Prandtl number than by modifying the Reynolds or Peclet number. It was, however, argued by Turner (1973) that E is a *slowly* varying function of Pe (thus also of Re), in the heating experiment, a property that he detected experimentally for a much wider range of Pe and Re.

In order to determine how turbulent entrainment across a density step-structure is governed by the Prandtl number, it was necessary to extend the range of coefficients of molecular diffusivities to values much smaller than the molecular diffusivity of salt in water. For that purpose, the density stratification was created, in order of decreasing molecular diffusivity, by differences in temperature, by solutions of dissolved salt and sugar, and by suspensions of an idealized sediment consisting of monodisperse silica sphere[1] in the hundredth of a micron size

[1] The silica spheres were synthesized by a system of chemical reactions first developed by Stöber et al. (1968), and further studied by the writers.

Table 1. *Coefficients of molecular diffusivity in water at 25°C*

	$\varkappa$(cm²/sec)
Heat	1.4×10^{-3}
Salt	1.1×10^{-5}
Sugar	6.7×10^{-6}
Suspension of Silica spheres (mean diam. = 150 A°)	2.0×10^{-7}
Clay (kaolinite)	$\leqslant 10^{-9}$

range, and, finally, by suspensions of natural kaolinite clay.

In Table 1 are given the corresponding coefficients of molecular diffusivity in water, at 25°C, when $\Delta\varrho^*/\varrho_0^* = 5.0 \times 10^{-3}$. The silica particles in suspension, just like heat, salt and sugar, created the appropriate density differences while modifying by orders of magnitude the molecular diffusivity from one case to the other without increasing in the worst case the coefficient of molecular viscosity by more than 40% from its value for fresh water. The value of the coefficient of molecular diffusivity for an aqueous supension of these monodisperse silica spheres of a diameter of 150 A° was estimated from the Einstein's theory of the Brownian motion.

Fig. 3 shows a plot of the entrainment velocity E, as function of the Richardson number. Experiments were also carried out (Wolanski, 1972) that demonstrated that the slope of the entrainment curve in Fig. 3 was independent of l and of a. This was expected because one can relate the entrainment rates to the properties of the turbulence in the quasi-homogeneous layer but near the stable layer, and because Thompson (1969) apparently found no dependence of these on the Reynolds numbers. Hence, for the range of parameters investigated here,

$$E \equiv \frac{\mathrm{Ve}}{u_*} = \frac{Q_0}{u_* g \Delta\varrho^*/\varrho_0^*} = A\mathrm{Ri}^{-n} \qquad (13)$$

where $A = A(l/a, D/a, c/a, b/a)$, $n = n(\mathrm{Pr})$.

If the geometry of the stirring grid is modified (say, cylindrical bars instead of square bars), Thompson (1969) suggested that the generation of the fluid motion at the level of the grid and the decay of the turbulent intensity away from the grid may not remain independent

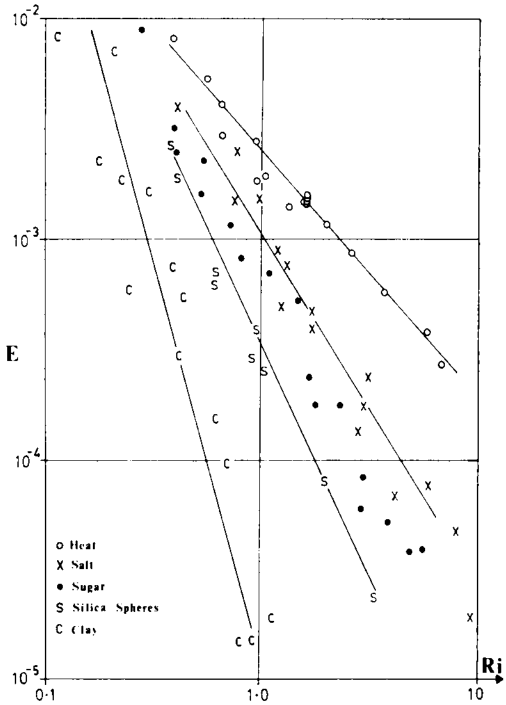

Fig. 3. The entrainment velocity as a function of Ri.

of the Reynolds numbers. In that case, the entrainment rate depends on Pe and Re, as found out by Rouse & Dodu (1955).

4. Buoyancy effects in the stable layer

Observations by the shadowgraph technique revealed the existence of a stable layer of constant thickness $\delta \lesssim 2$ cm for Ri larger than 1 in the salinity stratified case. Because δ is much larger than the effective radius of influence (0.25 mm) of our point conductivity probe, it was possible to record the value of the mean local buoyancy, ϱ, and thus to draw the mean buoyancy profile in the stable layer, as is shown in Fig. 4. From such a profile, one can compute δ/D, where D is the thickness of a quasi-homogeneous layer. This method of evaluating δ/D was repeated at different Ri and the results are given in Fig. 5.

The new notation, $A = O(B)$, is introduced here to express the fact that (A/B) is not a function of Ri. Thus, for a given geometry and a given fluid, eq. (13) implies that A/B is an absolute constant.

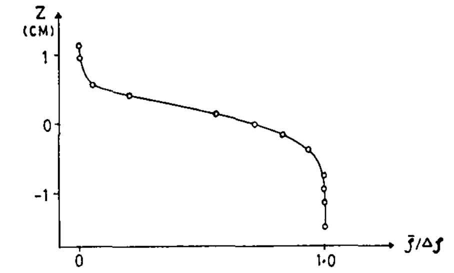

Fig. 4. The mean buoyancy profile in the stable layer.

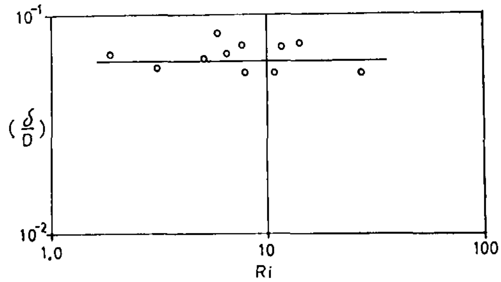

Fig. 5. The thickness of the stable layer.

It is then possible to write, for salt,

$$\frac{\delta}{D} = O(1) \qquad (14)$$

A similar relation was found by Long (1972). There is evidence (Turner, 1973) that eq. (14) is not correct at very large values of Ri and when velocities are so small that, for heat, molecular diffusion is not negligible.

It was also possible to record local buoyancy (salinity) fluctuations ϱ' with time anywhere in the stable layer. The fluctuations were quasiperiodic and corresponded, of course, to the visually observed gravity waves in the stable layer. As described earlier by Long (1973), these fluctuations also varied abruptly at times at the level of maximum curvature of the mean density profile, i.e. at one halfway between the mid-point of the stable layer and the location of the upper surface of the layer. Indeed, at that point, superimposed on the periodic fluctuations, the mean salinity regularly rose suddenly for periods of 1–2 min, then dropped down again to a lower value for longer periods of time. This behaviour was not observed at the mid-point of the stable layer (where $\bar{\varrho}_{zz} = 0$). The cause was probably a creation of a patch of mixed fluid of characteristic size l_p at a level of significant $\bar{\varrho}_{zz}$ and, as suggested by Long (1970), a lowering of that patch at a certain distance L_p,

$$L_p = \bar{\varrho}_{zz} l_p^2 / \bar{\varrho}_z \qquad (15)$$

Because the salinity had a different mean while the patch was identifiable, L_p could be measured from the traces $\varrho'_{(t)}$. Using eq. (15), it was then found that the characteristic dimension l_p of such a patch of mixed fluid was $\frac{1}{2}$ to $\frac{1}{3}$ of the thickness of the stable layer. Hence, the intermittent turbulence, responsible for maintaining the buoyancy flux in the stable layer, was probably created by the overturning of internal gravity waves. It is known (Bretherton, 1969) that the probability of wave breaking to occur from the interaction of packets of internal gravity waves is a function of Ri.

From the recordings of buoyancy fluctuations $\varrho'_{(t)}$ at level ($z = 0$) of maximum density gradient in the stable layer, the root mean square (r.m.s.) amplitude of the internal waves was calculated as follows. Let ξ be the instantaneous vertical displacement of a gravity wave from its level at rest, corresponding to a buoyancy departure ϱ' from the mean. Then, if there is no mixing, one can write for the stable layer,

$$\varrho' = \bar{\varrho}_z \xi \simeq \frac{\Delta\varrho}{\delta} \xi$$

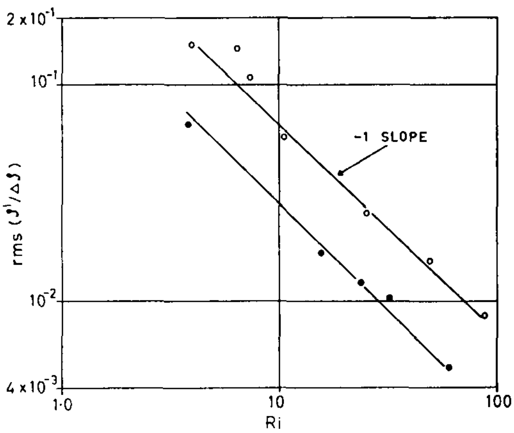

Fig. 6. The r.m.s. buoyancy fluctuations in the stable layer, when both (open circles) or only the bottom (full circles) quasi-homogeneous layers are turbulent.

where subscripts designate partial derivatives, and where it is assumed that $\bar{\varrho}_z \simeq \Delta\varrho/\delta$. The z axis is directed upwards, its origin is located at the level of maximum density gradient in the stable layer. Then, as shown by Fig. 6,

$$\text{r.m.s.}\left(\frac{\varrho'}{\Delta\varrho}\right) = \text{r.m.s.}\left(\frac{\xi}{\delta}\right) = O(\text{Ri}^{-1}) \tag{16}$$

A possible physical explanation for eq. (16) is that the stable layer is essentially passive and is convoluted by turbulent eddies of characteristic kinetic energy $T' \sim O(u_*^2)$ in the homogeneous layers on either side of the interface. For a thin interface, the corresponding increase V' in potential energy,

$$V' = O(\xi\Delta\varrho)$$

The above hypothesis can be written,

$$T' = O(V')$$

so that,

$$\xi/D = O(\text{Ri}^{-1}) \tag{17}$$

The experimental result (eq. 16) can then be found from eqs. (14) and (17).

It is also possible to derive the characteristic time scale T of the motion in the stable layer, at large Ri. One wants to filter out the small (turbulent) time scale components from any flow property, say ϱ', by defining a mean $\tilde{\varrho}'$,

$$\tilde{\varrho}'(\mathbf{x},t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} \varrho'(\mathbf{x},t)\,dt$$

From the recordings of buoyancy fluctuations, we calculated,

$$\eta = \frac{\text{r.m.s.}(\tilde{\varrho}'_t/\Delta\varrho)}{\text{r.m.s.}(\tilde{\varrho}'/\Delta\varrho)} \Bigg/ \left(\frac{\Delta\varrho}{\delta}\right)^{1/2} = O\left(\frac{1}{T\left(\dfrac{\Delta\varrho}{\delta}\right)^{1/2}}\right)$$

For large Richardson numbers, it appears, from the results in Fig. 7, that,

$$\eta = O(1)$$

Thus,

$$T = O\left(\left(\frac{\Delta\varrho}{\delta}\right)^{-1/2}\right) = O(\bar{\varrho}_z^{-1/2}) \tag{19}$$

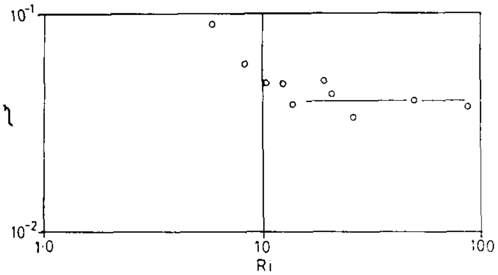

Fig. 7. Plot of η against Ri.

Hence, the dominant time scale of the motions in the stable layer is controlled by the gradient of the buoyancy through the interface. The constant of proportionality in eq. (19) is however, an order of magnitude larger than 2π, its value if the dominant time scale was exactly the Brünt–Vaisälä one. Eq. (19) can also be predicted on theoretical grounds by examining the order of magnitude of characteristic terms in the equations for the conservation of mass and of vertical momentum,

$$\tilde{\varrho}'_t + \tilde{w}'\bar{\varrho}_z + \nabla\cdot\overline{\mathbf{u}'\varrho'} = 0 \tag{20}$$

$$\tilde{w}'_t + \nabla\cdot\overline{\mathbf{u}'w'} = -(\tilde{P}/\varrho_0^*) - \tilde{\varrho}' + \nu\nabla^2\tilde{w}' \tag{21}$$

At large Ri, the buoyancy effects are important, and eqs. (20) and (21) become (Long, 1971),

$$\tilde{\varrho}'_t = O(\tilde{w}'\bar{\varrho}_z) = O(\tilde{w}'\Delta\varrho/\delta) = O(\tilde{\varrho}'/T) \tag{22}$$

$$\tilde{w}'_t = O(\tilde{\varrho}') = O(\tilde{w}'/T) \tag{23}$$

The experimental result (eq. 19 can be found by eliminating $\tilde{w}'$ and $\tilde{\varrho}'$ between eqs. (22) and (23).

5. Buoyancy effects in the quasi-homogeneous layers

The density gradient in the quasi-homogeneous layers is extremely small. Characteristically,

$$(\Delta\bar{\varrho}/\Delta\varrho) \leqslant 0.01$$

where $\Delta\varrho$ is the buoyancy difference across the stable layer, and $\Delta\bar{\varrho}$ is the buoyancy difference across a quasi-homogeneous layer. Because of this reason, and, also, because observations revealed that the quasi-homogeneous layers are fully turbulent, it was assumed initially that the density stratification was not an important parameter for the study of the turbulence in a quasi-homogeneous layer. Then, the buoyancy effects are not affecting the length scale of the energy containing eddies. Thus, the eddy size l_e is proportional to the thickness of a quasi-homogeneous layer (Kato & Phillips, 1969), i.e.,

$$\frac{\partial}{\partial z} = O\left(\frac{1}{D}\right) \tag{24}$$

Let ϱ' and w' be the r.m.s. fluctuations of buoyancy and vertical velocity. Assuming in eq. (21) that the effects of inertia (turbulence) are much larger than the buoyancy effects in the quasi-homogeneous layer, then,

$$\frac{\partial}{\partial z} w'^2 \gg \varrho' \tag{25}$$

As, $\frac{\partial}{\partial z} w'^2 = O(u_*^2/D)$, and, $\varrho' = O(\Delta\bar{\varrho})$

eq. (24) becomes,

$$u_*^2/D \gg O(\Delta\bar{\varrho})$$

or also,

$$\Delta\bar{\varrho}/\Delta\varrho \ll O(\mathrm{Ri}^{-1}) \tag{26}$$

In the thermally stratified case, one can use eq. (6) to write,

$$Q = C_1 \Delta\varrho u_* \mathrm{Ri}^{-1} = \gamma C_q \Delta\bar{\varrho} u_* \tag{27}$$

where $C_{1(z)}$ and $\gamma_{(z)}$ are independent of Ri, and where

$$C_q = \frac{\overline{w'\varrho'}}{(\overline{w'^2})^{\frac{1}{2}}(\overline{\varrho'^2})^{\frac{1}{2}}},$$

is the correlation coefficient in the quasi-homogeneous layer. Eq. (27) is based on the assumption that a characteristic turbulent velocity in the quasi-homogeneous layer near the stable layer is proportional to u_*. While this is true for a *homogeneous* fluid, as was shown by Thompson (1969), there exists no experimental confirmation of this hypothesis in this case where a slight but measurable mean stable density gradient is present. Eqs. (26) and (27) give

$$C_q \gg 1$$

which is physically impossible (Long, 1971). Thus, one must take,

$$C_q = 1$$

and conclude that,

$$\Delta\bar{\varrho}/\Delta\varrho = O(\mathrm{Ri}^{-1}) \tag{28}$$

Hence, parcels of fluid are not simply rising and falling in wave motion. This would imply small values of C_q. Indeed, in pure wave motion in which the density surfaces are not ruptured, C_q vanishes. Let V' be the characteristic potential energy, and T' the characteristic kinetic energy of an energy containing eddy in the quasi-homogeneous layer. Then,

$$\frac{V'}{T'} \equiv O\left(\frac{\bar{\varrho}_z l_e^2}{u_*^2}\right) = O(1) \equiv A^\circ \tag{29}$$

The buoyancy effects are then non-negligible in the quasi-homogeneous layer unless the value of the constant A° is much less than 1. The determination of A° was a complicated problem because the density gradient in the temperature-stratified quasi-homogeneous layers was too small to be measured accurately. However, because of the greater accuracy of our salinity probes that permit accurate reading of $\Delta\varrho^*/\varrho_0^*$ to the 2×10^{-7}, $\Delta\bar{\varrho}$ was measurable in the salinity-stratified case from the records of a salinity probe travelling vertically through the upper quasi-homogeneous layer. As, in the quasi-homogeneous layer,

$$C_q = 1$$

one predicts (for salt),

$$\Delta\bar{\varrho}/\Delta\varrho = O(\mathrm{Ri}^{-1.5}) \tag{30}$$

Eq. (30) was confirmed experimentally as is shown in Fig. 8. at 4 cm above the level of maximum density gradient of the stable layer, thus well inside the quasi-homogeneous turbulent layer, and at a Richardson number of 18, it was possible (see section 6) to evaluate experimentally, in the salinity-stratified case,

$$\frac{T'}{V'} = \frac{0.5\overline{u'^2 + v'^2 + w'^2}}{\bar{\varrho}_z l_e^2/2} \simeq 2.5 \times 10^{-4}$$

Hence,

$$A^\circ \simeq 1.9 \times 10^{-3} < 1$$

This result confirms the validity of eq. (24).

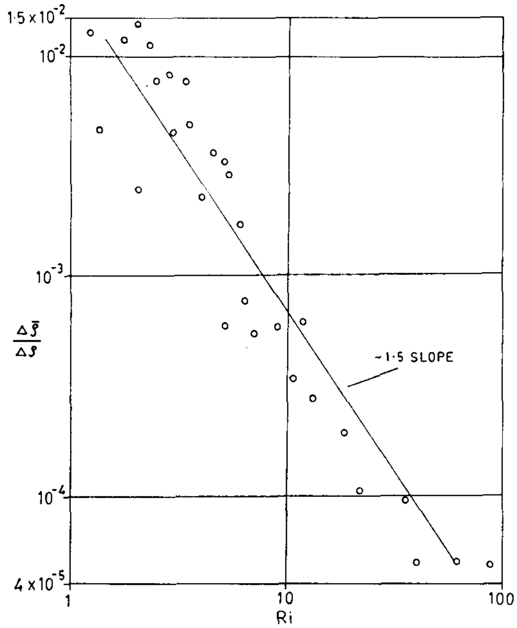

Fig. 8. The buoyancy differences across the quasi-homogeneous layers.

6. The balance of energy

Assuming horizontal homogeneity, the energy equation for a quasi-steady-state is,

$$0 = \underset{\text{I}}{-\frac{\partial}{\partial z}\left(\overline{w'p'/\varrho_0^*}\right)} \underset{\text{II}}{- \frac{1}{2}\frac{\partial}{\partial z}\left(\overline{w'(u'^2+v'^2+w'^2)}\right)} \underset{\text{III}}{+\overline{w'\varrho'}} \underset{\text{IV}}{-\varepsilon} \tag{31}$$

where p' is the pressure perturbation, $\mathbf{u} = (u', v', w')$ is the velocity, $Q = \overline{w'\varrho'}$ is the buoyancy flux, and ε is the viscous dissipation function. It was possible to measure the orders of magnitude of the terms II to IV of eq. (30) at one level ($z = \frac{1}{4}$ cm) in the stable layer, and at one level ($z = 4$ cm) in the quasi-homogeneous layer. The first term (I) could not be measured, but it can be tentatively argued (Stewart, 1969) that I < II for the case considered. Then, I + II $\simeq$ II $\equiv$ TEF is the turbulent energy flux. II was measured from 407 photographed displacements of non-buoyant tracers, a method that can however result in appreciable statistical errors. Thus, only rough estimates of orders of magnitude could be inferred. These data were also used to estimate IV, in the quasi-homogeneous layer only, from the equation (Batchelor, 1953),

$$\varepsilon = \frac{1}{3}\sum_{i=1}^{3} u_i^3/l_e$$

where u_i is the r.m.s. velocity along the i axis, and l_e is the characteristic size of the energy containing eddies, say, also, the integral length scale whose value was estimated from hot wire data by Thompson (1969). Because of the presence of the (weak) density stratification in the quasi-homogeneous layer, eq. (31) gave an upper bound to the actual value of ε.

Finally Q could be estimated from the equation for the conservation of buoyancy (neglecting molecular diffusion),

$$\frac{\partial\bar{\varrho}}{\partial t} + \frac{\partial Q}{\partial z} = 0$$

as the first term in this equation was measured.

It was then found that the buoyancy effects are, as expected from the results of section 5, small in the overall balance of energy in the quasi-homogeneous layers, where,

$$\text{TEF} \simeq \varepsilon \gg Q$$

On the other hand, the buoyancy effects are quite large in the stable layer, where it was found,

$$\beta\text{TEF} = Q$$

where,

$$\beta \sim 1$$

This latter result also agrees well with the data of Thompson (1969) who showed that, in eq. (13),

$$A \sim 1$$

When Ri was scaled with the genuine integral length scale and r.m.s. velocity (measured in a *homogeneous* fluid) of the turbulence in the quasi-homogeneous layer near the stable layer.

At a boundary between the stable layer and the bottom-homogeneous layer,

$$Q = C_Q w'_* \varrho'_*$$

where w'_* and ϱ'_* are the r.m.s. fluctuations in vertical velocity and buoyancy, and C_q is the correlation coefficient. By symmetry, when both quasi-homogeneous layers are turbulent (case II), there is no advection of turbulent kinetic energy from the upper quasi-homogeneous layer to the lower one. Therefore, there is no interaction between the quasi-homogeneous layers across the thin stable layer in case II. Obviously, in case I, there is no flux of energy from the upper quiescent homogeneous layer to the bottom turbulent quasi-homogeneous layer. Hence, there is no interaction between the qnasi-homogeneous layers across the thin stable layer in cases I and II, and w'_*, ϱ'_* and C_q are independent of what happens on the other side of the stable layer. Therefore, eq. (4) holds even though the characteristic motions in the stable layer are very different in case I from case II as is shown in Fig. 6.

7. Summary

An investigation was made of some properties of the entrainment rate across a stable density step structure, i.e. quasi-homogeneous layers of slightly different density separated by thin stable layers containing a strong density gradient. These systems are commonly observed in the oceanic thermocline structure, in the lower atmosphere, in temperature stratified lakes and reservoirs, and along turbidity currents of fine sediment suspensions.

The experiments were carried out in a tank identical to the one used by Turner (1968). Turbulence was generated by vertically oscillating grids and there was no mean velocity shear. The experiments were only carried out for large values of the stability parameters.

The data suggest a systematic dependence of the non-dimensional entrainment rate on the Richardson and Prandtl numbers. The Reynolds and Peclet numbers do not intervene *strongly* at least for the range of high stability investigated.

A determination was also made of the order of magnitude of the terms in the energy equation of the system. It was found that, as expected, the vertical flux of turbulent kinetic energy in the quasi-homogeneous layers is of the same order of magnitude as the rate of viscous dissipation of kinetic energy and much larger than the production of potential energy (buoyancy flux). In the stable layer, on the other hand, the buoyancy flux is an important sink of kinetic energy. It was possible to prove that the buoyancy flux across the surface of separation between a quasi-homogeneous layer and the stable layer, remained unchanged whether or not both quasi-homogeneous layers were stirred. The equality of these buoyancy fluxes was considered somewhat paradoxical in earlier studies.

It was possible to determine experimentally the relation between the amplitude of the gravity waves in the stable layer and the Richardson number. It is believed that this relation indicates that the stable layer was convoluted primarily by large gravity waves qualitatively similar to the waves at the sharp interface between two turbulent homogeneous layers. Because the thickness of the stable layer was finite, the density stratification introduced a Brünt–Vaisälä time scale. This time scale was found to be the dominant time scale of the wave motions in the stable layer where mixing occurs by overturning of patches of stratified fluid involving characteristically $\frac{1}{2}$ to $\frac{1}{3}$ of the thickness of the stable layer.

It was also possible to predict analytically and to verify experimentally the relation between the weak density gradients in the quasi-homogeneous layers and the overall Richardson number.

Acknowledgments

It is a pleasure to thank Professors R. R. Long and O. M. Phillips, of the Johns Hopkins University, for stimulating discussions, their kind advice and assistance. This research was supported by the National Science Foundation, grant number GK-18780- The authors are grateful to Ms J. Grech and Mr W. Kellas for typing the manuscript and preparing the figures.

REFERENCES

Batchelor, G. K. 1953. *The theory of homogeneous turbulence*. Cambridge Univ. Press, 197 pp.

Bretherton, F. P. 1969. Waves and turbulence in stably stratified fluids. *Radio Science 4*, 1279–87.

Browand, F. K. 1972. Theımocline microstructure: field measurements in a fresh water reservoir. In *1972 summer study program in Geophysical Fluid Dynamics at the Woods Hole Oceanographic Institution*, Ref. No. 72–79, Office of Naval Research, Springfield, Va.

Brush, L. M., Jr. 1970. *Artificial mixing of a stratified fluid formed by salt and heat in a laboratory reservoir*. New Jersey Water Res. Inst., 37 pp.

Buehler, W. E., King, C. H. & Lunden, C. D. 1969. Radar echoes from clear air inhomogeneities. In *Clear air turbulence and its detection*. pp. 425–33. Plenum Press, New York.

Ellison, T. H. & Turner, J. S. 1959. Turbulent entrainment in stratified flows. *J. Fluid Mech. 6*, 423–48.

Fleagle, R. G. 1969. The significance of clear air turbulence in large scale meteorology. In »*Clear air turbulence and its detection*, pp. 1–3. Plenum Press, New York.

Kato, H. & Phillips, O. M. 1969. On the penetration of a turbulent layer into a stratified fluid. *J. Fluid Mech. 37*, 643–56.

Long, R. R. 1970. A theory of turbulence in stratified fluids. *J. Fluid Mech. 42*, 349–65.

Long, R. R. 1971. Progress report on an experiment in turbulent, stratified, shearing flow. Unpublished manuscript, Dept. of Mechanics and Earth and Planetary Sciences, The Johns Hopkins Univ., Baltimore.

Long, R. R. 1972. Some aspects of turbulence in stratified fluids, Technical Rep. No. 1, Series C., Dept. of Mechanics and Earth and Planetary Sciences, The Johns Hopkins Univ., Baltimore.

Long, R. R. 1973. Some properties of horizontally homogeneous, statistically steady turbulence in a stratified fluid. *Boundary-Layer Meteorology 5*, 139–57.

Lumley, J. L. & Panofsky, H. A. 1964. *The structure of atmospheric turbulence*, p. 117. Interscience Publ., New York.

Middleton, G. V. 1966. Experiments on density and turbidity currents. I. Motion of the head. *Can. J. Earth Sci. 3*, 523–546.

Moore, M. J. & Long, R. R. 1971. An experimental investigation of turbulent stratified shearing flow. *J. Fluid Mech. 49*, 635–56.

Rouse, H. & Dodu, J. 1955. Diffusion turbulente à travers une discontinuité de densité. *La Houille Blanche 4*, 522–29.

Stewart, R. W. 1969. Turbulence and waves in a stratified atmosphere. *Radio Science 4*, 1269–78.

Stöber, W., Fink, A. & Bohn, E. 1968. Controlled growth of monodisperse silica spheres in the micron size range. *J. Colloid & Interface Science 26*, 62–9.

Thompson, S. M. 1969. Turbulent interfaces generated by an oscillating grid in a stably stratified fluid. Ph.D. thesis, Univ. of Cambridge.

Turner, J. S. 1968. The infleunce of molecular diffusivity on turbulent entrainment across a density interface. *J. Fluid Mech.*, 639–56.

Turner, J. S. 1973. *Buoyancy effects in fluids*. Cambridge Univ. Press.

Wolanski, E. J. 1972. Turbulent entrainment across stable density-stratified liquids and suspensions. Ph.D. Thesis, The Johns Hopkins Univ., Baltimore.

Woods, J. D. 1969. On Richardson's number as a criterion for laminar–turbulent–laminar transition in the ocean and atmosphere. *Radio Science 4*, No. 12, 1289–98.

ПРОНИКНОВЕНИЕ ТУРБУЛЕНТНОСТИ ЧЕРЕЗ СТУПЕНЧАТЫЕ СТРУКТУРЫ В УСТОЙЧИВОЙ СТРАТИФИКАЦИИ ПЛОТНОСТИ

Данная статья суммирует исследования свойств турбулентного проникновения через слои, устойчиво стратифицированные по плотности, явления, часто наблюдаемого в геофизической гидродинамике. Турбулентность создается осциллирующими решетками в квазиоднородных слоях, слегка различающихся по плотности, разделенных тонкими слоями жидкости с сильными градиентами плотности. Найдено, что поток плавучести поперек тонких устойчивых слоев является функцией чисел Ричардсона и Прандтля. Выполнены теоретические и экспериментальные исследования, ведущие к определению порядка величин четырех членов, выражающих баланс энергии в физическом пространстве. Показано, что поток турбулентной энергии того же порядка величины, что и вязкая диссипация кинетической энергии в квазиоднородных слоях, где малый градиент плотности не является полностью пренебрежимым, поскольку малая, но измеримая часть потока турбулентного потока энергии идет на поддержание потока плавучести. Данные, полученные в устойчивом слое, показывают одновременное существование больших внутренних волн и пятнистой турбулентности. Определены некоторые свойства этих волн в виде функций числа Ричардсона.